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T-Polyform Modules

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Abstract

We introduce the notion of t-polyform modules. The class of t- polyform modules contains the class of polyform modules and contains the class of t-essential quasi-Dedekind.

Many characterizations of t-polyform modules are given. Also many connections between these class of modules and other types of modules are introduced.

Keywords: Polyform, modules, essential submodule, t-essentially, quasi-Dedekind

المقاسات المتعددة الصيغ من النمط-T

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الخلاصه

قدمنا مفهوم المقاسات المتعدده الصيغ من النمط-T. هذا الصنف من المقاسات المتعدده الصيغ من النمط-T يحتوي على صنف المقاسات المتعددة الصيغ ويحتوي على المقاسات شبه الديدكانديه من النمط-T.عدة تشخيصات للمقاسات المتعدده الصيغ من النمط-T قد اعطيت وكذلك عدة روابط بين هذا الصنف من المقاسات وانواع اخرى من المقاسات قد قدمت .

Introduction

Throughout the paper, rings will have a nonzero identity element and modules will be unitary right modules. We first briefly review some background materials relevant to the topics discussed in this paper.

Recall that, a submodule N of an R-module M is called essential submodule of M(briefly $N \underset{ess}{\leq} M$) if for each nonzero submodule W of M , N∪W≠ 0 [1]. Equivalently $N \underset{ess}{\leq} M$ if whenever W ≤ M, N∪W=(0) implies W=(0) [1] A submodule N of M is called closed (denoted by $N \underset{c}{\leq} M$) if has no proper essential extension in M; that is, if $N \underset{ess}{\leq} W \leq M$, then N=W [1]. Ashari et. al in [2], introduced the concept of t-essential submodule, where a submodule N of M is called t-essential (briefly $N \underset{tes}{\leq} W$) if whenever W ≤ M, N∩W⊆Z_2(M), then W⊆ Z_2(M) where Z_2(M) is the second singular submodule of M and defined by $Z\left(\frac{M}{Z(M)}\right) = \frac{Z_2(M)}{Z(M)}$ [1]. It is well known that $Z(M) = \{m : mI = 0, \text{for some} I \underset{ess}{\leq} R\}$

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.Equivalententy $Z(M)=\{m\in M: ann(m) \underset{ess}{\leq} R\}$,[1] where $ann(m)=\{r \in R: mr = 0\}$ .Similarly $Z_2(M)=\{m\in M: mI=0, \text{for some} I \underset{tes}{\leq} R\}=\{m\in M: ann(m)\underset{tes}{\leq} R$

Obviously, every essential submodule is t-essential, but not conversely, for example the submodule $(\bar{4})$ of the z-module Z_{12} is t-essential but not essential.

However, the two concepts are equivalent if M is nonsingular (ie Z(M)=0). A module M is called singular if Z(M)= M and is called Z_2-torslon if $Z_2(M) = 0$. If $A\leq M$ then $Z_2(A) = Z_2(M) \cap A$

Asgari..etc, in [2], introduced the concept t-closed submodule where a submodule N of an R-Module M is t-closed (denoted by $N \underset{tc}{\leq} M$ if N has no proper t-essential extension in M. It is clear that every t –closed submodule is closed, but the converse is not true for example $(\bar{0})$ is closed in Z_8 as Z-module but it is not t-closed. The two concepts closed submodule and t-closed submodule are coincide in nonsignular modules.

An R-module M is called polyform if for each L≤ M and for each ϕ: L→M, $\text{Ker}\phi \underset{ess}{\leq} L$ implies ϕ=0 (i.e if$\phi\neq0$, then $\text{Ker}\phi \underset{ess}{\not\leq} L$). [3, 4].

Rizvi in [5] introduced the nation of k-nonsingular module, where an R-module M is called K-nonsingular if $\phi\in$End (M), $\text{Ker}\phi \underset{ess}{\leq} M$ implies ϕ=0, where End (M) means the ring of endomorphism on M.

It is clear that polyform module implies K-nonsingular but not conversely see [5].

Thaa'r in [4] gave the notion of essentially quasi- Dedekind modules as a generalization of quasi – Dedekind modules by restricting the definition of quasi-Dedekind modules (which is introduced in [6] on essential submodules, where an R-module M is called essentially quasi-Dedekind if $\text{Hom}(\frac{M}{N}, M)=0$ for each $N \underset{ess}{\leq} M$ (that is M is essentially quasi- Dedekind if every $N \underset{ess}{\leq} M$, N is quasi-invertible. Thaa'r in [7]proved that k-nonsingular modules and essentially quasi-Dedekind are coincided.

F,S and Inaam in [8] introduced the notion of t-essentially quasi-Dedekind where an R- module M is called t-essentially quasi-Dedekind (Shortly t-ess.q-Ded) if $\text{Hom}(\frac{M}{N}, M)=0$ for each $N \underset{tes}{\leq} M$. Equivalently M is t-ess. q-Ded if for each $\phi\in$End (M) with $0\neq \text{Ker}\phi \underset{tes}{\leq} M$ implies ϕ = 0 [8].

It is obvious that every t-ess. q.Ded module is ess. q-Ded, but not conversely [8,Rem&Ex.2.2(2)].

In the present paper, motivated by these works, we introduce and study t-polyform modules as follows: An R-module M is called t-polyform if for each L≤M, and ϕ:L→M, $\text{Ker}\phi \underset{tes}{\leq} L$ implies ϕ=0. Then we have

If M is t-polyform then M is polyform model and if M is t-polyform then M is t-ess q-Ded module and none of these implications is reversible (see Rem& Ex.3.2(1),(3))

We give many properties and characterizations of t-polyform modules which are analogous to that of polyform modules (See Rem 3.2(3),Th.3.6,Th.4.7)

Also, many connections between t-polyform module and other types of modules are presented (see Theorems 3.3,3.4,4.1 and 4.4).

Next note that our notion ((t-polyform modules)) is different from (st-polyform modules) which is appeared recently in [9] as we explain that in S.3, Note 3.5

2-Preliminaries

We list some known results which are relevant for our work.

Lemma 2.1 [2]

The following statements are equivalent for a submodule A of an R-module M.

1. $A \underset{tes}{\leq} M$,
2. $A+ Z_2(M) \underset{ess}{\leq} M$,

3. $\frac{A+Z_2(M)}{Z_2(M)} \underset{ess}{\leq} \frac{M}{Z_2(M)}$

4. $\frac{M}{A}$ is Z_2-torsion (i.e $Z_2(\frac{M}{A}) = \frac{M}{A}$)

Lemma 2.2 [10]

Let A_λ be a submodule of M_λ for each $\lambda \in \wedge$. Then

1.If $\wedge$ is a finite set and $A_\lambda \underset{tes}{\leq} M_\lambda$, then $\bigcap_{\lambda\in\wedge} A_\lambda \underset{tes}{\leq} \bigcap_{\lambda\in\wedge} M_\lambda$,

2. $\underset{\lambda\in\wedge}{\oplus} A_\lambda \underset{tes}{\leq} \oplus M_\lambda$ if and only if $A_\lambda \underset{tes}{\leq} M_\lambda$, $\forall\lambda \in \wedge$

Lemma 2.3 [10]

Let A≤B≤M. Then $A \underset{tes}{\leq} M$ if and only if $A \underset{tes}{\leq} B$ and $B \underset{tes}{\leq} M$

Lemma 2.4 [2]

Let M be an R-module. Then

1. If $C \underset{tc}{\leq} M$ then $Z_2(M) \leq C$
2. $(0) \underset{tc}{\leq} M$ if and only if M is nonsingular.
3. If $A \leq C \leq M$, then $C \underset{tc}{\leq} M$ if and only if $\frac{C}{A} \underset{tc}{\leq} \frac{M}{A}$

Lemma 2.5 [2]

Let C be a submodule of an R-module M. Then the following statements are equivalent:

1. There exists a submodule S such that C is a maximal with respect to the property C∩S is Z_2-torsion.
2. $C \underset{tc}{\leq} M$.
3. C contain $Z_2(M)$ and $\frac{C}{Z_2(M)} \underset{c}{\leq} \frac{M}{Z_2(M)}$
4. C contains $Z_2(M)$ and $C \underset{c}{\leq} M$
5. C is a complement of a nonsingular submodule of M.
6. $\frac{M}{C}$ is nonsignular.

Lemma 2.6 [2]

Let $M = \underset{\alpha\in\wedge}{\oplus} M_\alpha$ Where $M_\alpha \leq M$ for each $\alpha \in\wedge$. Then $Z_2(M) = \underset{\alpha\in\wedge}{\oplus} Z_2(M_\alpha)$

3- t-polyform Modules

Definition 3.1: An R-module M is called t-polyform if for each $L \leq M$ and $\phi:L\to M$, $\phi \neq 0$, then $\text{Ker}\,\phi \underset{tes}{\not\leq} L$. A ring R is said to be right t-polyform if the module R_R is t-polyform.

Remarks and Examples 3.2

1.Every t-polyform module is polyform, since every essential submodule is t-essential. However, the converse is
not always true for example:
Let M be the Z-module Z_6 since M has no proper essential submodule, then M is polyform.
But M is singular hence M is Z_2-torsion and so every submodule, $0\neq L\leq M$ is Z_2-torsion and hence $Z_2(L)=L$. Now for each $0\neq \phi: L \to M$, $\text{Ker}\phi + Z_2(L)=\text{Ker}\phi=L$ and hence $\text{Ker}\,\phi \underset{tes}{\leq} L$ (by Lemma 2.1)
Thus $M=Z_6$ is not t-polyform

2. It is known that every semisimple module is polyform,but it is not necessary t-polyform, see the example
in(1).

3. It is clear that every t-polyform module is t-ess.q. Ded. However, the converse may be noted true in general, for
example: Let $M=Z_p$ as Z-module, where P is a prime number. For each $0\neq f: Z_p \to Z_p$

Since f$\neq$ 0, and M is simple so Kerf=(0) and hence by lemma 2.1, Kerf=(0) $\underset{tes}{\leq} M$,Since Kerf+ Z_2 (M)=M $\underset{ees}{\leq} M$. Thus M is not t.polyform. But M is t-ess.q.Ded. Since for each f:M $\rightarrow$ M, with 0$\neq$ Kerf $\underset{tes}{\leq} M$ implies Kerf = M and so f=0.

4. Recall that every nonsingular module M(i.e Z(M)=0) is polyform. Also every nonsingular module M is t-polyform

Proof : Let L$\leq$ M,ϕ:L$\rightarrow$M and $\phi\neq$ 0. Since M is polyform Ker$\phi \underset{tes}{\nleq} M$. But M is nonsingular, hence Ker$\phi \underset{tes}{\nleq} M$.

In particular each of the Z- module: Z,Q,Z$\oplus$Z,Q$\oplus$ Q, Z[X] is t-polyform module, also for each prime number P, Z_p as Z_p- module is t-polyform.

5. Every singular M(hence M is Z_2-torsion (Z_2(M)=M)) is not t-polyform module

Proof: Let L$\leq$ M, ϕ:L$\rightarrow$M and $\phi\neq$ 0. Hence Kerϕ+$Z_2(M)$ = Kerϕ + M = $M \underset{ees}{\leq} M$, so Kerϕ $\underset{tes}{\leq} M$ by lemma 2.1. Thus M is not t-polyform.

6. Prime module need not be t-polyform, for example M= $Z_2 \oplus Z_2$ as Z-module is prime and M is not t-polyform since M is singular. However evey prime faithful module is nonsingular, hence it is t-polyform by part (4).

7. Every submodule N$\neq$ 0 of t-polyform module M is t-polyform.

Proof: Let $0 \neq L \leq$ N and let f:L$\rightarrow$N, $f \neq 0$. Then, 0$\neq$ $i$of: L$\rightarrow$ M where i is the inclusion mapping from N into M.

Since M is t-polyform then Ker($i$of) $\underset{tes}{\nleq} L$.

But it is easy to check that Kerf=Ker(iof) and hence Kerf $\underset{tes}{\nleq}$L. Thus N is t-polyform.

In particular if $\overline{M}$ (quasi-injective hull of M) or E(M) (injective hull of M), then M is t-polyform.

8. A homomorphic image of t-polyform module is not necessarily t-polyform, for example the Z-module Z is t-polyform. Let $\pi : Z \rightarrow Z/(6) \approx Z_6$ where π is the natural epimorphism, but Z_6 is not t-polyform by part(1).

9. If M is a t-polyform R-module and $N \underset{tc}{\leq} M$ then $\frac{M}{N}$ is t-polyform.

Proof: Since $N \underset{tc}{\leq} M$, $\frac{M}{N}$ is nonsignular by lemma (2.5). Hence $\frac{M}{N}$ is t-polyform by part (4).

10.Recall that an R-module is Co-epi-retractable if for each N$\leq$ M, there exists K$\leq$ M such that $\frac{M}{N} \simeq$ K [11, 12].

If M is t-polyform and Co-epi-retractable, then $\frac{M}{N}$ is t-polyform, for each N$\leq$ M.

Proof: it follows directly

The following theorem is a characterization of t-polyform modules.

Theorem 3.3 An R-module M is t-polyform if for each 0$\neq$ L $\leq$ M and 0$\neq$ ϕ: $L \rightarrow$ M, $\mathrm{Ker}\varphi \underset{tc}{\leq} L$

Proof: Suppose there exist 0$\neq$ $L \leq$ M and 0$\neq$ $\phi : L \rightarrow$ M, but $\mathrm{Ker}\varphi \underset{tc}{\nleq} L$. By definition of t-closed submodule, there exists U$\leq$ L such that U is a proper t-essential extension of kerf.

Then $\phi \circ i : U \to M$ where *i* is the inclusion mapping from U into L. Clearly $Ker(i \circ \phi) \underset{tes}{\leq} Ker\phi$, so that $Ker(\phi \circ i) \underset{tes}{\leq} U$. Hence $\phi \circ i$ =0 since M is t-polyform. It follows that $\phi(U)=0$; that is U≤ Kerf which is a contradiction. Thus $Ker\phi \underset{tc}{\leq} L$

Conversely, suppose there exist $L \leq M$ and $0 \neq \phi: L \to M$ with $Ker\, f \underset{tes}{\leq} L$. But $Ker\phi \underset{tc}{\leq} L$ by hypothesis, so Ker ϕ=L which implies $\phi = 0$ which is a contradiction. Thus $Ker\, f \underset{tes}{\not\leq} L$ and So M is t-polyform.

The following is another characterization of t-polyform modules

Theorem 3.4 Let M be an R-module. Then M is t-polyform if and only if for each $0 \neq N \underset{tes}{\leq} M$ and for nonzero f∈Hom (N,M), then $kerf \underset{tes}{\not\leq} N$

Proof:(□) it is clear

(□)Let N≤M, If $N \underset{tes}{\leq} M$ then nothing to prove if $N \underset{tes}{\not\leq} M$, let $f: N \to M, f \neq 0$. Since $N \underset{tes}{\not\leq} M$ then $N \underset{ess}{\not\leq} M$. Hence there exists K (a relative complement) of N and so that $N \oplus K \underset{ess}{\leq} M$. which implies $N \oplus K \underset{tes}{\leq} M$. Define g: N⊕k → M by g(n+k)= f(n), n∈N, k∈k.g is well-defined and $g \neq 0$. By hypothesis, $ker\, g \underset{tes}{\not\leq} N \oplus K$ But Kerg=Kerf⊕K and so that $ker\, f \underset{tes}{\not\leq} N$ by lemma 2.2 (2). Thus M is t-polyform.

The notion of ((st-polyform modules)) appeared in [9], where an R-module M is called st-polyform if for each $0 \neq L \leq M, 0 \neq \phi: L \to M$ $ker\, f \underset{st.c}{\leq} L$. A submodule U of M is called st-closed($U \underset{st.c}{\leq} M$) if U has no proper semiessentiall extension of U , and a submodule U of M is called semi-essential in M if U has nonzero intersection with any nonzero prime submodule

Note 3.5

The two concepts (t-polyform modules) and (st-polyform modules) are independent as we can see by the following examples.

1. Z_6 as Z-module is not t-polyform (see Rem 3.2(1)) and it is is st-poly by [5,Rem.3(vii)]
2. Z as Z-module is t-polyform (See Rem 3.2.(4)), and it is not st-polyform [see 5, Ex.5(ii)]

[4] gave the following; An R-module M is polyform if and only if every essential submodule is rational, where a submodule N of M is called rational in M(briefly $N \underset{r}{\leq} M$) if Hom $(\frac{V}{N}, M)$ =0for each $N \leq V \leq M$ [1].

Note that every rational submodule is essential but not conversely [1]

We give the following:

Theorem 3.6 An R-module M is t-polyform implies every nonzero t-essential submodule of M is rational.

Proof: Assume $0 \neq N \underset{tes}{\leq} M$ and f∈Hom $(\frac{V}{N}, M)$,where $N \leq V \leq M$. Then $f \circ \pi \in Hom(v, M)$ where π is the natural epimorphism from V onto $\frac{V}{N}$. Hence $N \leq ker(f \circ \pi)$, but $N \underset{tes}{\leq} M$ implies $ker(f \circ \pi) \underset{tes}{\leq} M$ by lemma (2.3).So that $ker(f \circ \pi) \underset{tes}{\leq} V$ (since $ker(f \circ \pi) \subseteq V$) . Since M is t-polyform , $f \circ \pi$ =0, and hence f=0. Thus Hom $(\frac{V}{N}, M)$ =0 that is $N \underset{r}{\leq} M$.

Remark 3.7 The converse of theorem (3.6) is not true in general, for example:

The Z-module Z_6 is not t-polyform, but Z_6 has only Z_6 as t-essential submodule of Z_6 and $Z_6 \underset{r}{\leq} Z_6$.

However, we have:

Theorem 3.8 if M is an R-module such that every nonzero t-essential submodule is rational, then M is polyform.

Proof: Let $N \underset{ess}{\leq} M$, hence 0≠ $N \underset{ess}{\leq} M$. Then by hypothesis is $N \underset{r}{\leq} M$. Thus every essential submodule is rational. It follows that M is polyform.

Recall a nonzero R-module M is called monoform if for each 0≠ N ≤ M and for each 0≠ f ∈ Hom(N,M),then ker f=0 , [9].
Equivalently a nonzero R-module M is monoform if for each nonzero submodule N of M, $N \underset{r}{\leq} M$, [9].

It is known that every monoform is polyform . Now we ask the following: Is there any relation between t-polyform modules and monoform?

Consider the following remarks

Remarks 3.9

1. t-polyform modules need not be monoform, for example: The Z-module $Z \oplus Z$ is t-polyform (Rem 3.2.(4)), but it is not monoform since there exists $f: Z \oplus 2Z \to Z \oplus Z$ such that f(x,y)=(y,0) for each x∈Z, y∈2Z then Kerf= $Z \oplus (0)$ ≠zero submodule.
2. Monoform module may be not t-polyform module, for example: The Z-module Z_p, where p is a prime number, is monoform but it is not t-polyform.

We introduce the following

Definition3.10 An R-module M is called t-essentialy monoform (shortly t-ess- mono) if for each 0≠ $N \underset{tes}{\leq} M$ and 0 ≠ f ∈ Hom(N,M) then kerf=0 .

Every simple module is t-ess mono and every monoform module is t-ess. mono.

Proposition 3.11: Let M be a t-ess-mono. module. Then M is quasi-Dedekind and hence M is t-ess.q.-Ded.

Proof: Since $M \underset{tes}{\leq} M$ and M is t-ess-mono, the for each 0 ≠ f ∈End (M) implies kerf=0 Thus M is quasi-Dedekind by [6 ,Th1.5,p.26] and hence M is t-ess-q-Ded.

By th.(3.6), We have: If M is t-polyform, then for each 0≠ $N \underset{tes}{\leq} M$ implies $N \underset{r}{\leq} M$.

Now we give the following

Proposition 3.12: If M is t-ess-mono. R-module, then for each 0≠ $N \underset{tes}{\leq} M$ implies $N \underset{r}{\leq} M$.

Proof: Suppose there exists 0≠ $N \underset{tes}{\leq} M$ but $N \underset{r}{\not\leq} M$ Hence there exists V⊐N such that $Hom(\frac{V}{N}, M) \neq 0$,so Let f∈ $Hom(\frac{V}{N}, M)$,f≠0.It follows that foπ∈ Hom(V, M), where π is natural epimorphism from V onto $\frac{V}{N}$, and foπ≠0 (Since f ≠ 0). But N⊆V, hence $V \underset{tes}{\leq} M$ and since M is t-ess-mono, Ker(foπ)=0. Since N⊆Ker(foπ)=0 thus N=0 which is a contradiction therefore $N \underset{r}{\leq} M$.

Corollary 3.13: Let M be a t-ess-mono. Then M is polyform.

Proof: It follows by prop.(3.12) and Th.(3.8)

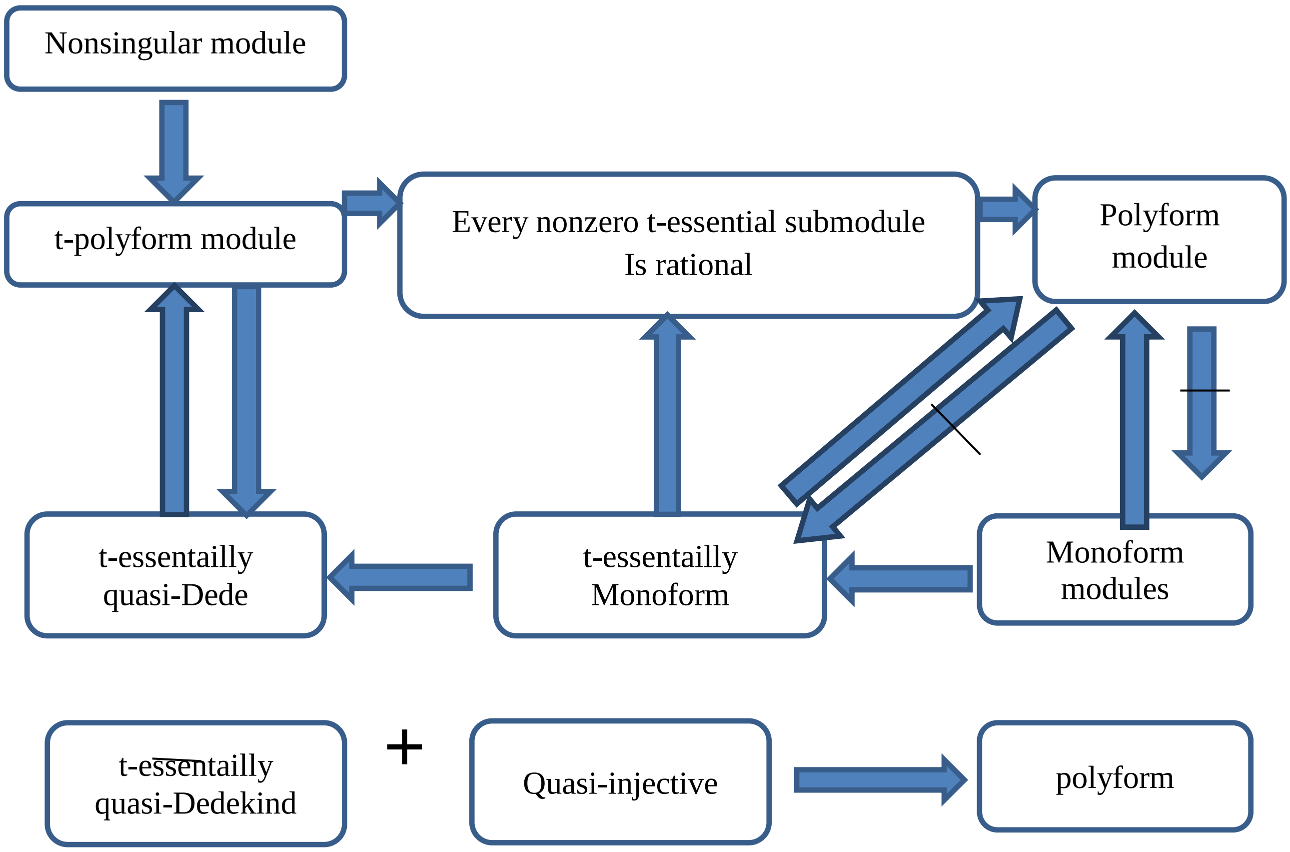
Proposition 3.14: Let M a quasi-injective R-module. I f M is t-ess.q.Ded, then for each $0 \neq N \underset{tes}{\leq} M$ implies $N \underset{r}{\leq} M$

Proof: Let $0 \neq N \underset{tes}{\leq} M$ Since M is t-ess,q-Ded, Hom $(\frac{M}{N}, M)$ =0;that is N is a quasi-invertible submodule of M. Since M is quasi-injective, then by [6,Th3.5 p.16], M is a rational extension of N; that is $N \underset{r}{\leq} M$.

Corollary 3.15: Let M be a quasi-injective if M is t-ess.q.Ded, then M is polyform

Proof: It follows by prop .(3.14) and Th.(3.7)

We can Summarize results of S.3 by the following tables



$.4 More about t-polyform module

It is known that, for an R-module M, the following are equivalent:-

1. Every essential submodule is rational (i.e. M is polyform)
2. For each $0 \neq N \leq M, f: N \to M,\ f \neq 0$, then $\ker f \underset{c}{\leq} N$ (i.e. All partial endomorphism of M have closed kernels in their domains)
3. End($\overline{M}$) is vonneuman regular
4. For each $N \underset{ess}{\leq} M$, $\mathrm{Hom}(\frac{M}{N}, \overline{M}) = 0$

Proof

(1) ⇔ (2) ⇔ (3) [2 , 4.9.P.34] .

(2)⇔ (3)⇔ (4) [13].

Our aim is to give analogize property for t-polyform module.

In S.3 we prove that an R- module M is t-polyform if and only if for each $0 \neq N \leq M$, f: N→M, $f \neq 0$ implies $\ker f \underset{tc}{\leq} M$.

Now we prove the following:

Theorem 4.1 An R- module M is t-polyform if and only if for each $0 \neq N \underset{tes}{\leq} M$, $\mathrm{Hom}(\frac{M}{N}, \overline{M}) = 0$ **.**

Proof:(□) suppose there exists $(N \underset{tes}{\leq} M) \neq 0$ such that $\mathrm{Hom}(\frac{M}{N}, \overline{M}) \neq 0$. Hence there exists $f: \frac{M}{N} \to \overline{M}$ and $f \neq 0$, and so there exists $m + N \in \frac{M}{N}$, $m + N \neq 0$ such that $f(m + N) = m' \neq 0$. Since $M \underset{ess}{\leq} \overline{M}$, there exists r∈R with $0 \neq m'r \in M$ let $m'r = x$ .Define $\phi : N + Rm \to Rx \subseteq M$, $\phi(n + tm) = tx$ for each $n \in N, t \in R$. To show that ϕ is well – defined: if $n_1 + t_1 m = n_2 + t_2 m$, then $n_1 - n_2 =$

$(t_2-t_1)m \in N$. Hence $(t_2-t_1)\, f(m+N) = f[(t_2-t_1)m+N] = 0$, this implies $(t_2-t_1)m' = 0$ and so $(t_2-t_1)m'r = 0$ .Thus $(t_2-t_1)x = 0$

So that $a_2x = a_1x$. It is clear that $\phi \neq 0$

Now $i o\phi$: N + Rm → M where i: Rx→M is the inclusion $i\circ\phi \neq 0$. Hence Ker($i\circ\phi$)= Kerϕ .But N ⊆ Kerϕ and $N \underset{tes}{\leq} M$ implies $\mathrm{Ker}\,\phi \underset{tes}{\leq} M$, so that $\mathrm{Ker}(i\circ\phi) \underset{tes}{\leq} M$. But $\mathrm{Ker}(i\circ\phi) \underset{tes}{\leq} N+Rm$ .Hence $\mathrm{Ker}(i\circ\phi) \underset{tes}{\leq} N+Rm$ which is a contradiction with Th.(3.4).

(□)Suppose that M is not t-polyform. Then there exists K≤M, f∈ Hom(K, M) ,f≠0 and $\mathrm{Kerf} \underset{tes}{\leq} K$. Since M is quasi – injective there exist $g \in \mathrm{End}(\overline{M})$ such that $g\circ i = j\circ f$ where $i: K \to \overline{M}$, $j: M \to \overline{M}$ be the inclusion mappings

Since $f \neq 0$, then $g \neq 0$. It is clear that kerf ⊆ kerg

Define by $\bar{g}: \frac{\overline{M}}{\mathrm{kerf}} \to \overline{M}$ by $\bar{g}(\overline{m}+\mathrm{kerf}) = g(\overline{m})$ for each $\overline{m} \in M$.Then it is easy to see that $\bar{g}$ is well – defined it follows that $g\circ i_1 \in \mathrm{Hom}\,(\frac{M}{\mathrm{Kerf}}, \overline{M})$, where i_1: $\frac{M}{\mathrm{kerf}} \to \frac{\overline{M}}{\mathrm{kerf}}$ by hypothesis $\bar{g}\circ i_1 = 0$.

That is for each $m \in M$, $\bar{g}\circ i(m+\mathrm{Kerf}) = \bar{g}(m+\mathrm{Kerf}) = g(m) = 0$. Thus $g = 0$ which is a contradiction. Therefore $\mathrm{Ker\,f} \underset{tes}{\not\leq} K$ and M is a t-polyform module.

Recall that an R-module M is called Rickart if for each f∈End(M), $\mathrm{Kerf} \overset{\otimes}{\leq} M$ [14 ,Def 2.11,P.20]. The following results is given in [14,Lemma 2.4.21.P.59].

Lemma 4.2

The following condition are equivalent for a right R-module M:

1. M is a polyform module
2. $\overline{M}$ is K-nonsingular (where $\overline{M}$ is the quassi-injective hull of M.
3. $\overline{M}$ is a Rickart module.

We prove the following characterization for t-polyform modules

Theorem 4.3

An R-module M is t-polyform if and only if $\overline{M}$ is t-ess. q-Ded.

Proof:(□) suppose there exists $K \underset{tes}{\leq} M$ and ϕ∈ Hom (K, M) with $\ker\phi \underset{tes}{\leq} K$. To prove ϕ=0. Since $M \underset{ess}{\leq} \overline{M}$, hence $M \underset{tes}{\leq} \overline{M}$ .Thus $\mathrm{Ker}\,\phi \underset{tes}{\leq} K \underset{tes}{\leq} M \underset{tes}{\leq} \overline{M}$ which implies $\mathrm{Ker}\,\phi \underset{tes}{\leq} \overline{M}$ and $K \underset{tes}{\leq} \overline{M}$ by lemma (2.30. Now $K = K\cap E(K) \underset{tes}{\leq} \overline{M}\cap E(K)$ by lemma 2.2(1), where E(K) is the injective hull of K. Hence $\overline{M}\cap E(K) \overset{\oplus}{\leq} \overline{M}$,so $\overline{M} = (\overline{M}\cap E(K)) \oplus X$ for some $X \leq \overline{M}$ .Define $\psi: K\oplus X \to \overline{M}$ by $\psi = \phi$ on k and $\phi = 0$ on X. Since $\overline{M}$ is quasi – injective, there exist $\overline{\psi}: \overline{M} \to \overline{M}$ such that $\overline{\psi} o i = \psi$ where i: K⊕X → M be the inclusion mapping. Since $\overline{\psi} = \psi$ on K⊕X, then $\overline{\psi} = \phi$ on K and $\overline{\psi} = 0$ on X.

We can easily see that: $\mathrm{Ker}\,\psi = \mathrm{Ker}\,\phi \oplus X$ but $\mathrm{Ker}\,\phi \underset{tes}{\leq} K$ and $X \underset{tes}{\leq} X$, hence by lemma 2.2(1), $\mathrm{Ker}\,\psi \underset{tes}{\leq} K\oplus X$ .On the other hand, $K \underset{tes}{\leq} \overline{M}$, so $K\oplus X \underset{tes}{\leq} \overline{M}$. Therefore $\mathrm{Ker}\,\psi \underset{tes}{\leq} \overline{M}$ .Also $\mathrm{Ker}\overline{\psi} \supseteq \mathrm{Ker}\psi$.

It follows that $\mathrm{Ker}\,\overline{\psi} \underset{tes}{\leq} \overline{M}$, hence $\bar{\psi} = 0$ since $\overline{M}$ is t-ess. q-Ded. However $\bar{\psi} = 0$ implies ϕ =0. Thus M is t-poly form.

(□)To prove $\overline{M}$ is t-ess. q-Ded. Let f∈End (M) and f≠0. To show that $\mathrm{Kerf} \underset{tes}{\not\leq} \overline{M}$, we shall prove that $\mathrm{Kerf} \underset{tc}{\leq} \overline{M}$ and hence $\mathrm{Kerf} \underset{tes}{\not\leq} \overline{M}$ By [3, Lemma 2.3], there exists $K \underset{tc}{\leq} \overline{M}$ such that $\mathrm{Kerf} \underset{tes}{\leq} K$. Hence $K \underset{c}{\leq} \overline{M}$ by Lemma 2.5, so that $\overline{M} = K\oplus A$ for some $A \leq \overline{M}$. Define $h: \overline{M} \longrightarrow \overline{M}$ by $h|_A = 0$ and $h|_k = f|_k$. Hence Kerh = kerf$\oplus A$. But $\mathrm{Kerf} \underset{tes}{\leq} K$, $A \underset{tes}{\leq} A$, implies $\mathrm{Kerh} = \mathrm{Kerf}\oplus A \underset{tes}{\leq} K\oplus A$ by

Lemma 2.2(2). Now for any $\alpha \in$ End $\bar{M}$, Ker(h$\circ\alpha$) $=\alpha^{-1}$(Kerh). Since $\text{Kerh} \underset{\text{tes}}{\leq} \bar{M}$, then $\alpha^{-1}\text{Kerh} \underset{\text{tes}}{\leq} \bar{M}$ by [10, 2014, cor. 1.2]. Thus $\text{Ker}(h \circ \alpha)$ $\cap M \underset{\text{tes}}{\leq} \bar{M} \cap M = M$ by Lemma (2.2). Then by Theorem (4.1), $0 = \text{Hom}(\frac{M}{\text{Ker}(h \circ \alpha) \cap M}, \bar{M}) \approx \text{Hom}((h \circ \alpha)$ $(M), \bar{M})$ and so $h(\alpha(M) = 0$. Since $\alpha \in$ End (M) is arbitrary, $h(\bar{M}) = \sum_{\alpha \in \text{End}(\bar{M})} h\alpha(M) = 0$ .Thus h=0 and $\text{Kerf} = K \underset{\text{tc}}{\leq} \bar{M}$. Thus $\text{Kerf} \underset{\text{tes}}{\not\leq} \bar{M}$.

Corollary 4.4

Let M be a quasi-injective module then M is t-poly form if and only if M is t-ess-q-Ded

Proof: It follows directly by Th. (4.3).

Recall that an R-module M is called a t-Rickart if $t_M(\phi) = \phi^{-1}(Z_2(M))$ is a direct summand of M for every $\phi \in$End (M)[1,Def2.1].

Note that every nonsingular Rickart module is t-Rickart, every extending module and every Z_2 −torsion module (i.e a module M for which $Z_2(M) = M$) is t-Rickart. A Rickart module need not be t-Rickart, see[1, Ex.2.10]

We prove that

Theorem 4.5

If M is a t-polyform module, then $\bar{M}$ is t-Rickart

Proof:

Since $\frac{\bar{M}}{Z_2(\bar{M})}$ is nonsingular, $Z_2(\bar{M}) \underset{\text{tc}}{\leq} \bar{M}$ and hence $Z_2(\bar{M}) \underset{C}{\leq} \bar{M}$. But $\bar{M}$ is quasi-injective (hence extending) so that $Z_2(\bar{M})$ is a direct summand of $\bar{M}$ Thus $\bar{M} = Z_2(\bar{M}) \oplus C$ for some $C \leq \bar{M}$. But $C \approx \frac{\bar{M}}{Z_2(\bar{M})}$ which is nonsingular, so C is nonsingular. But M is t-polyform, hence $\bar{M}$ is t-ess. Quasi-Ded by Theorem 4.3. Thus $\bar{M}$ is K-nonsingular (i.e ess. q-Ded) . On other, $\bar{M}$ is quasi-injective, so $\bar{M}$ is extending. But $\bar{M}$ is K-nonsingular extending module implies $\bar{M}$ is Baer which implies Rickart by [15, Lemma 2.2.4, r.13].

Since $C \overset{\oplus}{\leq} \bar{M}$, then C is Rickart. Thus $\bar{M}$ is t- Rickart by [11,Th2.6.1 (1→2)]

Remarks 4.6

1. The converse of Th.(4.5) is not true if $Z_2(\bar{M}) \neq 0$

Proof:

Since $\bar{M}$ is t-Rickart, $\bar{M} = Z_2(\bar{M}) \oplus C$, for some nonsingular Rickart submodule C of $\bar{M}$.If $Z_2(\bar{M}) \neq 0$, then there $i: Z_2(\bar{M}) \longrightarrow M$, where i is the inclusion mapping, and i≠0. Thus ker f = (0). But $(0) + Z_2(\bar{M}) \underset{\text{ess}}{\leq} Z_2(\bar{M})$,thus $(0) \underset{\text{ess}}{\leq} Z_2(\bar{M})$. That is $\text{Kerf} \underset{\text{tes}}{\leq} Z_2(\bar{M})$ and so $\bar{M}$ is not t-polyform therefore $(\bar{M})$ is not t-ess.q-Ded by cor (4.4)

2. If $Z_2(\bar{M}) = 0$ and $\bar{M}$ is t-Rickart, then M is t-polyform.

Proof: As in (1) , $\bar{M} = Z_2(\bar{M}) \oplus C$ where C is nonsingular Rickart. Since $Z_2(\bar{M}) = 0$, then $\bar{M} = C$; that is $\bar{M}$ is nonsingular, hence $\bar{M}$ is t-polyform thus M is t-polyform by Rem & Ex.2.2.(7)

Now we have:

Theorem 4.7

Let M be a t-polyform extending module. Then $\bar{M} + M$ is t-Rickart module.

Proof: Since M is extending, M is t-Rickart. Also M is t-polyform implies $\bar{M}$ is t-Rickart by (4.4). By [1, Th.2.6.1] $M = Z_2(M) \oplus A$, A is nonsingular Rickart sub- module of M, $\bar{M} = Z_2(\bar{M}) \oplus B$, B is a nonsingular Rickart sub module of $\bar{M}$. Hence $\bar{M} \oplus M = Z_2(\bar{M}) \oplus Z_2(M) \oplus (B \oplus A) = Z_2(\bar{M} \oplus M) \oplus (B \oplus A)$ by Lemma 2.6 hence B⊕A is a nonsingular submodule of $\bar{M} \oplus M$ since $A \overset{\oplus}{\leq} M$, then A is t-polyform and extending and so A is polyform and extending $B \overset{\oplus}{\leq} M$ and $\overline{M}$ is quasi-injective, hence B is a quasi-injective. On the other hand, $M = Z_2(M) \oplus A$ implies $\overline{M} = \overline{Z_2(M)} \oplus \overline{A} = Z_2(\overline{M}) \oplus \overline{A}$. But $\overline{M} = Z_2(\overline{M}) \oplus B$, So $B = \overline{A}$. Thus $B \oplus A = \overline{A} \oplus A$ and hence by [14 , prop 2.4.22, p.60], $B \oplus A$ is Rickart and then by [11, Th 2.6.1], $\overline{M} \oplus M$ is t-Rickart .

It is well-known that a sub module N of M is fully invariant if for each $f \in End\ (M), f(N) \subseteq N$. Also recall the following basic fact: if N is a fully invariant sub module of $M = M_1 \oplus M_2$ then $N = (N \cap M_1) \oplus (N \cap M_2)$

Proposition 4.8

For an R-module M. if E (M) (injective hull of M) is t-poly form, then $Z_2(M)$ is a direct summand of M.

Proof: Since E(M) is t-polyform, then $\overline{E(M)}$ is t-Rickart by Th.(4.5). But $\overline{E(M)} = E(M)$, hence E (M) is t-Rickart. Then by [1 , Th.2.6.1] $E(M) = Z_2(E(M) \oplus A$, A is a nonsingular Rickart submodule of E (M) since M is a fully invariant submodule of E(M), then $M = (Z_2(E(M) \cap M) \oplus (A \cap M)$, but $Z_2(M) = Z_2(E(M)) \cap M$. Thus $M = Z_2(M) \oplus (A \cap M)$ therefore $Z_2(M) \overset{\oplus}{\leq} M$.

References

1. Gooderl K.R. **1976.** *Ring theory, Nonsingular Rings and Module Theorey*, Marcel Dekker, Inc. New York and Basel.
2. Asgari, Sh., and Haghany, A. **2011.** t-extending Modules and t-Baer Modules, *Comm. Algebra*, **39**: 1605 – 1623.
3. Dung, N. V., Huynh, D. V. and Smith P.F. **1996.** *Wisbauer Extending Modules*, John wiely & sons, Inc. New York 1996.
4. Zelmanowitze, J. M **1986.** Representation of ring with faithful polyform modules, *Comm In Algebra*, **14**(6):1141 – 1169.
5. Rizvi, S. T. and Roman, C. S. **2007.** On K-nonsingular modules and applications, Comm Algebra, **35**: 2960.
6. Mijbass A.S. **1997.** Quasi-Dedkind modules, Ph. D. thesis, college of science, university of Baghdad.
7. Ghawi, Th. Y**. 2010.** Some Generalizations of Quasi-Dedekind modules. M.Sc. Thesis, College of Education Ibn AL-Haitham,University of Baghdad
8. Shyaa, F. D., Alaeashi, S. N. and Hadi, I. M. A. **2018.**T-essentially Quasi-Dedekind modules, accepted for publication in *Journal of AL-Qadisya*, 2018.
9. Ahmed, M. A. **2018.** St-Polyform Modules and Related Concepts, *Baghdad Science Journal*, **5**: 335 – 343.
10. Asgari, Sh., and Haghany, A. **2014.** Modules whose t-closed sub modules have summand as a complement, *Comm. Algebra*, **42**(2014): 5299 – 5318.
11. Asgari, Sh., and Haghany A. **2015.** t-Rick art and Dual t-Rick art Modules, *Algebra colloquium*, **22**: 849 – 870.
12. Ghorbani, A. **2010.** Co-epi-retractable modules and Co-pri-rings, Comm Algebra, **38**: 3589 – 3596.
13. Weakly W. D. **1983.** modules whose proper sub modules are finitely generated, *J. Algebra*, 189 – 219.
14. Gongyoung. Lee, M. S. **2010.** Theory of Rickart modules, Ph. D. Thesis. The ohio state university.
15. Rizvi, S. T. and Roman, C. S. **2004.** Baer and quasi-Baer modules. *Comm. Algebra*, **32**(1): 103 – 123.