Alhaidary and Majeed Iraqi Journal of Science, 2024, Vol. xx, No. x, pp: xx
DOI: 10.24996/ijs.2024.65.12.29

ISSN: 0067-2904

Results on Multiplicative (Generalized) (α,β)-reverse Derivation on Prime Rings

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Received: 20/10/2023 Accepted: 5/2/2024 Published: xx

Abstract

Let $\mathcal{R}$ be a 2-torsion-free prime ring, $\mathbb{U}$ be non-zero square closed Lie ideal of $\mathcal{R}$, α and β be automorphisms of $\mathcal{R}$.A mapping $\mathrm{F}: \mathcal{R} \rightarrow \mathcal{R}$ is called a multiplicative (generalized) (α, β)-reverse derivation if $\mathrm{F}(ab) = \mathrm{F}(b)\alpha(a) + \beta(b)d(a)$ for all $a, b \in \mathcal{R}$ where $d: \mathcal{R} \rightarrow \mathcal{R}$ is any map . The purpose of this paper, is to give some important results of multiplicative (generalized) (α,β)-reverse derivation on square closed Lie ideals F that satisfying any one of the properties:
(i) $\mathrm{F}(uv) \pm \alpha(uv) = 0$, (ii) $\mathrm{F}(uv) \pm \alpha(vu) = 0$, (iii) $\mathrm{F}(u)\mathrm{F}(v) \pm \alpha(uv) = 0$, (iv) $\mathrm{F}(u)\mathrm{F}(v) \pm \alpha(vu) = 0$,
(v) $\mathrm{F}(uv) = \mathrm{F}(u)\mathrm{F}(v)$, (vi) $\mathrm{F}(uv) = \mathrm{F}(v)\mathrm{F}(u)$ and (vii) $\mathrm{F}[u, v] = 0$ for all $u, v \in \mathbb{U}$.

Keywords: Prime Ring, Multiplicative (Generalized) (α, β) Reverse-Derivation, Lie ideal.

نتائج عن مشتقات المعكوسة–(α,β) الضربيه (المعممه) على الحلقات الاولية

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الخلاصة

لتكن $\mathcal{R}$ حلقة اولية طليقة الالتواء من النوع $\mathbb{U}$,2 مثالي لي مربع مغلق غير صفري من $\mathcal{R}$ ، βو α تشاكل على $\mathcal{R}$. نسمي التطبيق $\mathrm{F}: \mathcal{R} \rightarrow \mathcal{R}$ مشتق معكوس ضربي (α, β)(معمم) اذا $\mathrm{F}(ab) = \mathrm{F}(b)\alpha(a) + \beta(b)d(a)$ لكل $a, b \in \mathcal{R}$ عندما $d: \mathcal{R} \rightarrow \mathcal{R}$ اي تطبيق. غرض هذا البحث اعطاء بعض النتائج المهمه للمشتق المعكوس الضربي (α, β)(معمم) على مثالي لي مربع مغلق غير صفري والتي تحقق احد هذه الخواص
(i)$\mathrm{F}(uv) \pm \alpha(uv) = 0$, (ii) $\mathrm{F}(uv) \pm \alpha(vu) = 0$, (iii) $\mathrm{F}(u)\mathrm{F}(v) \pm \alpha(uv) = 0$, (iv) $\mathrm{F}(u)\mathrm{F}(v) \pm \alpha(vu) = 0$,(v) $\mathrm{F}(uv) = \mathrm{F}(u)\mathrm{F}(v)$, (vi) $\mathrm{F}(uv) = \mathrm{F}(v)\mathrm{F}(u)$ and (vii) $\mathrm{F}[u, v] = 0$ $\forall u, v \in \mathbb{U}$.

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1. Introduction

Let Ɽ will be denoted a ring with center $Z(Ɽ)$. For all $a, b \in Ɽ$, we denote the commentator $ab - ba$ by $[a, b]$, and anti-commentator $ab + ba$ by $a \circ b$ [1]. A ring Ɽ is called a prime if $a\, Ɽ\, b = 0$ either $a = 0$ or $b = 0$. A ring Ɽ is said to be 2-torsion free, if $2a = 0, a \in Ɽ$ implies $a = 0$, [1]. An additive subgroup $\mathbb{U}$ of Ɽ is said to be Lie ideal of Ɽ if for all $a \in \mathbb{U}, r \in Ɽ$ then $[a, r] \in \mathbb{U}$, and the Lie ideal of $\mathbb{U}$ is known as square closed Lie ideal if $a^2 \in \mathbb{U}$, where $a \in U$ [2]. If $\mathbb{U}$ is a square closed Lie ideal of Ɽ, then $2ab \in \mathbb{U}\ \forall a, b \in \mathbb{U}$. A derivation is an additive mapping with the property $d(ab) = d(a)b + ad(b)$ where $a, b \in Ɽ$.

An additive mapping $\mathrm{F}: Ɽ \to Ɽ$ is called a generalized derivation related with d if there exists a derivation $d: Ɽ \to Ɽ$ such that $\mathrm{F}(ab) = \mathrm{F}(a)b + ad(b)$ where $a, b \in Ɽ$ [3]. The concept of multiplicative derivation was first introduced by Daif [4]. A mapping $d:: Ɽ \to Ɽ$ is called a multiplicative derivation if it satisfies $d(ab) = d(a)b + ad(b)$ for all $a, b \in Ɽ$. Of course, these maps need not to be additive

In [5], a mapping $\mathrm{F}: Ɽ \to Ɽ$, is said to be a multiplicative generalized derivation if $\mathrm{F}(ab) = \mathrm{F}(a)b + ad(b)$ where $a, b \in Ɽ$, where d is a derivation from Ɽ to Ɽ. In [6], F is referred to a multiplicative (generalized) derivation if $\mathrm{F}(ab) = \mathrm{F}(a)b + ad(b)$ for all $a, b \in Ɽ$, where d is any map that is not an additive for necessity. Herstein, first time introduced the concept of reverse derivation in [7]; let Ɽ be a ring an additive mapping $d: Ɽ \to Ɽ$ is called a reverse derivation if $d\,(ab) = d(b)a + bd(a)$ for all $a, b \in Ɽ$. In [8], let Ɽ be a ring a mapping $d: Ɽ \to Ɽ$ will be said to a multiplicative left centralizer if $d(ab) = d(a)b$ for all $a, b \in Ɽ$, where d is not necessary additive. In [9]; a multiplicative left reverse α-centralizer of a ring Ɽ is a mapping that satisfies the condition $d(ab) = d(b)\alpha(a)$ and has the form $d: Ɽ \to Ɽ$ for each $a, b \in Ɽ$, where α be a mapping of Ɽ.

In [10], a mapping $\mathrm{F}: Ɽ \to Ɽ$ is said to be a multiplicative (generalized)-reverse derivation of Ɽ, if $\mathrm{F}(ab) = \mathrm{F}(b)a + bd(a)$, for all $a, b \in Ɽ$, where d is any map. From the idea of a multiplicative (generalized)-reverse derivation, the author of [10] created a multiplicative (generalized) (α, β)-reverse derivation. If $\mathrm{F}(ab) = \mathrm{F}(b)\alpha(a) + \beta(b)d(a)$ for all $a, b \in Ɽ$, where α, β are automorphisms of Ɽ, the mapping $\mathrm{F}: Ɽ \to Ɽ$ is known as a multiplicative (generalized) (α, β)-reverse derivation of Ɽ associated with a map d on Ɽ.

The following identities related to multiplicative (generalized) derivation on Lie ideals in prime rings is investigated in [11]:

1) $F(uv) \pm uv = 0$,
2) $F(uv) \pm vu = 0$,
3) $F(u)F(v) \pm uv = 0$,
4) $F(u)F(v) \pm vu = 0$, for all $u, v \in \mathbb{U}$.

The purpose of this paper is finding an important results for a 2-torsion free prime rings admitting a multiplicative (generalized) (α, β)-reverse derivation on square closed Lie ideal [for short we use m(g)(α, β)- rd].

Through this paper, we study the following identities:

(i) $\mathrm{F}(uv) \pm \alpha(uv) = 0$,
(ii) $\mathrm{F}(uv) \pm \alpha(vu) = 0$,
(iii) $\mathrm{F}(u)\mathrm{F}(v) \pm \alpha(uv) = 0$,
(iv) $\mathrm{F}(u)\mathrm{F}(v) \pm \alpha(vu) = 0$,
(v) $\mathrm{F}(uv) = \mathrm{F}(u)\mathrm{F}(v)$,
(vi) $\mathrm{F}(uv) = \mathrm{F}(v)\mathrm{F}(u)$ and
(vii) $\mathrm{F}[u, v] = 0\ \forall u, v \in \mathbb{U}$.

Where $\mathbb{U}$ be a non-zero square closed Lie ideal of 2-torsion free prime ring, α and β be automorphisms of Ɽ.

We need the following lemmas to proof of our main results.

Lemma 1.1. [12]: Suppose $\mathcal{R}$ is a prime ring and F is an m (g)(α,β)- rd of $\mathcal{R}$ associated with a map d:$\mathcal{R}\to\mathcal{R}$, then either $\mathcal{R}$ is commutative or d is the multiplicative left α-centralizer.

Lemma 1.2. [13]: Let $\mathbb{U}$ be a Lie ideal of $\mathcal{R}$ with $\mathbb{U}\nsubseteq Z(\mathcal{R})$and $\mathcal{R}$ be a 2-torsion free prime ring. If $m,n\in\mathcal{R}$ and $m\,U\,n=0$, then either $m=0$ or $n=0$.

Lemma 1.3. [14]: Let $\mathcal{R}$ be a 2-torsionfree prime ring, $\mathbb{U}$ be a Lie ideal of $\mathcal{R}$. If $\mathbb{U}$ is a commutative Lie ideal of $\mathcal{R}$, then $\mathbb{U}\subseteq Z(\mathcal{R})$.

2. Main Results

Theorem 2.1

Let F be an m (g)(α,β)- rd on $\mathcal{R}$ related with a map d: $\mathcal{R}\to\mathcal{R}$. If $F(uv)\pm\alpha(uv)=0$ for all $u,v\in\mathbb{U}$, then $d(\mathbb{U})=0$.

Proof:

Suppose that $\mathcal{R}$ is non-commutative and $\mathbb{U}\nsubseteq Z(\mathcal{R})$.

We have

$$\mathrm{F}(uv)-\alpha(uv)=0\,. \tag{1}$$

Exchange v by $2vz$ in (1), where $z\in\mathbb{U}$, gives

$$\mathrm{F}(z)\alpha(uv)+\beta(z)d(uv)-\alpha(uvz)=0.$$

By using Lemma 1.1, it gives

$$\mathrm{F}(z)\alpha(u)\alpha(v)+\beta(z)d(u)\alpha(v)-\alpha(uvz)=0\,. \tag{2}$$

On the other hand $\mathrm{F}(vz)\alpha(u)+\beta(vz)d(u)-\alpha(uvz)=0$,

$$\mathrm{F}(z)\alpha(v)\alpha(u)+\beta(z)d(v)\alpha(u)+\beta(v)\beta(z)d(u)-\alpha(uvz)=0. \tag{3}$$

Subtract Equation (3) from Equation (2), gives

$$\mathrm{F}(z)\alpha[u,v]+\beta(z)(d(u)\alpha(v)-d(v)\alpha(u))-\beta(v)\beta(z)d(u)=0. \tag{4}$$

Substituting v with $2vu$ in (4), and applying Lemma 1.1, this implies

$$\mathrm{F}(z)\alpha[u,v]\alpha(u)+\beta(z)(d(u)\alpha(v)\alpha(u)-d(v)\alpha(u)\alpha(u))-\beta(v)\beta(u)\beta(z)d(u)=0. \tag{5}$$

By multiplying Equation (4) by α(u) on the right we have

$$\mathrm{F}(z)\alpha[u,v]\alpha(u)+\beta(z)(d(u)\alpha(v)-d(v)\alpha(u))\alpha(u)-\beta(v)\beta(z)d(u)\alpha(u)=0. \tag{6}$$

Subtract Equation (6) from Equation (5), gives

$$\beta(v)(\beta(z)d(u)\alpha(u)-\beta(u)\beta(z)d(u))=0\,. \tag{7}$$

By putting $v=[v,r]$ in (7), and use it, where $r\in\mathcal{R}$, we obtain

$$\beta(v)\,\mathcal{R}\,(\beta(z)d(u)\alpha(u)-\beta(u)\beta(z)d(u))=0.$$

$$\beta(z)d(u)\alpha(u)-\beta(u)\beta(z)d(u)=0\,. \tag{8}$$

We substitute z by $2vz$ in (8), then

$$\beta(v)\beta(z)d(u)\alpha(u)-\beta(u)\beta(v)\beta(z)d(u)=0\,. \tag{9}$$

Left multiply Equation (8), by $\beta(v)$ and subtract it from Equation (9), then

$$\beta[v,u]\beta(z)d(u)=0.$$

Taking β^{-1} in above relation, gives

$$[v,u]\,z\,\beta^{-1}(d(u))=0\,.$$

By Lemma 1.2, since $\mathbb{U}\nsubseteq Z(\mathcal{R})$ becomes, $d(\mathbb{U})=0$.

On the other hand, if $\mathcal{R}$ is commutative, then $\mathbb{U}\subseteq Z(\mathcal{R})$.

It is clear $\alpha(u),\beta(u)\subseteq Z(\mathcal{R})$.

From Equation (7), we have $\beta(v)(\beta(z)\alpha(u)d(u)-\beta(z)\beta(u)d(u))=0$.

$$\beta(v)\beta(z)(\alpha(u)d(u)-\beta(u)d(u))=0\,. \tag{10}$$

Exchange v by$[v,r]$, where $r\in\mathcal{R}$ in (10), and using it, this yield

$$\beta(v)\,\mathcal{R}\,\beta(z)(\alpha(u)d(u)-\beta(u)d(u))=0.$$

$$\beta(z)(\alpha(u)d(u)-\beta(u)d(u))=0\,.$$

Change z by $[z,r]$ in above relation and using the same technique as above, we finally obtain

$$((\alpha-\beta)u)d(u)=0\,.$$

In the relationship above, if we multiply it by r on the left, we obtain,

$$r(\alpha-\beta)\,(u)d(u)=0.$$
$$(\alpha-\beta)u\,\mathcal{R}\,d(u)=0.$$
Since $\mathcal{R}$ is a prime ring, we get either $(\alpha-\beta)=0$ or $d(\mathbb{U})=0$.
If $\alpha=\beta$ on $\mathbb{U}$, from Equation (4), we obtain
$$\beta(z)d(u)\beta(v)-\beta(z)d(v)\beta(u)-\beta(v)\beta(z)d(u)=0.$$
Let $v=u$ in above relation, we have $\beta(u)\beta(z)d(u)=0$.
By taking β^{-1}in above equation, we get $u\,z\,\beta^{-1}\,(d(u))=0$.
Multiplying above equation by r on the left, then
$$r\,u\,z\beta^{-1}\,(d(u))=0\,.$$
$$\beta^{-1}\,(d(u))=0.$$
Again, multiplying above equation by r on the left, we find $r\,z\,\beta^{-1}\,(d(u))=0$.
$$d(\mathbb{U})=0.$$
For the case $\mathrm{F}(uv)+\alpha(uv)=0$ for all $u,v\in\mathbb{U}$,the same conclusion is reached using a similar approach.

Theorem 2.2
Let F be an m (g)(α,β)- rd on $\mathcal{R}$ related with a map $d{:}\,\mathcal{R}\to\mathcal{R}$. If $\mathrm{F}(uv)\pm\alpha(vu)=0$ for all $u,v\in\mathbb{U}$, then $d(\mathbb{U})$=0.
Proof:
Suppose that $\mathbb{U}\not\subseteq Z(\mathcal{R})$.
$$F(uv)-\alpha(vu)=0\,. \quad (11)$$
Replace u by u^2 in Equation (11), and using it, and taking β^{-1},we get
$$u\,v\,\beta^{-1}\,(d(u))=0. \quad (12)$$
Using Lemma 1.2, implies β^{-1} $(d(u))$=0. Since β is an automorplism of $\mathcal{R}$, then becomes $d(\mathbb{U})=0$.
Now, if $\mathbb{U}\subseteq Z(\mathcal{R})$. By multiplying the left side of Equation (12), by r, we obtain
$$r\,u\,v\,\beta^{-1}(d(u))=0.$$
Since $\mathbb{U}\subseteq Z(\mathcal{R})$, we get $u\,r\,v\,\beta^{-1}\,(d(u))=0$ and then
$$v\,\beta^{-1}\,(d(u))=0\,. \quad (13)$$
Place $v=[v,r]$ in (13), for all $r\in\mathcal{R}$, and use Equation (13), then $\beta^{-1}\,(d(u))=0$.
Since β is an automorphism of $\mathcal{R}$, we have $d(\mathbb{U})=0$.
For the case $\mathrm{F}(uv)+\alpha(vu)=0$ for all $u,v\in\mathbb{U}$, the same conclusion is reached by using a similar approach.

Theorem 2.3
Let F be an m (g)(α,β)- rd on $\mathcal{R}$ related with a map d: $\mathcal{R}\to\mathcal{R}$. If $F(u)F(v)\pm\alpha(uv)=0$ for all $u,v\in\mathbb{U}$, then $\mathbb{U}\subseteq Z(\mathcal{R})$.
Proof:
Assume that $\mathcal{R}$ is non-commutative.
$$\mathrm{F}(u)\mathrm{F}(v)-\alpha(uv)=0\,. \quad (14)$$
Substituting $2vz$ for v in Equation (14), where $z\in\mathbb{U}$, implies
$$\mathrm{F}(u)\mathrm{F}(z)\alpha(v)+\mathrm{F}(u)\beta(z)d(v)-\alpha(uvz)=0\,. \quad (15)$$
Substitute $2vg$ for v in Equation (15), where $g\in\mathbb{U}$, and by applying Lemma 1.1, we get
$$\mathrm{F}(u)\mathrm{F}(z)\alpha(v)\alpha(g)+\mathrm{F}(u)\beta(z)d(v)\alpha(g)-\alpha(uvgz)=0. \quad (16)$$
Multiplying right side of Equation (15), by $\alpha(g)$, this gives
$$\mathrm{F}(u)F(z)\alpha(v)\alpha(g)+\mathrm{F}(u)\beta(z)d(v)\alpha(g)-\alpha(uvz)\alpha(g)=0. \quad (17)$$
Comparing Equation (16) and Equation (17), to get
$$\alpha(uv)\alpha[z,g]=0\,.$$
By taking α^{-1}in above Equation, we find
$$u\,v\,[z,g]=0. \quad (18)$$

Replace u by $[u, r]$ in (18), and use it, where $r \in R$, we get

$$v\,[z, g] = 0\,. \tag{19}$$

Exchange v by $[v, r]$ in (19), and use it, implies

$$vR[z, g] = 0.$$

By using Lemma 1.3, we get $\mathbb{U} \subseteq Z(R)$.
If R is commutative, we get our result.
For the case $\mathrm{F}(u)\mathrm{F}(v) + \alpha(uv) = 0$ for all $u, v \in \mathbb{U}$, the same conclusion is reached by using the similar approach.

Theorem 2.4
Let F be an m (g)(α, β)- rd on R related with a map d: $R \to R$. If $\mathrm{F}(u)\mathrm{F}(v) \pm \alpha(vu) = 0$ for all $u, v \in \mathbb{U}$, then $\mathbb{U} \subseteq Z(R)$.
Proof:
Assume that R is non-commutative

$$\mathrm{F}(u)\mathrm{F}(v) - \alpha(vu) = 0. \tag{20}$$

Taking $2vz$ instead of v in (20), where z$\in \mathbb{U}$, we obtain

$$\mathrm{F}(u)\mathrm{F}(z)\alpha(v) + \mathrm{F}(u)\beta(z)d(v) - \alpha(vzu) = 0. \tag{21}$$

By substituting $2vg$ for v in Equation (21), where $g \in \mathbb{U}$, by using Lemma 1.1, then

$$\mathrm{F}(u)\mathrm{F}(z)\alpha(v)\alpha(g) + \mathrm{F}(u)\beta(z)d(v)\alpha(g) - \alpha(vgzu) = 0. \tag{22}$$

Multiplying right side of Equation (21), by $\alpha(g)$, then

$$\mathrm{F}(u)\mathrm{F}(z)\alpha(v)\alpha(g) + \mathrm{F}(u)\beta(z)d(v)\alpha(g) - \alpha(vzu)\alpha(g) = 0. \tag{23}$$

Comparing (22) and (23), we get
$\alpha(v)\alpha[zu, g] = 0.$
By taking α^{-1} in above relation, gives

$$v\,[zu, g] = 0\,. \tag{24}$$

Replacing v by $[v, r]$, in (24), and using (24), becomes

$$z[u,g] + [z, g]u = 0\,. \tag{25}$$

Exchange u by $2ui$ in (25), where $i \in \mathbb{U}$, implies

$$zu[i, g] + z[u, g]i + [z, g]ui = 0 \tag{26}$$

Multiplying Equation (25), by i on the right then

$$z[u, g]i + [z, g]ui = 0\,. \tag{27}$$

Comparing (27) and (26), becomes

$$z\,u\,[i, g] = 0\,.$$

By putting $z = [z, r]$ in the relationship above and using it, we have $u\,[i, g] = 0$.
Putting $u = [u, r]$ in above equation and using it, for all $r \in R$, then $[i, g] = 0$.
By using Lemma 1.3, we conclude $\mathbb{U} \subseteq Z(R)$.
If R is commutative, we get our result.
For the case $\mathrm{F}(u)\mathrm{F}(v) + \alpha(vu) = 0$ for all $u, v \in \mathbb{U}$, the same conclusion is reached by using the similar approach.

Theorem 2.5
Let F acts as a homomorphism and be an m (g)(α, β)- rd on R related with a map $d: R \to R$, then $d(\mathbb{U}) = 0$ or $\mathbb{U} \subseteq Z(R)$.
Proof:
Since $\mathbb{U} \nsubseteq Z(R)$, then R is non-commutative.
Since F acts as a homomorphism on R, then $\mathrm{F}(uv) = \mathrm{F}(u)\mathrm{F}(v)$.

$$\mathrm{F}(v)\alpha(u) + \beta(v)d(u) = F(u)F(v)\,. \tag{28}$$

We substitute v by $2zv$ in (28), where $z \in \mathbb{U}$, to get

$$\mathrm{F}(v)\alpha(z)\alpha(u) + \beta(v)d(z)\alpha(u) + \beta(z)\beta(v)d(u) - \mathrm{F}(u)F(v)\alpha(z) - \mathrm{F}(u)\beta(v)d(z) = 0\,. \tag{29}$$

Multiplying (28), by α(z) on the right, gives

$$\mathrm{F}(v)\alpha(u)\alpha(z)+\beta(v)d(u)\alpha(z)-\mathrm{F}(u)F(v)\alpha(z)=0. \quad (30)$$

We subtracting Equation (30) from Equation (29), we have

$$\mathrm{F}(v)\alpha[z,u]+\beta(v)d(z)\alpha(u)+\beta(z)\beta(v)d(u)-\mathrm{F}(u)\beta(v)d(z)-\ \beta(v)d(u)\alpha(z)=0. \quad (31)$$

Subsisting $2zu$ in the place of z in Equation (31), by applying Lemma 1.1, then

$$\mathrm{F}(v)\alpha[z,u]\alpha(u)+\beta(v)d(z)\alpha(u)\alpha(u)+\beta(z)\beta(u)\beta(v)d(u)-\ \mathrm{F}(u)\beta(v)d(z)\alpha(u)-\beta(v)d(u)\alpha(z)\alpha(u)=0. \quad (32)$$

Multiplying right side of Equation (31), by $\alpha(u)$, gives

$$\mathrm{F}(v)\alpha[z,u]\alpha(u)+\beta(v)d(z)\alpha(u)\alpha(u)+\beta(z)\beta(v)d(u)\alpha(u)-\mathrm{F}(u)\beta(v)d(z)\alpha(u)-\beta(v)d(u)\alpha(z)\alpha(u)=0\,. \quad (33)$$

From Equation (32) and Equation (33), we get

$$\beta(z)(\beta(v)d(u)\alpha(u)-\beta(u)\beta(v)d(u))=0\,.$$

Putting $z=[z,r]$ in the relationship above and using it, results in

$$\beta(z)\beta(r)(\beta(v)d(u)\alpha(u)-\beta(u)\beta(v)d(u))=0.$$

Since β is an automorphism of $\mathcal{R}$, then

$$\beta(v)d(u)\alpha(u)-\beta(u)\beta(v)d(u)=0\,. \quad (34)$$

Substituting v for $2nv$ in (34), where $n\in\mathbb{U}$, we get

$$\beta(n)\beta(v)d(u)\alpha(u)-\beta(u)\beta(n)\beta(v)d(u)=0\,. \quad (35)$$

Multiplying Equation (34), by β(n), on the left and compare Equation (35), gives

$$\beta[u,n]\beta(v)d(u)=0\,.$$

By taking β^{-1}in above relation, we get $[u,n]v\ \beta^{-1}\ (d(u))=0$.
By using Lemma 1.2, and since $\mathbb{U}\not\subseteq Z(\mathcal{R})$, we have $\beta^{-1}\ (d(u))=0$, we get $d(\mathbb{U})=0$.
If $\mathbb{U}\subseteq Z(\mathcal{R})$, we achieve our goal.

Theorem 2.6

Let F acts as anti- homomorphism and be an m (g)(α,β)- rd on $\mathcal{R}$ related with a map d: $\mathcal{R}\to\mathcal{R}$, then either $d(\mathbb{U})=0$ or $\mathbb{U}\subseteq Z(\mathcal{R})$.

Proof:

Since $\mathbb{U}\not\subseteq Z(\mathcal{R})$, then $\mathcal{R}$ is non-commutative.
Since F acts as anti-homomorphism of $\mathcal{R}$, then

$$\mathrm{F}(uv)=\mathrm{F}(v)\mathrm{F}(u)\,. \quad (36)$$

Substitute $2zu$ for u in Equation (36), where $z\in\mathbb{U}$, to give

$$\mathrm{F}(v)\alpha(zu)+\beta(v)d(zu)=\mathrm{F}(v)\mathrm{F}(u)\alpha(z)+\mathrm{F}(v)\beta(u)d(z).$$

By applying Lemma 1.1

$$\mathrm{F}(v)\alpha(z)\alpha(u)+\beta(v)d(z)\alpha(u)-\mathrm{F}(v)\mathrm{F}(u)\alpha(z)-\mathrm{F}(v)\beta(u)d(z)=0. \quad (37)$$

On the other hand

$$\mathrm{F}(v)\alpha(u)\alpha(z)+\beta(v)d(u)\alpha(z)+\beta(u)\beta(v)d(z)-\mathrm{F}(v)\mathrm{F}(u)\alpha(z)-\mathrm{F}(v)\beta(u)d(z)=0\,. \quad (38)$$

Subtract Equation (38) from Equation (37), gives

$$\mathrm{F}(v)\alpha[z,u]+\beta(v)d(z)\alpha(u)-\beta(v)d(u)\alpha(z)-\beta(u)\beta(v)d(z)=0. \quad (39)$$

By putting $u=2uz$ in (39), so by applying Lemma 1.1, we have

$$\mathrm{F}(v)\alpha[z,u]\alpha(z)+\beta(v)d(z)\alpha(u)\alpha(z)-\beta(v)d(u)\alpha(z)\alpha(z)-\beta(u)\beta(z)\beta(v)d(z)=0. \quad (40)$$

Equation (39) is multiplied by $\alpha(z)$ on the right, it implies

$$\mathrm{F}(v)\alpha[z,u]\alpha(z)+\beta(v)d(z)\alpha(u)\alpha(z)-\beta(v)d(u)\alpha(z)\alpha(z)-\beta(u)\beta(v)d(z)\alpha(z)=0 \quad (41)$$

Subtracted Equation (41) from Equation (40), then

$$\beta(u)(\beta(v)d(z)\alpha(z)-\beta(z)\beta(v)d(z))=0\,.$$

Putting $u=[u,r]$ in the relationship above and using it, then

$$\beta(u)\ R\ (\beta(v)d(z)\alpha(z)-\beta(z)\beta(v)d(z)=0.$$

$$\beta(v)d(z)\alpha(z) - \beta(z)\beta(v)d(z) = 0\,.$$

Let $z = u$ in over equation, gives

$$\beta(v)d(u)\alpha(u) - \beta(u)\beta(v)d(u) = 0\,.$$

The proof follows from the Theorem 2.5, after the Equation (34) and we get the result then either $\mathbb{U} \subseteq Z(R)$ or $d(\mathbb{U}) = 0$.

Theorem 2.7

Let F be an m (g)(α, β)- rd on R related with a map $d: R \to R$. If $F[u, v] = 0$ for all $u, v \in \mathbb{U}$, then either $d(\mathbb{U}) = 0$ or $\mathbb{U} \subseteq Z(R)$.

Proof:

Suppose that $\mathbb{U} \nsubseteq Z(R)$.

$$F[u, v] = 0\,. \qquad (42)$$

Taking $2uv$ instead of v in (42), and using it, it gives

$$\beta[u, v]d(u) = 0\,. \qquad (43)$$

Substituting 2zv for v in (43), and using it, we have, $\beta[u, z]\beta(v)d(u) = 0$.

β^{-1} $(\beta[u, z]\beta(v)d(u)) = 0.$

$$[u, z]\ \mathrm{U}\beta^{-1}(d(u)) = 0.$$

By using Lemma 1.2, and because $\mathbb{U} \nsubseteq Z(R)$ implies that β^{-1} $(d(u)) = 0$.

Since β is an automorphism of R, we get $d(\mathbb{U}) = 0$.

If $\mathbb{U} \subseteq Z(R)$ then, we get our result.

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