T-Polyform Modules
Inaam Mohammed Ali¹, Alaa A. Elewi²

¹Department of Mathematics University of Baghdad, College of Education for Pure Sciences (Ibn-Al-Haitham), University Of Baghdad, Baghdad, Iraq
²Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq

Abstract
We introduce the notion of t-polyform modules. The class of t-polyform modules contains the class of polyform modules and contains the class of t-essential quasi-Dedekind.

Many characterizations of t-polyform modules are given. Also many connections between these class of modules and other types of modules are introduced.

Keywords: Polyform, modules, essential submodule, t-essentially, quasi-Dedekind

المقاسات المتعددة الصيغ من النمط-Τ

انعام محمد علي، عباس عليهي

¹قسم الرياضيات، كلية التربية ابن الهيثم، جامعة بغداد، بغداد، العراق
²قسم الرياضيات، كلية العلوم، جامعة بغداد، بغداد، العراق

الخلاصه
قدمنا مفهوم المقاسات المتعددة الصيغ من النمط-Τ. هذا الرنف من مقاسات متعددة الصيغ من النمط-Τ يحتوي على صف المقاسات المتعددة الصيغ ويفتح على المقاسات شبه الديدكانية من النمط-Τ. عدة تشخيصات لمقاسات متعددة الصيغ من النمط-Τ قد اعطيت وكذلك عدة روابط بين هذا الصف من المقاسات وأنواع أخرى من المقاسات قد قدمت.

Introduction
Throughout the paper, rings will have a nonzero identity element and modules will be unitary right modules. We first briefly review some background materials relevant to the topics discussed in this paper.

Recall that, a submodule N of an R-module M is called essential submodule of M (briefly N ≤ M) if for each nonzero submodule W of M, N ∩ W 0 [1]. Equivalently N ≤ M if whenever W ≤ M, N ∩ W = (0) implies W = (0) [1] A submodule N of M is called closed (denoted by N ≤ M) if has no proper essential extension in M; that is, if N ≤ W ≤ M, then N = W [1]. Ashari et. al in [2], introduced the concept of t-essential submodule, where a submodule N of M is called t-essential (briefly N ≤ M) if whenever W ≤ M, N ∩ W ≤ Z₂(M) where Z₂(M) is the second singular submodule of M and defined by Z₂(M) = Z₂(M) = Z₂(M) [1]. It is well known that Z(M) = {m: mI = 0, for some I ≤ R}

Email: alaa.abbas.math@gmail.com
Equivalentl
\[ Z(M) = \{ m \in M : \text{ann}(m) \leq R \} \text{,} \]
where \( \text{ann}(m) = \{ r \in R : mr = 0 \} \). Similarly
\[ Z_2(M) = \{ m \in M : m_1 = 0, \text{for some } l \leq R \} = \{ m \in M : \text{ann}(m) \leq R \} \text{,} \]

Obviously, every essential submodule is t-essential, but not conversely, for example the submodule (4) of the \( \mathbb{Z} \)-module \( \mathbb{Z}_8 \) is t-essential but not essential.

However, the two concepts are equivalent if \( M \) is nonsingular (i.e. \( Z(M) = 0 \)). A module \( M \) is called singular if \( Z(M) = M \) and is called \( Z_2 \)-torsion if \( Z_2(M) = 0 \). If \( A \leq M \) then \( Z_2(A) = Z_2(M) \cap A \).

Asgari, etc., in [2], introduced the concept t-closed submodule where a submodule \( N \) of an \( R \)-module \( M \) is t-closed (denoted by \( N \leq_M \) if \( N \) has no proper t-essential extension in \( M \). It is clear that every t-closed submodule is closed, but the converse is not true for example (0) is closed in \( Z_8 \) as \( \mathbb{Z} \)-module but it is not t-closed. The two concepts closed submodule and t-closed submodule are coincide in nonsingular modules.

An \( R \)-module \( M \) is called polyform if for each \( L \leq M \) and for each \( \phi : L \rightarrow M \), \( \ker \phi \leq_L \text{ implies } \phi = 0 \)
(i.e if \( \phi \neq 0 \), then \( \ker \phi \leq L \). [3, 4].

Rizvi in [5] introduced the nation of k-nonsingular module, where an \( R \)-module \( M \) is called \( k \)-nonsingular if \( \phi \in \text{End} \ (M) \), \( \ker \phi \leq_M \text{ implies } \phi = 0 \), where \( \text{End} \ (M) \) means the ring of endomorphism on \( M \).

It is clear that polyform module implies \( k \)-nonsingular but not conversely see [5].

Tha’r in [4] gave the notion of essentially quasi-Dedekind modules as a generalization of quasi-Dedekind modules by restricting the definition of quasi-Dedekind modules (which is introduced in [6] on essential submodules, where an \( R \)-module \( M \) is called essentially quasi-Dedekind if \( \text{Hom}(\frac{M}{N}, \text{M}) = 0 \) for each \( N \leq M \) (that is \( M \) is essentially quasi-Dedekind if every \( N \leq M \), \( N \) is quasi-invertible. Tha’r in [7] proved that \( k \)-nonsingular modules and essentially quasi-Dedekind are coincid.

F, S and Inaam in [8] introduced the notion of t-essentially quasi-Dedekind where an \( R \)-module \( M \) is called t-essentially quasi-Dedekind (shortly \( t \)-ess.q-Ded) if \( \text{Hom}(\frac{M}{N}, \text{M}) = 0 \) for each \( N \leq M \).

Equivalently \( M \) is \( t \)-ess. q-Ded if for each \( \phi \in \text{End} \ (M) \) with \( \phi \neq 0 \), \( \ker \phi \leq_M \text{ implies } \phi = 0 \) [8].

It is obvious that every \( t \)-ess. q-Ded module is ess. q-Ded, but not conversely [8, Rem& Ex.2.2(2)].

In the present paper, motivated by these works, we introduce and study \( t \)-polyform modules as follows: An \( R \)-module \( M \) is called \( t \)-polyform if for each \( L \leq M \), and, \( \phi : L \rightarrow M \), \( \ker \phi \leq_L \text{ implies } \phi = 0 \).

Then we have
If \( M \) is \( t \)-polyform then \( M \) is polyform model and if \( M \) is \( t \)-polyform then \( M \) is \( t \)-ess q-Ded module and none of these implications is reversible (see Rem& Ex.3.2(1),(3)).

We give many properties and characterizations of \( t \)-polyform modules which are analogous to that of polyform modules (See Rem 3.2(3), Th.3.6, Th.4.7)
Also, many connections between \( t \)-polyform module and other types of modules are presented (see Theorems 3.3,3.4.4.1 and 4.4).

Next note that our notion ((t-polyform modules)) is different from (st-polyform modules) which is appeared recently in [9] as we explain that in S.3, Note 3.5

**2-Preliminaries**
We list some known results which are relevant for our work.

**Lemma 2.1** [2]
The following statements are equivalent for a submodule \( A \) of an \( R \)-module \( M \).

1. \( A \leq_M \)
2. \( A + Z_2(M) \leq_M \)

2044
3. \( \frac{A + Z_2(M)}{Z_2(M)} \leq \frac{M}{\text{ess } Z_2(M)} \)

4. \( \frac{M}{A} \) is \( Z_2 \)-torsion (i.e. \( Z_2(M) A \) = \( \frac{M}{A} \))

**Lemma 2.2** [10]

Let \( A_\lambda \) be a submodule of \( M_\lambda \) for each \( \lambda \in \Lambda \). Then
1. If \( \Lambda \) is a finite set and \( A_\lambda \leq M_\lambda \), then \( \bigcap_{\lambda \in \Lambda} A_\lambda \leq \bigcap_{\lambda \in \Lambda} M_\lambda \).
2. \( \bigoplus_{\lambda \in \Lambda} A_\lambda \leq \bigoplus_{\lambda \in \Lambda} M_\lambda \) if and only if \( A_\lambda \leq M_\lambda \), \( \forall \lambda \in \Lambda \)

**Lemma 2.3** [10]

Let \( A \leq B \leq M \). Then \( A \leq M \) if and only if \( A \leq B \) and \( B \leq M \)

**Lemma 2.4** [2]

Let \( M \) be an \( R \)-module. Then
1. If \( C \leq M \) then \( Z_2(M) C \leq C \)
2. \( (0) \leq M \) if and only if \( M \) is nonsingular.
3. If \( A \leq C \leq M \), then \( C \leq M \) if and only if \( C \leq \frac{M}{A} \)

**Lemma 2.5** [2]

Let \( C \) be a submodule of an \( R \)-module \( M \). Then the following statements are equivalent:
1. There exists a submodule \( S \) such that \( C \) is a maximal with respect to the property \( C \cap S \) is \( Z_2 \)-torsion.
2. \( C \leq M \).
3. \( C \) contains \( Z_2(M) \) and \( \frac{C}{Z_2(M)} \leq \frac{M}{Z_2(M)} \)
4. \( C \) contains \( Z_2(M) \) and \( C \leq M \)
5. \( C \) is a complement of a nonsingular submodule of \( M \).
6. \( \frac{M}{C} \) is nonsignular.

**Lemma 2.6** [2]

Let \( M = \bigoplus_{\alpha \in \Lambda} M_\alpha \) Where \( M_\alpha \leq M \) for each \( \alpha \in \Lambda \). Then \( Z_2(M) = \bigoplus_{\alpha \in \Lambda} Z_2(M_\alpha) \)

**3- t-polyform Modules**

**Definition 3.1:** An \( R \)-module \( M \) is called t-polyform if for each \( L \leq M \) and \( \phi : L \rightarrow M \), \( \phi \neq 0 \), then \( \text{Ker } \phi \leq L \). A ring \( R \) is said to be right t-polyform if the module \( R \) is t-polyform.

**Remarks and Examples 3.2**

1. Every t-polyform module is polyform, since every essential submodule is t-essential. However, the converse is not always true for example:
   - Let \( M \) be the \( Z \)-module \( Z_6 \) since \( M \) has no proper essential submodule, then \( M \) is polyform.
   - But \( M \) is singular hence \( M \) is \( Z_2 \)-torsion and so every submodule, \( 0 \neq L \leq M \) is \( Z_2 \)-torsion and hence \( Z_2(L) = L \). Now for each \( 0 \neq \phi : L \rightarrow M, \text{Ker } \phi + Z_2(L) = \text{Ker } \phi = L \) and hence \( \text{Ker } \phi \leq L \) (by Lemma 2.1)

Thus \( M = Z_6 \) is not t-polyform

2. It is known that every semisimple module is polyform, but it is not necessary t-polyform, see the example in (1).

3. It is clear that every t-polyform module is t-ess.q. Ded. However, the converse may be noted true in general, for example: Let \( M = Z_p \) as \( Z \)-module, where \( P \) is a prime number. For each \( 0 \neq f : Z_p \rightarrow Z_p \)
Since \( f \neq 0 \), and \( M \) is simple so \( \text{Ker} f = (0) \) and hence by lemma 2.1, \( \text{Ker} f = (0) \leq M \), since \( \text{Ker} f \oplus Z_2 \)(M) = \( M \) implies \( \text{Ker} f = (0) \) and hence by lemma 2.1, \( \text{Ker} f = (0) \leq M \). Thus \( M \) is not t-polyform. But \( M \) is t-ess.q.Ded. Since for each \( f: M \to M \), with \( 0 \neq \text{Ker} f \leq M \), implies \( \text{Ker} f = M \) and so \( f = 0 \).

4. Recall that every nonsingular module \( M \) (i.e \( Z(M) = 0 \)) is polyform. Also every nonsingular module \( M \) is t-polyform.

\[ \textbf{Proof:} \] Let \( L \leq M, \phi: L \to M \) and \( \phi \neq 0 \). Since \( M \) is polyform \( \text{Ker} f \leq M \). But \( M \) is nonsingular, hence \( \text{Ker} f = (0) \leq M \).

In particular each of the \( Z \)-module: \( ZQ, Z\oplus ZQ \oplus Q, Z[X] \) is t-polyform module, also for each prime number \( p \), \( Z_p \) as \( Z_p \)-module is t-polyform.

5. Every singular \( M \) (hence \( M \) is \( Z_2 \)-torsion \( (Z_2(M) = M) \)) is not t-polyform module.

\[ \textbf{Proof:} \] Let \( L \leq M, \phi: L \to M \) and \( \phi \neq 0 \). Hence \( \text{Ker} \phi + Z_2(M) = \text{Ker} f + M = M \leq M \), so \( \text{Ker} f \leq M \) by lemma 2.1. Thus \( M \) is not t-polyform.

6. Prime module need not be t-polyform, for example \( M \) as \( Z \)-module is prime and \( M \) is not t-polyform since \( M \) is singular. However every prime faithful module is nonsingular, hence it is t-polyform by part (4).

7. Every submodule \( N \neq 0 \) of t-polyform module \( M \) is t-polyform.

\[ \textbf{Proof:} \] Let \( 0 \neq L \leq N \) and let \( f: L \to N, f \neq 0 \). Then, \( 0 \neq \text{i of}: L \to M \) where \( i \) is the inclusion mapping from \( N \) into \( M \).

Since \( M \) is t-polyform then \( \text{Ker} (i \circ f) \leq L \).

But it is easy to check that \( \text{Ker} f = \text{Ker} (i \circ f) \) and hence \( \text{Ker} f \leq L \). Thus \( N \) is t-polyform.

8. In particular if \( \overline{M} \) (quasi-injective hull of \( M \)) or \( E(M) \) (injective hull of \( M \)), then \( M \) is t-polyform.

9. A homomorphic image of t-polyform module is not necessarily t-polyform, for example the \( Z \)-module \( Z \) is t-polyform module. Let \( \pi: Z \to Z/6) \approx Z_{6n} \) where \( \pi \) is the natural epimorphism, but \( Z_{6n} \) is not t-polyform by part (1).

10. If \( M \) is a t-polyform \( R \)-module and \( N \leq M \) then \( \frac{M}{N} \) is t-polyform.

\[ \textbf{Proof:} \] Since \( N \leq M \), \( \frac{M}{N} \) is nonsignular by lemma (2.5). Hence \( \frac{M}{N} \) is t-polyform by part (4).

10. Recall that an \( R \)-module is Co-epi-retractable if for each \( N \leq M \), there exists \( K \leq M \) such that \( \frac{M}{N} \approx K \) [11, 12].

If \( M \) is t-polyform and Co-epi-retractable, then \( \frac{M}{N} \) is t-polyform, for each \( N \leq M \).

\[ \textbf{Proof:} \] it follows directly

The following theorem is a characterization of t-polyform modules.

\[ \textbf{Theorem 3.3} \] An \( R \)-module \( M \) is t-polyform if for each \( 0 \neq L \leq M \) and \( 0 \neq \phi: L \to M, \text{Ker} \phi \leq L \).

\[ \textbf{Proof:} \] Suppose there exist \( 0 \neq L \leq M \) and \( 0 \neq \phi: L \to M \), but \( \text{Ker} \phi \leq L \). By definition of t-closed submodule, there exists \( U \leq L \) such that \( U \) is a proper t-essential extension of kerf.
Then \( \phi \circ i: U \to M \) where \( i \) is the inclusion mapping from \( U \) into \( L \). Clearly \( \text{Ker}(i \circ \phi) \leq \text{Ker} \phi \), so that \( \text{Ker}(\phi \circ i) \leq U \). Hence \( \phi \circ i = 0 \) since \( M \) is t-polyform. It follows that \( \phi(U) = 0 \); that is \( U \leq \text{Ker} \phi \) which is a contradiction. Thus \( \text{Ker} \phi \leq L \).

Conversely, suppose there exists \( L \leq M \) and \( 0 \neq \phi: L \to M \) with \( \text{Ker} \phi \leq L \). But \( \text{Ker} \phi \leq L \) by hypothesis, so \( \phi = 0 \) which implies \( \phi = 0 \) which is a contradiction. Thus \( \text{Ker} \phi \leq L \) and so \( M \) is t-polyform.

The following is another characterization of t-polyform modules

**Theorem 3.4** Let \( M \) be an \( R \)-module. Then \( M \) is t-polyform if and only if for each \( 0 \neq N \leq M \) and for nonzero \( f \in \text{Hom}(N,M) \), then \( \text{kerf} \leq N \).

**Proof:** (\( \Rightarrow \)) it is clear

(\( \Leftarrow \)) Let \( N \leq M \). If \( N \leq M \) then nothing to prove if \( N \leq M \). Since \( N \leq M \), then \( N \leq M \).

Define \( g: N \not\leq K \to M \) by \( g(n+k) = f(n), n \in N, k \in K \). It is well-defined and \( g \neq 0 \). By hypothesis, \( \text{ker} g \leq N \otimes K \). But \( \text{Ker} g = \text{Ker} f \otimes K \) and so \( \text{ker} f \leq N \) by lemma 2.2 (2). Thus \( M \) is t-polyform.

The notion of (st-polyform modules) appeared in [9], where an \( R \)-module \( M \) is called st-polyform if for each \( 0 \neq L \leq M, 0 \neq f: L \to M \) \( \text{ker} f \leq L \). A submodule \( U \) of \( M \) is called st-closed \( (U \leq M) \) if \( U \) has no proper semiessential extension of \( U \), and a submodule \( U \) of \( M \) is called semi-essential in \( M \) if \( U \) has nonzero intersection with any nonzero prime submodule.

**Note 3.5** The two concepts (t-polyform modules) and (st-polyform modules) are independent as we can see by the following examples.

1. \( Z_n \) as \( Z \)-module is not t-polyform (see Rem 3.2(1)) and it is is t-poly by [5, Rem.3(viii)])
2. \( Z \) as \( Z \)-module is t-polyform (See Rem 3.2.(4)), and it is not st-polyform [see 5, Ex.5(ii)]

[4] gave the following: An \( R \)-module \( M \) is polyform if and only if every essential submodule is rational, where a submodule \( N \) of \( M \) is called rational in \( M \) (briefly \( N \leq M \)) if \( \text{Hom}(N,M) = 0 \) for each \( N \leq V \leq M \).

Note that every rational submodule is essential but not conversely [1].

We give the following:

**Theorem 3.6** An \( R \)-module \( M \) is t-polyform implies every nonzero t-essential submodule of \( M \) is rational.

**Proof:** Assume \( 0 \neq N \leq M \) and \( f \in \text{Hom}(V_N,M) \), where \( N \leq V \leq M \). Then \( f \circ \pi \in \text{Hom}(V,N) \) where \( \pi \) is the natural epimorphism from \( V \) onto \( V_N \). Hence \( N \leq \text{Ker}(f \circ \pi) \), but \( N \leq M \) implies \( \text{Ker}(f \circ \pi) \leq M \) by lemma (2.3). So that \( \text{ker}(f \circ \pi) \leq V \) (since \( \text{ker}(f \circ \pi) \subseteq V \)). Since \( M \) is t-polyform, \( f \circ \pi = 0 \), and hence \( f = 0 \). Thus \( \text{Hom}(V_N, M) = 0 \) that is \( N \leq M \).

**Remark 3.7** The converse of theorem (3.6) is not true in general, for example:
The \( Z \)-module \( Z_6 \) is not t-polyform, but \( Z_6 \) has only \( Z_6 \) as t-essential submodule of \( Z_6 \) and \( Z_6 \leq Z_6 \).

However, we have:

**Theorem 3.8** if \( M \) is an \( R \)-module such that every nonzero t-essential submodule is rational, then \( M \) is polyform.
Proof: Let \( N \leq M \), hence \( 0 \not= N \leq M \). Then by hypothesis is \( N \leq M \). Thus every essential submodule is \( r \) rational. It follows that \( M \) is polyform.

Recall a nonzero \( R \)-module \( M \) is called monoform if for each \( 0 \not= N \leq M \) and for each \( 0 \not= f \in \text{Hom}(N,M) \), then \( \text{ker} f = 0 \), [9].

Equivalently a nonzero \( R \)-module \( M \) is monoform if for each nonzero submodule \( N \) of \( M \), \( N \leq M \), [9].

It is known that every monoform is polyform. Now we ask the following: Is there any relation between \( t \)-polyform modules and monoform?

Consider the following remarks

Remarks 3.9
1. \( t \)-polyform modules need not be monoform, for example: The \( Z \)-module \( Z \oplus Z \) is \( t \)-polyform (Rem 3.2.(4)), but it is not monoform since there exists \( f : Z \oplus 2Z \rightarrow Z \oplus Z \) such that \( f(x,y) = (y,0) \) for each \( x \in Z \), \( y \in 2Z \) then \( \text{Ker} f = Z \oplus (0) \neq \text{zero submodule} \).

2. Monoform module may be not \( t \)-polyform module, for example: The \( Z \)-module \( Z_p \), where \( p \) is a prime number, is monoform but it is not \( t \)-polyform.

We introduce the following

Definition 3.10 An \( R \)-module \( M \) is called \( t \)-essentialy monoform (shortly \( t \)-ess- mono) if for each \( 0 \not= N \leq M \) and \( 0 \not= f \in \text{Hom}(N,M) \) then \( \text{ker} f = 0 \).

Every simple module is \( t \)-ess mono and every monoform module is \( t \)-ess. mono.

Proposition 3.11: Let \( M \) be a \( t \)-ess- mono. module. Then \( M \) is quasi-Dedekind and hence \( M \) is \( t \)-ess.q.-Ded.

Proof: Since \( M \leq M \) and \( M \) is \( t \)-ess- mono, the for each \( 0 \not= f \in \text{End} (M) \) implies \( \text{ker} f = 0 \) Thus \( M \) is quasi-Dedekind by [6, Th.1.5, p.26] and hence \( M \) is \( t \)-ess-q.-Ded.

By th.(3.6), We have: If \( M \) is \( t \)-polyform, then for each \( 0 \not= N \leq M \) implies \( N \leq M \).

Now we give the following

Proposition 3.12: If \( M \) is \( t \)-ess- mono. \( R \)-module, then for each \( 0 \not= N \leq M \) implies \( N \leq M \).

Proof: Suppose there exists \( 0 \not= N \leq M \) but \( N \not\leq M \). Hence there exists \( V \rhd N \) such that \( \text{Hom}(\frac{V}{N}, M) \neq 0 \), so \( \exists f \in \text{Hom}(\frac{V}{N}, M), f \neq 0 \). It follows that \( f_\pi \in \text{Hom}(V, M) \), where \( \pi \) is natural epimorphism from \( V \) onto \( \frac{V}{N} \), and \( f_\pi \neq 0 \) (Since \( f \neq 0 \)). But \( N \not\leq V \), hence \( V \leq M \) and since \( M \) is \( t \)-ess- mono, \( \text{Ker}(f_\pi) = 0 \).

Since \( N \not\leq \text{Ker}(f_\pi) = 0 \) thus \( N = 0 \) which is a contradiction therefore \( N \leq M \).

Corollary 3.13: Let \( M \) be a \( t \)-ess- mono. Then \( M \) is polyform.

Proof: It follows by prop.(3.12) and Th.(3.8)

Proposition 3.14: Let \( M \) a quasi-injective \( R \)-module. If \( M \) is \( t \)-ess.q.Ded, then for each \( 0 \not= N \leq M \) implies \( N \leq M \).

Proof: Let \( 0 \not= N \leq M \). Since \( M \) is \( t \)-ess,q-Ded, \( \text{Hom}(\frac{M}{N}, M) = 0 \); that is \( N \) is a quasi-invertible submodule of \( M \). Since \( M \) is quasi-injective, then by [6, Th.3.5, p.16], \( M \) is a rational extension of \( N \); that is \( N \leq M \).

Corollary 3.15: Let \( M \) be a quasi-injective if \( M \) is \( t \)-ess.q.Ded, then \( M \) is polyform

Proof: It follows by prop.(3.14) and Th.(3.7)

We can Summarize results of S.3 by the following tables
§4 More about t-polyform module

It is known that, for an R-module M, the following are equivalent:
1. Every essential submodule is rational (i.e. M is polyform)
2. For each $0 \neq N \leq M$, $f: N \to M$, $f \neq 0$, then $\ker f \leq N$ (i.e. All partial endomorphism of M have closed kernels in their domains)
3. $\text{End}(M)$ is vonneuman regular
4. For each $N \leq M$, $\text{Hom}(\frac{M}{N}, \overline{M})=0$

Proof

(1) $\iff$ (2) $\iff$ (3) [2, 4.9.P.34], (2)$\iff$ (3) $\iff$ (4) [13].

Our aim is to give analogize property for t-polyform module.

In S.3 we prove that an R-module M is t-polyform if and only if for each $0 \neq N \leq M$, $f: N \to M$, $f \neq 0$ implies $\ker f \leq M$.

Now we prove the following:

**Theorem 4.1** An R-module M is t-polyform if and only if for each $0 \neq N \leq M$, $\text{Hom}(\frac{M}{N}, \overline{M})=0$.

**Proof:** (1) suppose there exists $(N \leq M) \neq 0$ such that $\text{Hom}(\frac{M}{N}, \overline{M}) \neq 0$. Hence there exists $f: \frac{M}{N} \to \overline{M}$ and $f \neq 0$, and so there exists $m + N \in \frac{M}{N}, m + N \neq 0$ such that $f(m + N) = m' \neq 0$. Since $M \leq \overline{M}$, there exists $r \in R$ with $0 \neq m' \in M$ let $m' = x$. Define $\phi: N + Rm \to Rx \subseteq M$, $\phi(n + tm) = tx$ for each $n \in N, t \in R$. To show that $\phi$ is well-defined: if $n_1 + t_1 m = n_2 + t_2 m$, then $n_1 - n_2 = \ldots$
(t_2 - t_1)m \in N. Hence (t_2 - t_1) f(m + N) = f((t_2 - t_1)m + N) = 0, this implies (t_2 - t_1)m' = 0 and so \((t_2 - t_1)x = 0\) . Thus \((t_2 - t_1)x = 0\)

So that \(a_2x = a_1x\). It is clear that \(\phi \neq 0\)

Now \(i\circ\phi : N \oplus Rm \to M\) where \(i: Rx \to M\) is the inclusion \(i \circ \phi \neq 0\). Hence \(\text{Ker}(i \circ \phi) = \text{Ker}\phi\). But \(N \subseteq \text{Ker}\phi\) and \(N \leq M\) implies \(\text{Ker}\phi \leq M\), so that \(K \leq \text{Ker}(i \circ \phi) \leq M\). But \(\text{Ker}(i \circ \phi) \leq N + Rm\). Hence \(\text{Ker}(i \circ \phi) \leq N + Rm\) which is a contradiction with Th.(3.4).

(\(\square\))Suppose that \(M\) is not \(t\)-polyform. Then there exists \(K \leq M\), \(f \in \text{Hom}(K, M)\) \(f \neq 0\) and \(\text{Ker} f \leq K\). Since \(M\) is quasi - injective there exist \(g \in \text{End}(\overline{M})\) such that \(g \circ i = j \circ f\) where \(i: K \to \overline{M}\), \(j: M \to \overline{M}\) be the inclusion mappings

Since \(f \neq 0\), then \(g \neq 0\). It is clear that \(\text{ker} f \leq \text{ker}\).

Define by \(\overline{g}: \overline{M}/\text{ker}f \to \overline{M}\) by \(g(m + \text{ker}f) = g(m)\) for each \(m \in M\).Then it is easy to see that \(\overline{g}\) is well-defined it follows that \(g \circ i_1 \in \text{Hom}(\overline{M}/\text{ker}f, \overline{M})\), where \(i_1; \overline{M}/\text{ker}f \to \overline{M}, \text{by hypothesis} \overline{g} \circ i_1 = 0\).

That is for each \(m \in M\), \(\overline{g} \circ i(m + \text{ker}f) = \overline{g}(m + \text{ker}f) = \overline{g}(m) = 0\). Thus \(g = 0\) which is a contradiction. Therefore \(\text{Ker} f \leq K\) and \(M\) is a \(t\)-polyform module.

Recall that an \(R\)-module \(M\) is called Rickart if for each \(f \in \text{End}(M)\), \(\text{Ker} f \leq M\) \([14, \text{Def} 2.11, \text{P} 20]\).
The following results is given in \([14, \text{Lemma 2.4.21, P} 59]\).

**Lemma 4.2**

The following condition are equivalent for a right \(R\)-module \(M\):

1. \(M\) is a polyform module
2. \(\overline{M}\) is \(K\)-nonsingular (where \(\overline{M}\) is the quasi-injective hull of \(M\)).
3. \(\overline{M}\) is a Rickart module.

We prove the following characterization for \(t\)-polyform modules

**Theorem 4.3**

An \(R\)-module \(M\) is \(t\)-polyform if and only if \(\overline{M}\) is \(t\)-ess. q-Ded.

**Proof:** \(\square\) suppose there exists \(K \leq M\) and \(\phi \in \text{Hom}(K, M)\) with \(\text{ker}\phi \leq K\). To prove \(\phi = 0\). Since \(M \leq \overline{M}\), hence \(M \leq \overline{M}\). Thus \(\text{Ker} \phi \leq K \leq M \leq \overline{M}\) which implies \(\text{Ker} \phi \leq \overline{M}\) and \(K \leq \overline{M}\) by lemma (2.30).

Now \(K = K \cap \text{E}(K) \leq \overline{M} \cap \text{E}(K)\) by lemma 2.2(1), where \(E(K)\) is the injective hull of \(K\). Hence \(\overline{M} \cap \text{E}(K) \leq \overline{M}\), so \(\overline{M} = (\overline{M} \cap \text{E}(K)) \oplus X\) for some \(X \leq \overline{M}\). Define \(\psi : K \oplus X \to \overline{M}\) by \(\psi = \phi\) on \(K\) and \(\phi = 0\) on \(X\). Since \(\overline{M}\) is quasi - injective, there exist \(\overline{\psi}: \overline{M} \to \overline{M}\) such that \(\overline{\psi}i = \psi\) where \(i: K \oplus X \to M\) be the inclusion mapping. Since \(\overline{\psi} = \psi\) on \(K \oplus X\), then \(\overline{\psi} = \phi\) on \(K\) and \(\overline{\psi} = 0\) on \(X\).

We can easily see that: \(\text{Ker} \overline{\psi} = \text{Ker} \phi \oplus X\) but \(\text{Ker} \phi \leq K \oplus X \leq \overline{M}\). \(\text{Ker} \overline{\psi} \leq \overline{M}\). Also \(\text{Ker} \overline{\psi} \subseteq \text{Ker} \phi\).

It follows that \(\text{Ker} \overline{\psi} \leq \overline{M}\), hence \(\overline{\psi} = 0\) since \(\overline{M}\) is \(t\)-ess. q-Ded. However \(\overline{\psi} = 0\) implies \(\phi = 0\).

Thus \(M\) is \(t\)-polyform.

\(\square\)To prove \(\overline{M}\) is \(t\)-ess. q-Ded. Let \(f \in \text{End}(M)\) and \(f \neq 0\). To show that \(\text{Ker} f \leq \overline{M}\), we shall prove that \(\text{Ker} f \leq \overline{M}\) and hence \(\text{Ker} f \leq M\). By [3, Lemma 2.3], there exists \(K \leq M\) such that \(\text{Ker} f \leq K\). Hence \(K \leq \overline{M}\) by Lemma 2.5, so that \(\overline{M} = K \oplus A\) for some \(A \leq M\). Define \(h: \overline{M} \to M\) by \(h|A = 0\) and \(h|K = f|K\). Hence \(\text{Ker} h = \text{Ker} f \oplus A\). But \(\text{Ker} f \leq K\), \(A \leq A\), implies \(\text{Ker} h = \text{Ker} f \oplus A \leq K \oplus A\) by
Lemma 2.2(2). Now for any \( \alpha \in \text{End } \bar{M} \), \( \text{Ker}(h\circ \alpha) = \alpha^{-1}(\text{Ker} h) \). Since \( \text{Ker} h \lesssim \bar{M} \), then \( \alpha^{-1}\text{Ker} h \lesssim \bar{M} \) by [10, 2014, cor. 1.2]. Thus \( \text{Ker}(h \circ \alpha) \)

\[
\cap M \lesssim \bar{M} \cap M = M
\]

by Lemma (2.2). Then by Theorem (4.1), \( 0 = \text{Hom}(\frac{M}{\text{Ker}(h \circ \alpha) \cap M}, \bar{M}) = \text{Hom}(h \circ \alpha) (M, \bar{M}) \) and so \( h(\alpha(M)) = 0 \). Since \( \alpha \in \text{End}(M) \) is arbitrary, \( \text{tr}(\bar{M}) = \sum_{\alpha \in \text{End}(M)} h(\alpha(M)) = 0 \). Thus \( h = 0 \) and

\[\text{Ker} f = K \lesssim \bar{M} \quad \text{Thus} \quad \text{Ker} f \lesssim \bar{M} .\]

**Corollary 4.4**

Let \( M \) be a quasi-injective module then \( M \) is t-polynorn form if and only if \( M \) is t-ess-q-Ded

**Proof:** It follows directly by Th. (4.3).

Recall that an \( R \)-module \( M \) is a \( t \)-Rickart if \( t_M(\phi) = \phi^{-1}(Z_2(M)) \) is a direct summand of \( M \) for every \( \phi \in \text{End}(M) \)[1, Def2.1].

Note that every nonsingular Rickart module is \( t \)-Rickart, every extending module and every \( Z_2 \)-torsion module (i.e a module \( M \) for which \( Z_2(M) = M \)) is \( t \)-Rickart. A Rickart module need not be \( t \)-Rickart, see[1, Ex.2.10]

We prove that

**Theorem 4.5**

If \( M \) is a \( t \)-polyform module, then \( \bar{M} \) is \( t \)-Rickart

**Proof:**

Since \( \frac{M}{Z_2(\bar{M})} \) is nonsingular, \( Z_2(\bar{M}) \lesssim \bar{M} \), hence \( Z_2(\bar{M}) \lesssim \bar{M} \) and \( \bar{M} \) is quasi-injective (hence extending) so that \( Z_2(\bar{M}) \) is a direct summand of \( M \). Thus \( M = Z_2(\bar{M}) \oplus C \) for some \( C \lesssim \bar{M} \). But \( C \approx \frac{M}{Z_2(\bar{M})} \) which is nonsingular, so \( C \) is nonsingular. But \( M \) is \( t \)-polyform, hence \( \bar{M} \) is \( t \)-ess. Quasi-Ded by Theorem 4.3. Thus \( \bar{M} \) is \( K \)-nonsingular (i.e ess. q-Ded). On other, \( \bar{M} \) is quasi-injective, so \( \bar{M} \) is extending. But \( \bar{M} \) is \( K \)-nonsingular extending module implies \( \bar{M} \) is Baer which implies Rickart by [15, Lemma 2.2.4, r.13].

Since \( C \lesssim \bar{M} \), then \( C \) is Rickart. Thus \( \bar{M} \) is \( t \)-Rickart by [11, Th2.6.1 (1\( \rightarrow \)2)]

**Remarks 4.6**

1. The converse of Th.(4.5) is not true if \( Z_2(\bar{M}) \neq 0 \)

**Proof:**

Since \( \bar{M} \) is \( t \)-Rickart, \( \bar{M} = Z_2(\bar{M}) \oplus C \), for some nonsingular Rickart submodule \( C \) of \( \bar{M} \). If \( Z_2(\bar{M}) \neq 0 \), then there \( i: Z_2(\bar{M}) \rightarrow M \), where \( i \) is the inclusion mapping, and \( i \neq 0 \). Thus ker \( f = (0) \). But \( (0) + Z_2(\bar{M}) \lesssim Z_2(M) \), thus

\[
(0) \lesssim Z_2(\bar{M}) \quad \text{That is} \quad \text{Ker} f \lesssim Z_2(\bar{M}) \quad \text{and so} \quad \bar{M} \text{ is not} \ t \text{-polyform therefore} \ (\bar{M}) \text{ is not} t \text{-ess.q-Ded by cor}(4.4)
\]

2. If \( Z_2(\bar{M}) = 0 \) and \( \bar{M} \) is \( t \)-Rickart, then \( M \) is \( t \)-polyform.

**Proof:** As in (1), \( \bar{M} = Z_2(\bar{M}) \oplus C \) where \( C \) is nonsingular Rickart. Since \( Z_2(\bar{M}) = 0 \), then \( \bar{M} = C \); that is \( \bar{M} \) is nonsingular, hence \( \bar{M} \) is \( t \)-polyform thus \( M \) is \( t \)-polyform by Rem & Ex.2.2.7.

Now we have:

**Theorem 4.7**

Let \( M \) be a \( t \)-polyform extending module. Then \( \bar{M} + M \) is \( t \)-Rickart module.

**Proof:** Since \( M \) is extending, \( M \) is \( t \)-Rickart. Also \( M \) is \( t \)-polyform implies \( \bar{M} \) is \( t \)-Rickart by (4.4). By [1, Th2.6.1] \( M = Z_2(M) \oplus A \), \( A \) is nonsingular Rickart sub-module of \( M \), \( \bar{M} = Z_2(\bar{M}) \oplus B \), \( B \) is a nonsingular Rickart sub-module of \( \bar{M} \). Hence \( \bar{M} \oplus M = Z_2(\bar{M}) \oplus Z_2(M) \oplus (B \oplus A) = Z_2(M) \oplus M \oplus (B \oplus A) \) by Lemma 2.6 hence \( B \oplus A \) is a nonsingular submodule of \( \bar{M} \oplus M \) since \( A \leq M \), then \( A \) is \( t \)-polyform and extending and so \( A \) is polyform and
extending \( B \leq M \) and \( \overline{M} \) is quasi-injective, hence \( B \) is a quasi-injective. On the other hand, \( M = Z_2(M) \oplus A \) implies \( \overline{M} = Z_2(M) \oplus \overline{A} = Z_2(M) \oplus A \). But \( \overline{M} = Z_2(M) \oplus B \), so \( B = \overline{A} \). Thus \( B \oplus A = \overline{A} \oplus A \) and hence by [14, prop 2.4.22, p.60], \( B \oplus A \) is Rickart and then by [11, Th 2.6.1], \( \overline{M} \oplus M \) is t-Rickart.

It is well-known that a sub module \( N \) of \( M \) is fully invariant if for each \( f \in \text{End}(M), f(N) \subseteq N \). Also recall the following basic fact: if \( N \) is a fully invariant sub module of \( M = M_1 \oplus M_2 \) then \( N = (N \cap M_1) \oplus (N \cap M_2) \).

**Proposition 4.8**

For an \( R \)-module \( M \), if \( E(M) \) (injective hull of \( M \)) is t-polyform, then \( Z_2(M) \) is a direct summand of \( M \).

**Proof:** Since \( E(M) \) is t-polyform, then \( \overline{E(M)} \) is t-Rickart by Th.(4.5). But \( E(M) = \overline{E(M)} \), hence \( E(M) \) is t-Rickart. Then by [1, Th.2.6.1] \( E(M) = Z_2(E(M)) \oplus A \). A is a nonsingular Rickart submodule of \( E(M) \) since \( M \) is a fully invariant submodule of \( E(M) \), then \( M = (Z_2(E(M)) \cap M) \oplus (A \cap M) \), but \( Z_2(M) = Z_2(E(M)) \cap M \). Thus \( M = Z_2(M) \oplus (A \cap M) \) therefore \( Z_2(M) \leq M \).

**References**