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Results on Multiplicative (Generalized) (α,β)-reverse Derivation on Prime Rings

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Abstract
Let $\mathcal{R}$ be a 2-torsion-free prime ring, $\mathbb{U}$ be non-zero square closed Lie ideal of $\mathcal{R}$, α and β be automorphisms of $\mathcal{R}$.A mapping $\mathrm{F}:\mathcal{R}\to\mathcal{R}$ is called a multiplicative (generalized) (α,β)-reverse derivation if $\mathrm{F}(ab)=\mathrm{F}(b)\alpha(a)+\beta(b)d(a)$ for all $a,b\in\mathcal{R}$ where $d:\mathcal{R}\to\mathcal{R}$ is any map . The purpose of this paper, is to give some important results of multiplicative (generalized) (α,β)-reverse derivation on square closed Lie ideals F that satisfying any one of the properties:
(i) $\mathrm{F}(uv)\pm\alpha(uv)=0$, (ii) $\mathrm{F}(uv)\pm\alpha(vu)=0$, (iii) $\mathrm{F}(u)\mathrm{F}(v)\pm\alpha(uv)=0$, (iv) $\mathrm{F}(u)\mathrm{F}(v)\pm\alpha(vu)=0$,
(v) $\mathrm{F}(uv)=\mathrm{F}(u)\mathrm{F}(v)$, (vi) $\mathrm{F}(uv)=\mathrm{F}(v)\mathrm{F}(u)$ and (vii) $\mathrm{F}[u,v]=0$ for all $u,v\in\mathbb{U}$.

Keywords: Prime Ring, Multiplicative (Generalized) (α,β) Reverse-Derivation, Lie ideal.

نتائج عن مشتقات المعكوسة-(α,β) الضربيه (المعممه) على الحلقات الاولية

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الخلاصة
لتكن $\mathcal{R}$ حلقة اولية طليقة الالتواء من النوع 2,$\mathbb{U}$ مثالي لي مربع مغلق غير صفري من $\mathcal{R}$, α وβ تشاكل على $\mathcal{R}$. نسمي التطبيق $\mathrm{F}:\mathcal{R}\to\mathcal{R}$ مشتق معكوس ضربي (α,β)(معمم) اذا $\mathrm{F}(ab)=\mathrm{F}(b)\alpha(a)+\beta(b)d(a)$ لكل $a,b\in\mathcal{R}$ عندما $d:\mathcal{R}\to\mathcal{R}$ اي تطبيق. غرض هذا البحث اعطاء بعض النتائج المهمه للمشتق المعكوس الضربي (α,β)(معمم) على مثالي لي مربع مغلق غير صفري والتي تحقق احد هذه الخواص
(i)$\mathrm{F}(uv)\pm\alpha(uv)=0$, (ii) $\mathrm{F}(uv)\pm\alpha(vu)=0$, (iii) $\mathrm{F}(u)\mathrm{F}(v)\pm\alpha(uv)=0$, (iv) $\mathrm{F}(u)\mathrm{F}(v)\pm\alpha(vu)=0$,(v) $\mathrm{F}(uv)=\mathrm{F}(u)\mathrm{F}(v)$, (vi) $\mathrm{F}(uv)=\mathrm{F}(v)\mathrm{F}(u)$ and (vii) $\mathrm{F}[u,v]=0\ \forall u,v\in\mathbb{U}$.

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1. Introduction

Let $\mathcal{R}$ will be denoted a ring with center $Z(\mathcal{R})$. For all $a, b \in \mathcal{R}$, we denote the commentator $ab - ba$ by$[a, b]$, and anti-commentator $ab + ba$ by $a \circ b$ [1]. A ring $\mathcal{R}$ is called a prime if $a\,\mathcal{R}\,b = 0$ either $a = 0$ or $b = 0$. A ring $\mathcal{R}$ is said to be 2-torsion free, if $2a = 0, a \in \mathcal{R}$ implies $a = 0$, [1]. An additive subgroup $\mathbb{U}$ of $\mathcal{R}$ is said to be Lie ideal of $\mathcal{R}$ if for all $a \in \mathbb{U}, r \in \mathcal{R}$ then $[a, r] \in \mathbb{U}$, and the Lie ideal of $\mathbb{U}$ is known as square closed Lie ideal if $a^2 \in \mathbb{U}$, where $a \in U$ [2]. If $\mathbb{U}$ is a square closed Lie ideal of $\mathcal{R}$, then $2ab \in \mathbb{U}\ \forall a, b \in \mathbb{U}$. A derivation is an additive mapping with the property $d(ab) = d(a)b + ad(b)$ where $a, b \in \mathcal{R}$.

An additive mapping $\mathrm{F}: \mathcal{R} \to \mathcal{R}$ is called a generalized derivation related with d if there exists a derivation $d: \mathcal{R} \to \mathcal{R}$ such that $\mathrm{F}(ab) = \mathrm{F}(a)b + ad(b)$ where $a, b \in \mathcal{R}$ [3]. The concept of multiplicative derivation was first introduced by Daif [4]. A mapping $d:: \mathcal{R} \to \mathcal{R}$ is called a multiplicative derivation if it satisfies $d(ab) = d(a)b + ad(b)$ for all $a, b \in \mathcal{R}$. Of course, these maps need not to be additive

In [5], a mapping $\mathrm{F}: \mathcal{R} \to \mathcal{R}$, is said to be a multiplicative generalized derivation if $\mathrm{F}(ab) = \mathrm{F}(a)b + ad(b)$ where $a, b \in \mathcal{R}$, where d is a derivation from $\mathcal{R}$ to $\mathcal{R}$. In [6], F is referred to a multiplicative (generalized) derivation if $\mathrm{F}(ab) = \mathrm{F}(a)b + ad(b)$for all $a, b \in \mathcal{R}$, where d is any map that is not an additive for necessity. Herstein, first time introduced the concept of reverse derivation in [7]; let $\mathcal{R}$ be a ring an additive mapping $d: \mathcal{R} \to \mathcal{R}$ is called a reverse derivation if $d\,(ab) = d(b)a + bd(a)$ for all $a, b \in \mathcal{R}$. In [8], let $\mathcal{R}$ be a ring a mapping $d: \mathcal{R} \to \mathcal{R}$ will be said to a multiplicative left centralizer if $d(ab) = d(a)b$ for all $a, b \in \mathcal{R}$, where d is not necessary additive. In [9]; a multiplicative left reverse α-centralizer of a ring $\mathcal{R}$ is a mapping that satisfies the condition $d(ab) = d(b)\alpha(a)$ and has the form $d: \mathcal{R} \to \mathcal{R}$ for each $a, b \in \mathcal{R}$, where α be a mapping of $\mathcal{R}$.

In [10], a mapping $\mathrm{F}: \mathcal{R} \to \mathcal{R}$ is said to be a multiplicative (generalized)-reverse derivation of $\mathcal{R}$, if $\mathrm{F}(ab) = \mathrm{F}(b)a + bd(a)$, for all $a, b \in \mathcal{R}$, where d is any map. From the idea of a multiplicative (generalized)-reverse derivation, the author of [10] created a multiplicative (generalized) (α, β)-reverse derivation. If $\mathrm{F}(ab) = \mathrm{F}(b)\alpha(a) + \beta(b)d(a)$ for all $a, b \in \mathcal{R}$, where α, β are automorphisms of $\mathcal{R}$, the mapping $\mathrm{F}: \mathcal{R} \to \mathcal{R}$ is known as a multiplicative (generalized) (α, β)-reverse derivation of $\mathcal{R}$ associated with a map d on $\mathcal{R}$.

The following identities related to multiplicative (generalized) derivation on Lie ideals in prime rings is investigated in [11]:

1) $F(uv) \pm uv = 0$,
2) $F(uv) \pm vu = 0$,
3) $F(u)F(v) \pm uv = 0$,
4) $F(u)F(v) \pm vu = 0$, for all $u, v \in \mathbb{U}$.

The purpose of this paper is finding an important results for a 2-torsion free prime rings admitting a multiplicative (generalized) (α, β)-reverse derivation on square closed Lie ideal [for short we use m(g)(α, β)- rd].

Through this paper, we study the following identities:

(i) $\mathrm{F}(uv) \pm \alpha(uv) = 0$,
(ii) $\mathrm{F}(uv) \pm \alpha(vu) = 0$,
(iii) $\mathrm{F}(u)\mathrm{F}(v) \pm \alpha(uv) = 0$,
(iv) $\mathrm{F}(u)\mathrm{F}(v) \pm \alpha(vu) = 0$,
(v) $\mathrm{F}(uv) = \mathrm{F}(u)\mathrm{F}(v)$,
(vi) $\mathrm{F}(uv) = \mathrm{F}(v)\mathrm{F}(u)$ and
(vii)$\mathrm{F}[u, v] = 0\ \forall u, v \in \mathbb{U}$.

Where $\mathbb{U}$ be a non-zero square closed Lie ideal of 2-torsion free prime ring, α and β be automorphisms of $\mathcal{R}$.

We need the following lemmas to proof of our main results.

Lemma 1.1. [12]: Suppose $\mathcal{R}$ is a prime ring and F is an m (g)(α,β)- rd of $\mathcal{R}$ associated with a map d:$\mathcal{R}\to\mathcal{R}$, then either $\mathcal{R}$ is commutative or d is the multiplicative left α-centralizer.

Lemma 1.2. [13]: Let $\mathbb{U}$ be a Lie ideal of $\mathcal{R}$ with $\mathbb{U}\nsubseteq Z(\mathcal{R})$and $\mathcal{R}$ be a 2-torsion free prime ring. If $m,n\in\mathcal{R}$ and $m\,U\,n=0$, then either $m=0$ or $n=0$.

Lemma 1.3. [14]: Let $\mathcal{R}$ be a 2-torsionfree prime ring, $\mathbb{U}$ be a Lie ideal of $\mathcal{R}$. If $\mathbb{U}$ is a commutative Lie ideal of $\mathcal{R}$, then $\mathbb{U}\subseteq Z(\mathcal{R})$.

2. Main Results

Theorem 2.1

Let F be an m (g)(α,β)- rd on $\mathcal{R}$ related with a map d: $\mathcal{R}\to\mathcal{R}$. If $F(uv)\pm\alpha(uv)=0$ for all $u,v\in\mathbb{U}$, then $d(\mathbb{U})=0$.

Proof:

Suppose that $\mathcal{R}$ is non-commutative and $\mathbb{U}\nsubseteq Z(\mathcal{R})$.

We have

$$\mathrm{F}(uv)-\alpha(uv)=0\,. \tag{1}$$

Exchange v by $2vz$ in (1), where $z\in\mathbb{U}$, gives

$$\mathrm{F}(z)\alpha(uv)+\beta(z)d(uv)-\alpha(uvz)=0.$$

By using Lemma 1.1, it gives

$$\mathrm{F}(z)\alpha(u)\alpha(v)+\beta(z)d(u)\alpha(v)-\alpha(uvz)=0\,. \tag{2}$$

On the other hand $\mathrm{F}(vz)\alpha(u)+\beta(vz)d(u)-\alpha(uvz)=0$,

$$\mathrm{F}(z)\alpha(v)\alpha(u)+\beta(z)d(v)\alpha(u)+\beta(v)\beta(z)d(u)-\alpha(uvz)=0. \tag{3}$$

Subtract Equation (3) from Equation (2), gives

$$\mathrm{F}(z)\alpha[u,v]+\beta(z)(d(u)\alpha(v)-d(v)\alpha(u))-\beta(v)\beta(z)d(u)=0. \tag{4}$$

Substituting v with $2vu$ in (4), and applying Lemma 1.1, this implies

$$\mathrm{F}(z)\alpha[u,v]\alpha(u)+\beta(z)(d(u)\alpha(v)\alpha(u)-d(v)\alpha(u)\alpha(u))-\beta(v)\beta(u)\beta(z)d(u)=0. \tag{5}$$

By multiplying Equation (4) by α(u) on the right we have

$$\mathrm{F}(z)\alpha[u,v]\alpha(u)+\beta(z)(d(u)\alpha(v)-d(v)\alpha(u))\alpha(u)-\beta(v)\beta(z)d(u)\alpha(u)=0. \tag{6}$$

Subtract Equation (6) from Equation (5), gives

$$\beta(v)(\beta(z)d(u)\alpha(u)-\beta(u)\beta(z)d(u))=0\,. \tag{7}$$

By putting $v=[v,r]$ in (7), and use it, where $r\in\mathcal{R}$, we obtain

$$\beta(v)\,\mathcal{R}\,(\beta(z)d(u)\alpha(u)-\beta(u)\beta(z)d(u))=0.$$

$$\beta(z)d(u)\alpha(u)-\beta(u)\beta(z)d(u)=0\,. \tag{8}$$

We substitute z by $2vz$ in (8), then

$$\beta(v)\beta(z)d(u)\alpha(u)-\beta(u)\beta(v)\beta(z)d(u)=0\,. \tag{9}$$

Left multiply Equation (8), by $\beta(v)$ and subtract it from Equation (9), then

$$\beta[v,u]\beta(z)d(u)=0.$$

Taking β^{-1} in above relation, gives

$$[v,u]\,z\,\beta^{-1}(d(u))=0\,.$$

By Lemma 1.2, since $\mathbb{U}\nsubseteq Z(\mathcal{R})$ becomes, $d(\mathbb{U})=0$.

On the other hand, if $\mathcal{R}$ is commutative, then $\mathbb{U}\subseteq Z(\mathcal{R})$.

It is clear $\alpha(u),\beta(u)\subseteq Z(\mathcal{R})$.

From Equation (7), we have $\beta(v)(\beta(z)\alpha(u)d(u)-\beta(z)\beta(u)d(u))=0$.

$$\beta(v)\beta(z)(\alpha(u)d(u)-\beta(u)d(u))=0\,. \tag{10}$$

Exchange v by$[v,r]$, where $r\in\mathcal{R}$ in (10), and using it, this yield

$$\beta(v)\,\mathcal{R}\,\beta(z)(\alpha(u)d(u)-\beta(u)d(u))=0.$$

$$\beta(z)(\alpha(u)d(u)-\beta(u)d(u))=0\,.$$

Change z by $[z,r]$ in above relation and using the same technique as above, we finally obtain

$$((\alpha-\beta)u)d(u)=0\ .$$

In the relationship above, if we multiply it by r on the left, we obtain,

$$r(\alpha-\beta)\ (u)d(u)=0.$$

$$(\alpha-\beta)u\ Ʀ\ d(u)=0.$$

Since $Ʀ$ is a prime ring, we get either $(\alpha-\beta)=0$ or $d(\mathbb{U})=0$.
If $\alpha=\beta$ on $\mathbb{U}$, from Equation (4), we obtain

$$\beta(z)d(u)\beta(v)-\beta(z)d(v)\beta(u)-\beta(v)\beta(z)d(u)=0.$$

Let $v=u$ in above relation, we have $\beta(u)\beta(z)d(u)=0$.
By taking β^{-1}in above equation, we get $u\ z\ \beta^{-1}\ (d(u))=0$.
Multiplying above equation by r on the left, then

$$r\ u\ z\beta^{-1}\ (d(u))=0\ .$$

$$\beta^{-1}\ (d(u))=0.$$

Again, multiplying above equation by r on the left, we find $r\ z\ \beta^{-1}\ (d(u))=0$.

$$d(\mathbb{U})=0.$$

For the case $\mathrm{F}(uv)+\alpha(uv)=0$ for all $u,v\in\mathbb{U}$,the same conclusion is reached using a similar approach.

Theorem 2.2
Let F be an m (g)(α,β)- rd on $Ʀ$ related with a map $d{:}\,Ʀ\rightarrow Ʀ$. If $\mathrm{F}(uv)\pm\alpha(vu)=0$ for all $u,v\in\mathbb{U}$, then $d(\mathbb{U})$=0.
Proof:
Suppose that $\mathbb{U}\nsubseteq Z(Ʀ)$.

$$F(uv)-\alpha(vu)=0\ . \quad (11)$$

Replace u by u^2 in Equation (11), and using it, and taking β^{-1},we get

$$u\ v\ \beta^{-1}\ (d(u))=0. \quad (12)$$

Using Lemma 1.2, implies β^{-1} $(d(u))$=0. Since β is an automorplism of $Ʀ$, then becomes $d(\mathbb{U})=0$.
Now, if $\mathbb{U}\subseteq Z(Ʀ)$. By multiplying the left side of Equation (12), by r, we obtain

$$r\ u\ v\ \beta^{-1}(d(u))=0.$$

Since $\mathbb{U}\subseteq Z(Ʀ)$, we get $u\ r\ v\ \beta^{-1}\ (d(u))=0$ and then

$$v\ \beta^{-1}\ (d(u))=0\ . \quad (13)$$

Place $v=[v,r]$ in (13), for all $r\in Ʀ$, and use Equation (13), then $\beta^{-1}\ (d(u))=0$.
Since β is an automorphism of $Ʀ$, we have $d(\mathbb{U})=0$.
For the case $\mathrm{F}(uv)+\alpha(vu)=0$ for all $u,v\in\mathbb{U}$, the same conclusion is reached by using a similar approach.

Theorem 2.3
Let F be an m (g)(α,β)- rd on $Ʀ$ related with a map d: $Ʀ\rightarrow Ʀ$. If $F(u)F(v)\pm\alpha(uv)=0$ for all $u,v\in\mathbb{U}$, then $\mathbb{U}\subseteq Z(Ʀ)$.
Proof:
Assume that $Ʀ$ is non-commutative.

$$\mathrm{F}(u)\mathrm{F}(v)-\alpha(uv)=0\ . \quad (14)$$

Substituting $2vz$ for v in Equation (14), where $z\in\mathbb{U}$, implies

$$\mathrm{F}(u)\mathrm{F}(z)\alpha(v)+\mathrm{F}(u)\beta(z)d(v)-\alpha(uvz)=0\ . \quad (15)$$

Substitute $2vg$ for v in Equation (15), where $g\in\mathbb{U}$, and by applying Lemma 1.1, we get

$$\mathrm{F}(u)\mathrm{F}(z)\alpha(v)\alpha(g)+\mathrm{F}(u)\beta(z)d(v)\alpha(g)-\alpha(uvgz)=0. \quad (16)$$

Multiplying right side of Equation (15), by $\alpha(g)$, this gives

$$\mathrm{F}(u)F(z)\alpha(v)\alpha(g)+\mathrm{F}(u)\beta(z)d(v)\alpha(g)-\alpha(uvz)\alpha(g)=0. \quad (17)$$

Comparing Equation (16) and Equation (17), to get

$$\alpha(uv)\alpha[z,g] = 0 .$$

By taking α^{-1}in above Equation, we find

$$u\, v\, [z,g] = 0. \quad (18)$$

Replace u by $[u,r]$ in (18), and use it, where $r \in R$, we get

$$v\, [z,g] = 0 . \quad (19)$$

Exchange v by $[v,r]$ in (19), and use it, implies

$$vR[z,g] = 0.$$

By using Lemma 1.3, we get $\mathbb{U} \subseteq Z(R)$.

If R is commutative, we get our result.

For the case $F(u)F(v) + \alpha(uv) = 0$ for all $u, v \in \mathbb{U}$, the same conclusion is reached by using the similar approach.

Theorem 2.4

Let F be an m (g)(α,β)- rd on R related with a map $d: R \to R$. If $F(u)F(v) \pm \alpha(vu) = 0$ for all $u, v \in \mathbb{U}$, then $\mathbb{U} \subseteq Z(R)$.

Proof:

Assume that R is non-commutative

$$F(u)F(v) - \alpha(vu) = 0. \quad (20)$$

Taking $2vz$ instead of v in (20), where z$\in \mathbb{U}$, we obtain

$$F(u)F(z)\alpha(v) + F(u)\beta(z)d(v) - \alpha(vzu) = 0. \quad (21)$$

By substituting $2vg$ for v in Equation (21), where $g \in \mathbb{U}$, by using Lemma 1.1, then

$$F(u)F(z)\alpha(v)\alpha(g) + F(u)\beta(z)d(v)\alpha(g) - \alpha(vgzu) = 0. \quad (22)$$

Multiplying right side of Equation (21), by $\alpha(g)$, then

$$F(u)F(z)\alpha(v)\alpha(g) + F(u)\beta(z)d(v)\alpha(g) - \alpha(vzu)\alpha(g) = 0. \quad (23)$$

Comparing (22) and (23), we get

$\alpha(v)\alpha[zu,g] = 0.$

By taking α^{-1} in above relation, gives

$$v\, [zu,g] = 0 . \quad (24)$$

Replacing v by $[v,r]$, in (24), and using (24), becomes

$$z[u,g] + [z,g]u = 0 . \quad (25)$$

Exchange u by $2ui$ in (25), where $i \in \mathbb{U}$, implies

$$zu[i,g] + z[u,g]i + [z,g]ui = 0 \quad (26)$$

Multiplying Equation (25), by i on the right then

$$z[u,g]i + [z,g]ui = 0 . \quad (27)$$

Comparing (27) and (26), becomes

$$z\, u\, [i,g] = 0 .$$

By putting $z = [z,r]$ in the relationship above and using it, we have $u\, [i,g] = 0$.

Putting $u = [u,r]$ in above equation and using it, for all $r \in R$, then $[i,g] = 0$.

By using Lemma 1.3, we conclude $\mathbb{U} \subseteq Z(R)$.

If R is commutative, we get our result.

For the case $F(u)F(v) + \alpha(vu) = 0$ for all $u, v \in \mathbb{U}$, the same conclusion is reached by using the similar approach.

Theorem 2.5

Let F acts as a homomorphism and be an m (g)(α,β)- rd on R related with a map $d: R \to R$, then $d(\mathbb{U}) = 0$ or $\mathbb{U} \subseteq Z(R)$.

Proof:
Since $\mathbb{U} \nsubseteq Z(\mathcal{R})$, then $\mathcal{R}$ is non-commutative.
Since F acts as a homomorphism on $\mathcal{R}$, then $\mathrm{F}(uv) = \mathrm{F}(u)\mathrm{F}(v)$.

$$\mathrm{F}(v)\alpha(u) + \beta(v)d(u) = F(u)F(v)\,. \tag{28}$$

We substitute v by $2zv$ in (28), where $z \in \mathbb{U}$, to get

$$\mathrm{F}(v)\alpha(z)\alpha(u) + \beta(v)d(z)\alpha(u) + \beta(z)\beta(v)d(u) - \mathrm{F}(u)F(v)\alpha(z) - \mathrm{F}(u)\beta(v)d(z) = 0\,. \tag{29}$$

Multiplying (28), by α(z) on the right, gives

$$\mathrm{F}(v)\alpha(u)\alpha(z) + \beta(v)d(u)\alpha(z) - \mathrm{F}(u)F(v)\alpha(z) = 0. \tag{30}$$

We subtracting Equation (30) from Equation (29), we have

$$\mathrm{F}(v)\alpha[z,u] + \beta(v)d(z)\alpha(u) + \beta(z)\beta(v)d(u) - \mathrm{F}(u)\beta(v)d(z) - \beta(v)d(u)\alpha(z) = 0. \tag{31}$$

Subsisting $2zu$ in the place of z in Equation (31), by applying Lemma 1.1, then

$$\mathrm{F}(v)\alpha[z,u]\alpha(u) + \beta(v)d(z)\alpha(u)\alpha(u) + \beta(z)\beta(u)\beta(v)d(u) - \mathrm{F}(u)\beta(v)d(z)\alpha(u) - \beta(v)d(u)\alpha(z)\alpha(u) = 0. \tag{32}$$

Multiplying right side of Equation (31), by $\alpha(u)$, gives

$$\mathrm{F}(v)\alpha[z,u]\alpha(u) + \beta(v)d(z)\alpha(u)\alpha(u) + \beta(z)\beta(v)d(u)\alpha(u) - \mathrm{F}(u)\beta(v)d(z)\alpha(u) - \beta(v)d(u)\alpha(z)\alpha(u) = 0\,. \tag{33}$$

From Equation (32) and Equation (33), we get

$$\beta(z)(\beta(v)d(u)\alpha(u) - \beta(u)\beta(v)d(u)) = 0\,.$$

Putting $z = [z,r]$ in the relationship above and using it, results in

$$\beta(z)\beta(r)(\beta(v)d(u)\alpha(u) - \beta(u)\beta(v)d(u)) = 0.$$

Since β is an automorphism of $\mathcal{R}$, then

$$\beta(v)d(u)\alpha(u) - \beta(u)\beta(v)d(u) = 0\,. \tag{34}$$

Substituting v for $2nv$ in (34), where $n \in \mathbb{U}$, we get

$$\beta(n)\beta(v)d(u)\alpha(u) - \beta(u)\beta(n)\beta(v)d(u) = 0\,. \tag{35}$$

Multiplying Equation (34), by β(n), on the left and compare Equation (35), gives

$$\beta[u,n]\beta(v)d(u) = 0\,.$$

By taking β^{-1}in above relation, we get $[u,n]v\,\beta^{-1}\,(d(u)) = 0$.
By using Lemma 1.2, and since $\mathbb{U} \nsubseteq Z(\mathcal{R})$, we have $\beta^{-1}\,(d(u)) = 0$, we get $d(\mathbb{U}) = 0$.
If $\mathbb{U} \subseteq Z(\mathcal{R})$, we achieve our goal.

Theorem 2.6
Let F acts as anti- homomorphism and be an m (g)(α,β)- rd on $\mathcal{R}$ related with a map d: $\mathcal{R}\rightarrow\mathcal{R}$, then either $d(\mathbb{U}) = 0$ or $\mathbb{U} \subseteq Z(\mathcal{R})$.
Proof:
Since $\mathbb{U} \nsubseteq Z(\mathcal{R})$, then $\mathcal{R}$ is non-commutative.
Since F acts as anti-homomorphism of $\mathcal{R}$, then

$$\mathrm{F}(uv) = \mathrm{F}(v)\mathrm{F}(u)\,. \tag{36}$$

Substitute $2zu$ for u in Equation (36), where $z \in \mathbb{U}$, to give

$$\mathrm{F}(v)\alpha(zu) + \beta(v)d(zu) = \mathrm{F}(v)\mathrm{F}(u)\alpha(z) + \mathrm{F}(v)\beta(u)d(z).$$

By applying Lemma 1.1

$$\mathrm{F}(v)\alpha(z)\alpha(u) + \beta(v)d(z)\alpha(u) - \mathrm{F}(v)\mathrm{F}(u)\alpha(z) - \mathrm{F}(v)\beta(u)d(z) = 0. \tag{37}$$

On the other hand

$$\mathrm{F}(v)\alpha(u)\alpha(z) + \beta(v)d(u)\alpha(z) + \beta(u)\beta(v)d(z) - \mathrm{F}(v)\mathrm{F}(u)\alpha(z) - \mathrm{F}(v)\beta(u)d(z) = 0\,. \tag{38}$$

Subtract Equation (38) from Equation (37), gives

$$\mathrm{F}(v)\alpha[z,u] + \beta(v)d(z)\alpha(u) - \beta(v)d(u)\alpha(z) - \beta(u)\beta(v)d(z) = 0. \tag{39}$$

By putting $u = 2uz$ in (39), so by applying Lemma 1.1, we have

$$\mathrm{F}(v)\alpha[z,u]\alpha(z) + \beta(v)d(z)\alpha(u)\alpha(z) - \beta(v)d(u)\alpha(z)\alpha(z) - \beta(u)\beta(z)\beta(v)d(z) = 0. \tag{40}$$

Equation (39) is multiplied by $\alpha(z)$ on the right, it implies

$$\mathrm{F}(v)\alpha[z,u]\alpha(z)+\beta(v)d(z)\alpha(u)\alpha(z)-\beta(v)d(u)\alpha(z)\alpha(z)-\beta(u)\beta(v)d(z)\alpha(z)=0 \quad (41)$$

Subtracted Equation (41) from Equation (40), then

$$\beta(u)(\beta(v)d(z)\alpha(z)-\beta(z)\beta(v)d(z))=0\,.$$

Putting $u=[u,r]$ in the relationship above and using it, then

$$\beta(u)\ R\ (\beta(v)d(z)\alpha(z)-\beta(z)\beta(v)d(z)=0.$$

$$\beta(v)d(z)\alpha(z)-\beta(z)\beta(v)d(z)=0\,.$$

Let $z=u$ in over equation, gives

$$\beta(v)d(u)\alpha(u)-\beta(u)\beta(v)d(u)=0\,.$$

The proof follows from the Theorem 2.5, after the Equation (34) and we get the result then either $\mathbb{U}\subseteq Z(R)$ or $d(\mathbb{U})=0$.

Theorem 2.7

Let F be an m (g)(α,β)- rd on R related with a map $d: R\to R$. If $\mathrm{F}[u,v]=0$ for all $u,v\in\mathbb{U}$, then either $d(\mathbb{U})=0$ or $\mathbb{U}\subseteq Z(R)$.

Proof:

Suppose that $\mathbb{U}\nsubseteq Z(R)$.

$$\mathrm{F}[u,v]=0\,. \quad (42)$$

Taking $2uv$ instead of v in (42), and using it, it gives

$$\beta[u,v]d(u)=0\,. \quad (43)$$

Substituting $2zv$ for v in (43), and using it, we have, $\beta[u,z]\beta(v)d(u)=0$.

$\beta^{-1}\ (\beta[u,z]\beta(v)d(u))=0.$

$$[u,z]\ \mathrm{U}\beta^{-1}(d(u))=0.$$

By using Lemma 1.2, and because $\mathbb{U}\nsubseteq Z(R)$ implies that $\beta^{-1}\ (d(u))=0$.

Since β is an automorphism of R, we get $d(\mathbb{U})=0$.

If $\mathbb{U}\subseteq Z(R)$ then, we get our result.

References

[1] I. N. Herstein, "Topics in ring theory," University of Chicago Press, 1969.

[2] M . Ashraf, N. Rehman, and Sh. Ali, "On Generalized (α, β)-Derivations in prime rings," *Algebra Colloquium*, vol. 17, pp. 865-874, 2010.

[3] M. Brešar, "On the distance of the composition of two derivation to the generalized derivations, " *Glasgow. Math. J*., vol. 33, no. 1, pp. 89-93, 1991.

[4] M. N. Daif, "When is a multiplicative derivation additive," *Int. J. Math. Math. Sci.,* vol. 14, no. 3, pp. 615-618, 1991.

[5] M. N. Daif, and M.S. Tammam El-Sayiad, "Multiplicative generalized derivation, which are additive," *East-west J. Math*., vol. 9, no. 1, pp. 31-37, 1997.

[6] B. Dhara, and S. Ali, "On multiplicative (generalized) derivation in prime and semi prime rings," *Aequat. Math.*, vol. 86, no. (1-2), pp. 65-79, 2013.

[7] I. N. Herstein, "Jordan derivation of prime rings," *Proc. Amer.Math.Soc.,* vol. 8, pp. 1104-1110, 1957.

[8] M. N. Daif, and M.S. Tammam El-Sayiad and V. D. Filippis, "Multiplicative of left centralizers forcing additivity," *Bol. Soc. Parana. Mat*., vol. 32, no. 1, pp. 61-69, 2014.

[9] Z.S.M. Alhaidary and A.H. Majeed, " Square Closed Lie Ideals and Multiplicative (Generalized) (α, β) Reverse Derivation of Prime Rings ," *journal of Discrete Mathematical Science & cryptography*, 2021.

[10] S.k. Tiwari, R.K. Sharma, and B. Dhara, "Some theorems of commutativity on semiprime ring with mapping," *Southeast Asian Bull. Math*., vol. 42, no. 2, pp. 279-292, 2018.

[11] S.Ali, B.Dhara, N. Dar and A.N.Khan, "On Lie ideals with multiplicative (generalized)-derivation in prime and semiprime rings," *Beitr Algebra Geon*., 56(2),pp. 325-337, 2015.

[12] Z.S.M. Alhaidary and A.H. Majeed, "Commutatively Results for Multiplicative (Generalized) (α, β) Reverse Derivations on Prime Rings," *Iraq journal of science*, vol. 62, no. 9, pp.3102-

3113, 2021.

[13] J. Bergen, I. N. Herstein, and J. W. Kerr, "Lie ideals and derivations of prime rings," *Journal of Algebra*, vol. 71, pp. 259-267, 1981.

[14] N. Rehman, "On Commutativity of prime rings with generalized derivations," *Math. J. Okayama University*, vol. 44, pp. 43-49, 2002.