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Sumudu A domian Decomposition Method for Solving SEIVR Epidemic Model

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Abstract
In this paper, we modify an SEIVR model concerning COVID-19 from the first-order system into a multi-fractional order system of differential equations. We also find an approximate solution by using the Sumudu Adomian decomposition method. Furthermore, we try to give qualitative results rather than qualitative results. Numerical simulation is given through a table and graphs which show the efficiency of the method.

Keywords: SEIVR epidemic, Fractional calculus, Caputo derivative, Sumudu, Adomian.

طريقة تحليل Sumudu Adomian لحل نموذج SEIVR الوبائي

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الخلاصة
تم في هذا البحث تعديل نموذج SEIVR لجائحة كورونا من نموذج معادلات تفاضلية من الرتبة الأولى الى نموذج تفاضلي ذو رتب كسرية مختلفة و إيجاد حل تقريبي لهذا النموذج باستخدام طريقة تحويل سومودو مع طريقة أدومين لأننا نريد إعطاء نتائج كمية و ليست نوعية. تم تقديم محاكاة عددية و تم توضيحها من خلال جدول و رسوم لبيان كفاءة الطريقة.

1.Introduction:

The world health organization(WHO) declared that COVID-19 is a pandemic in March 2020. This disease spread very fast in more countries around the world. The governments impose limitations on traveling inside and outside them. The WHO also issued a series of preliminary regulatory determinations for healthcare services against the emerging disease and called on all nations to cooperate in its control. The disease then spreads to the rest of the world, it causes health crises in one region after another. Despite public vaccination in a limited number of countries, programs such as social isolation, physical distancing, and wearing masks are employed as the main control strategies to reduce the exponential growth

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of COVID-19. Mathematicians together with others submit a large number of models such as SEIR, SEIVR, SEQAIJRE. A modeling SEQIR for Italy was submitted [1]. Fractional mathematical models are given to investigate SEI_1I_2R and others [2]. The authors introduce the extended SEIVR model into a limit-state function and define the model parameters including transmission, recovery, and mortality rates as random variables, the problem is transformed into a reliability model and analyzed by the Monte Carlo sampling [3]. The numerical study of a deterministic mathematical model of an SEIVR is also given. This model incorporates a temporary immune recovery class which involves subsequent dose vaccination for the infants, Hypothetical values are chosen for the parameters to test the validity of the mathematical model. The parameter with the greatest impact on the model is computed using the eigenvalue elasticity and sensitivity analyses and it is found that the parameter of the rate at which the vaccine wanes in the infants has the greatest impact on the mathematical model [4]. In [5] a suitable numerical simulation method is used to solve a nonlinear system that contains multi-variables and multi-parameters with absent real data. A susceptible-vaccinated-exposed-infectious-recovered (SEIVR) epidemic model is investigated for an infectious disease that spreads in the host population through horizontal transmission. It is shown that the model exhibits two equilibria, namely, the disease-free equilibrium and the endemic equilibrium, by constructing a suitable Lyapunov function. It is also observed that the global asymptotic stability of the disease-free equilibrium depends on R_o as well as on the treatment rate, if $R_0 > 1$, then the endemic equilibrium is globally asymptotically stable with the help of the Li and Muldowney geometric approach applied to four-dimensional systems [6]. In the references of [7-19], the authors studied the stability of criticality of epidemiological systems for Covid-19. Rather than the local study of the equilibrium points by studying their stability [20]. Many researchers [21-48] studied the stability of the equilibrium points of various medical and environmental models in the world. In [21], the authors also searched and investigated the global behavior of the system so we solve it by the Sumudu Adomian decomposition method. This method is effective and its computations can be done recursively in simple procedure. The basic concepts are discussed in section 2. In section 3, the mathematical model is discussed. The main results are discussed in section 4. The numerical simulation is presented in section 5. All figures are made in the MATLAB and MATHCAD.

2.Basic Concepts:

2.1) Fractional Calculus

Fractional calculus is the generalization of integrals and derivatives of any arbitrary real or complex order. It has a long history from 1695 when L.Hopital sent a letter to Leibniz asking about the meaning of $(\frac{d^{\frac{1}{2}}x}{dx^{\frac{1}{2}}})$. The fractional differential and fractional integration go back to many great mathematicians such as Liouville, Leibniz, Riemann, Letnikov, Able, Weyl, Riesz, and others. The derivatives and integrals of non-integer order and the fractional integrodifferential equations have been found in many applications in recent studies, for example, theoretical physics, mechanics, medicine, rheology, electrical networks, viscoelasticity, chemical physics and applied mathematics etc.

2.2) Riemmann-Liouville fractional integral [49]

Let f be a function defined as $f:[0,\infty) \to R$ then the fractional integral of order $\alpha > 0$ is given as follows:

${}_0I_u^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} f(s)ds.$

Where ${}_0I_u^{\alpha}$ is the Riemann-Liouville fractional integral.

2.3) The Riemann-Liouville Fractional Derivatives [49]
It is one of the most famous types to calculate the fractional derivative where suppose that $\alpha > 0, x > \alpha, \alpha, a \in R$ then:

$$D^{\alpha} f(x) := \begin{cases} \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^x \frac{f(\tau)}{(x-\tau)^{\alpha+1-n}} d\tau, & n-1 < \alpha < n \in \mathbb{N} \\ \frac{d^n}{dx^n} f(x) \alpha = n \in \mathbb{N} \end{cases}$$

Where D^{α} is the Riemann-Liouville fractional derivative.

2.4) The Caputo Fractional Derivatives
Suppose that $\alpha > 0, t > a, \alpha, a, x \in R$. Then the Caputo fractional operator:

$$D_*^{\alpha} f(x) := \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{f^{(n)}(\tau)}{(x-\tau)^{\alpha+1-n}} d\tau, & n-1 < \alpha < n \in \mathbb{N} \\ \frac{d^n}{dt^n} f(x) = \alpha n \in \mathbb{N} \end{cases}$$

It is clear that in the Caputo sense, the derivative of the constant function is equal to zero.
2.5) The Sumudu transform

Definition (2.5.1) [49] : The Sumudu transform is defined over the set of functions as follows:
$A = f(x) | \exists H, v_1, v_2 > 0, |f(x)| < He^{\frac{|t|}{v_j}}, if x \in (-1)^j x[0, \infty)$ is given by the following formula
Y(u) = $\int_0^{\infty} (ux) e^{-x} dx\, y, or Y(u) = \frac{1}{u} \int_0^{\infty} \quad e^{\frac{-x}{u}} y(x) dx$ for any function y(x) and -v_1< u < v_2.

2.6) Some important Properties of the Sumudu Transform [49]
1. Linearity:
S[$\alpha f(x) + \beta g(x)$]= $\alpha S[f(x)] + \beta S[g(x)]$**.**
2. The convolution of f(x), g(x) ∈ A with the Sumudu transform is given by:
S[f*g(x)]=uS[F(u)*G(u)].
3. The power series f(x) = $\sum_{a=0}^{\infty} (-1)^a \frac{\gamma x^{\tau}}{a!} = e^{-\gamma x}$ transforms to the geometric series
S[f(x)] = = $\sum_{a=0}^{\infty} (-1)^a (\gamma a)^a = \frac{1}{1+\gamma u}, u \in \left(-\frac{1}{\gamma}, \frac{1}{\gamma}\right)$.
4. The Sumudu and Laplace transforms exhibit a duality relation as follows:
$G\left(\frac{1}{s}\right) = sF(s), F(u) = uG(u)$.
5. Let f(x) = x^m then the Sumudu transform of it is
S[$\frac{x^m}{\Gamma(\alpha+1)}$] $= u^m, m > 0$.
then the Sumudu inverse transform of u^{α} is $S^{-1}[u^m] = \frac{x^m}{\Gamma(\alpha+1)}$.

Theorem (2.6.1)[49]: The Sumudu transform of the Caputo derivative is given by**:**
S[$D_X^{\alpha} f(x);\, u] = u^{-\alpha} G(u) - \sum_{k=0}^{n-1} u^{k-a} \left[f^{(k)}(x)\right]_{x=0}, n-1 < a \leq n$.

2.7) Using the suggested method with the Sumudu transform to find the solution to nonlinear fractional differential equations [49]:

The suggested method factors of the function g(x) are to arrive at the precise result. The form of the ordinary fractional differential equations of the nonhomogeneous nonlinear is given as follows.

$$D_*^{\alpha} y(x) + Lu(x) + Nu(x) = g(x), x \geq 0, n-1 \leq \alpha \leq n\,. \qquad (1)$$

Where $D_*^{\alpha} y(x)$ denotes to the Caputo fractional derivative of order α for u(x), L is the linear term operator, N is a nonlinear part of the previous equation and f (x) is the source function. Eq. (1) is correlated with the initial conditions

$$u^{(i)}(0)=u_0^{(i)}, i = 0,1,\dots,n-1. \qquad (2)$$

Also, the real numbers $u_0^{(i)}, i = 0,1,\dots,n-1$ are assumed to be given.

The Sumudu transform with the Adomian decomposition method consists of applying the Sumudu transform first to both sides of (1) to give:

$S[D_*^{\alpha} u(x)$ = S[u(x)] − S[Lu (x) + Nu(x)].

Using the property of the Sumudu transform and the initial condition in Eq. (2), we have

$S[y(x)]-\sum_{j=1}^{r-1} u^{j-\alpha}[u^{(k)}(0)] = u^{\alpha}(S[g(x)] - u^{\alpha}(S[Lu(x) + Nu(x)].$

Suppose $W=\sum_{j=1}^{r-1} u^{j-\alpha}[u^{(k)}(0)]$, then we will get

$$S[y(x)]-W = u^{\alpha}(S[g(x)] - u^{\alpha}(S[Lu(x) + Nu(x)]. \qquad (3)$$

Applying the inverse of the transformation to both sides of Eq. (3) to get

$$u(x)=g(u)-S^{-1}[u^{\alpha}(S[Lu(x) + Nu(x)])]. \qquad (4)$$

Where g(u) represents the terms arising from the inverse of the transformation for W and from the source term $u^{\alpha}(S[g(x)])$.

Assuming that the solution to Eq. (3) is in the form

$$u(x) = \sum_{n=0}^{\infty} u_n\,. \qquad (5)$$

And the nonlinear term Nu(x) is equated to an infinite series of polynomials:

i.e.

$$Nu(x)=\sum_{n=0}^{\infty} A_n. \qquad (6)$$

Where A_n's are the Adomian polynomials which can be evaluated by using the following expression

$$A_n=\frac{1}{N!}\frac{d^n}{d\vartheta^n}[N(\sum_{i=0}^{n} \vartheta^i u_i)]_{\vartheta} = 0, \qquad n=0,1,2,3,\dots \qquad (7)$$

Substituting Eqs.(5) and (6) into (4), it gives

$\sum_{n=0}^{\infty} u_n(x) = g(u) - S^{-1}[u^{\alpha}(S[L\sum_{n=0}^{\infty} u_n(x) + \sum_{n=0}^{\infty} A_n(x)])].$

The components $u_n(x)$ of the solution u can be determined by using the recursive relation

$u_0 = g(x),$

$u_{k+1} = - S^{-1}[u^{\alpha}(S[L(u_k(x) + A_n])], k = 0,1,2,\dots$

And we consider the modified form of Eq.(7) by factorizing g(x) as follows:

$$g(x)=g_1+g_2. \qquad (8)$$

Using Eq.(8), we introduce a qualitative change in the formula of the recursive relation Eq.(8). To reduce the size of calculations, we identify the zeroth approximate u_0, by one part of, namely g_1 or g_2. The other part of g can be added to the term y_1 among other terms. In other words, the modified recursive relation can be identified by :

$u_0 = g_1,$

$u_1 = g_2 - S^{-1}[u^{\alpha}(S[L(u_k(x) + A_k])],$

$u_{k+1} = - S^{-1}[u^{\alpha}(S[L(u_k(x) + A_k])].$

An important point can be made here which is that we suggest a change in the formula of the first two terms u_0, and u_1 only. Although, this variation in the formation of u_0, and u_1 is slight, however, it plays a major role in accelerating the convergence of the solution and in minimizing the size of calculations.

3. The modified mathematical model:

The model consists of five fractional equals where S, E, I, V, and R represent the susceptible, expand, infected, vaccinated, and recovered individuals, respectively.

Let N(t) be the total population size of the individuals with N(t) = S(t)+E(t)+I(t)+V(t)+R(t). The flow of parameters is depicted in Figure 1. The governing model is given by:

$$\begin{aligned}
\frac{d^{\alpha_1}s}{dt^{\alpha_1}} &= \Delta(1-k) - \mu s - \frac{\beta sI}{\Psi(I)} + \mathrm{v}_4 V,\\
\frac{d^{\alpha_2}E}{dt^{\alpha_2}} &= \frac{\beta sI}{\Psi(I)} - \mathrm{v}_1 E - \mu E,\\
\frac{d^{\alpha_3}I}{dt^{\alpha_3}} &= \mathrm{v}_1 E - (\mu + \mathrm{v}_0 + \mathrm{v}_2 + \mathrm{v}_3)I,\\
\frac{d^{\alpha_4}V}{dt^{\alpha_4}} &= \Delta\mathrm{k} + \mathrm{v}_3\mathrm{I} - \mu\mathrm{V} - \mathrm{v}_4\mathrm{V}\\
\frac{d^{\alpha_5}R}{dt^{\alpha_5}} &= \mathrm{v}_2\mathrm{I} - \mu\mathrm{R}\,,
\end{aligned} \tag{9}$$

The initial conditions of the above system are: S(0) = So $>$ 0, E(0) = Eo ≥ 0, I(0) = Io ≥ 0, V (0) = Vo ≥ 0, R(0) = Ro ≥ 0 and $0 < \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 < 1$.

Here, in the system (9), the growth rate of the population is denoted by Δ. The parameter β is the disease contact rate and κ represents the fraction of individuals to be vaccinated. The natural death rate is denoted by μ and the disease related death rate is shown by ν_0. The exposed individuals that are infected at the rate of ν_1 and ν_2 show the rate of recovery from infection. The infected individuals are vaccinated/treated at the rate of ν_3. The vaccinated individuals lose their immunity at the rate of ν_4. The same transmission rate in the form of βSI /ψ(I) is assumed where ψ represents a positive function such that ψ(0) = 1 and ψ(I) ≥ 0,which is used by [16]. This generalizes the mass action incidence (i.e. ψ(I) = 1), and the incidence rate βSI/ 1+kI . For small I, the function I/ ψ(I) is increasing while it is decreasing for large I, that is $\psi(I)=1+I^2$. This describes the psychological effect: for a very large number of infective individuals, the infection force may decrease as the number of infective individuals increases, because in the presence of the large number of infective individuals, the population may tend to reduce the number of contacts per unit time [50].

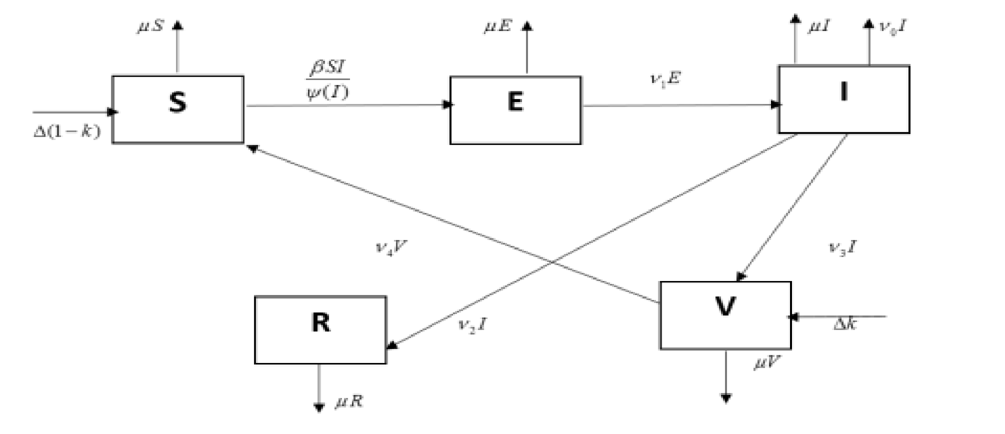

Fig. 1. Flow chart.

4. Main results :

In this section, we solve the fractional model (9) in the Caputo derivative sense. Taking the Sumudu transform to both sides of the system (9) to get:

$$\begin{aligned}
&S[D_*^{\alpha_1} s] = S\left[\Delta(1-k) - \mu s - \frac{\beta sI}{\Psi(I)} + v_4 V\right],\\
&S[D_*^{\alpha_2} E] = S[\beta SI/\Psi(I) - v_1 E - \mu E],\\
&S[D_*^{\alpha_3} I] = S[v_1 E - (\mu + v_0 + v_2 + v_3)I] \qquad (10)\\
&S[D_*^{\alpha_4} V] = S[\Delta k + v_3 I - \mu V - v_4 V],\\
&S[D_*^{\alpha_5} R] = S[v_2 I - \mu R].
\end{aligned}$$

Now we take the transform to both sides of Eq.(10) :

$$\begin{aligned}
&u^{-\alpha_1}[S(s) - s(0)] = S\left[\Delta(1-k) - \mu s - \frac{\beta sI}{\Psi(I)} + v_4 V\right]\\
&u^{-\alpha_2}[S(E) - E(0)] = S\left[\frac{\beta sI}{\Psi(I)} - v_1 E - \mu E\right]\\
&u^{-\alpha_3}[S(I) - I(0)] = S[v_1 - (\mu + v_0 + v_2 + v_3)I] \qquad (11)\\
&u^{-\alpha_4}[S(V) - V(0)] = S[\Delta k + v_3 I - \mu V - v_4 V]\\
&u^{-\alpha_5}[S(R) - R(0)] = S[v_2 I - \mu R]
\end{aligned}$$

Now we substitute the initial conditions $S(0)=S_0$, $E(0)=E_0$, $I(0)=I_0$, $V(0)=V_0$ and $R(0)=R_0$ to get the following:

$$\begin{aligned}
&S(t) = S_0 + S^{-1}\left[u^{\alpha_1} S\left\{\Delta - \Delta K - \mu S - \frac{\beta SI}{\Psi(I)} + v_4 V\right\}\right];\\
&E(t) = E_0 + S^{-1}\left[u^{\alpha_2} S\left\{\frac{\beta SI}{\Psi(I)} - v_1 E - \mu E\right\}\right];\\
&I(t) = I_0 + S^{-1}[u^{\alpha_3} S\{v_1 E - (\mu I + v_0 I + v_2 I + v_3 I)\}]; \qquad (12)\\
&V(t) = V_0 + S^{-1}[u^{\alpha_4} S\{\Delta k + v_3 I - \mu V - v_4 V\}];\\
&R(t) = R_0 + S^{-1}[u^{\alpha_5} S\{v_2 I - \mu R\}];
\end{aligned}$$

This system is nonlinear so the Sumudu transformation does not applicable, however, if we use the Adomian decomposition method, then the system (2) becomes linear. To do that we apply ADM as follows:

We suppose

$$S=\sum_{i=0}^{\infty} S_i;\ E = \sum_{i-0}^{\infty} E_i;\ I = \sum_{i=0}^{\infty} I_i;\ V = \sum_{i=0}^{\infty} V_i;\ R = \sum^{\infty}{}_{i=0} R_i;$$

Now suppose $SI=\sum_{i=0}^{\infty} P_i$; $where\ ,\ P_i = \sum_{j=0}^{\infty} I_j S_{i-j};$

Therefore, Eq.(12) becomes :

$$\begin{aligned}
&\sum_{i=1}^{\infty} S_i = S_0 + S^{-1}[u^{\alpha_1} S\{\Delta - \Delta k - \mu \sum_{i=0}^{\infty} S_i - \frac{\beta \sum_{i=0}^{\infty} P_i}{\Psi(\sum_{i=0}^{\infty} I_i)} + v_4 \sum_{i=0}^{\infty} V_i\}];\\
&\sum_{i=1}^{\infty} E_i = E_0 + S^{-1}[u^{\alpha_2} S\{\frac{\beta \sum_{i=0}^{\infty} P_i}{\Psi(\sum_{i=0}^{\infty} I_i)} - v_1 \sum_{i=0}^{\infty} E_i - \mu \sum_{i=0}^{\infty} E_i\}]; \qquad (13)\\
&\sum_{i=1}^{\infty} I_i = I_0 + S^{-1}[u^{\alpha_3} S\{v_1 \sum_{i=0}^{\infty} E_i - (\mu \sum_{i=0}^{\infty} I_i + v_0 \sum_{i=0}^{\infty} I_i + v_2 \sum_{i=0}^{\infty} I_i + v_3 \sum_{i=0}^{\infty} I_i)\}];\\
&\sum_{i=1}^{\infty} V_i = V_0 + S^{-1}[u^{\alpha_4} S\{\Delta k + v_3 \sum_{i=0}^{\infty} I_i - \mu \sum_{i=0}^{\infty} V_i - v_4 \sum_{i=0}^{\infty} V_i\}];\\
&\sum_{i=1}^{\infty} R_i = R_0 + S^{-1}[u^{\alpha_5} S\{v_2 \sum_{i=0}^{\infty} I_i - \mu \sum_{i=o}^{\infty} R_i\}];
\end{aligned}$$

Therefore, we get the following

$$S_{n+1} = S^{-1}[u^{\alpha_1} S\{\Delta - \Delta k - \mu S_n - \frac{\beta P_n}{\Psi(I_n)} + v_4 V_n\}];$$

$E_{n+1} = S^{-1}\left[u^{\alpha_2} S\left\{\frac{\beta P_n}{\Psi(I_n)} - \mathrm{v}_1 E_n - \mu E_n\right\}\right];$ (14)
$I_{n+1} = S^{-1}[u^{\alpha_3} S\{\mathrm{v}_1 E_n - (\mu I_n + \mathrm{v}_0 I_n + \mathrm{v}_2 I_n + \mathrm{v}_3 I_n)\}];$
$V_{n+1} = S^{-1}[u^{\alpha_4} S\{\Delta k + \mathrm{v}_3 I_n - \mu V_n - \mathrm{v}_4 V_4\}];$
$R_{n+1} = S^{-1}[u^{\alpha_5} S\{\mathrm{v}_2 I_n - \mu R_n\}];$

(1) If n = 0, 1, 2, 3, 4.

If n=0, then the first iteration of S_1, E_1, I_1, V_1 *and* R_1 is given by:-

$$S_1 = x_1 \frac{t^{\alpha_1}}{\Gamma(\alpha_1+1)}; where\ x_1 = \Delta - \Delta k - \mu S_0 - \frac{\beta S_0}{\Psi(I_0)} + \mathrm{v}_4 V_0$$
$$E_1 = x_2 \frac{t^{\alpha_2}}{\Gamma(\alpha_2+1)}; where\ x_2 = \frac{\beta P_0}{\Psi(I_0)} - \mathrm{v}_1 E_0 - \mu E_0; \quad (15)$$
$$I_1 = x_3 \frac{t^{\alpha_3}}{\Gamma(\alpha_3+1)}; where\ x_3 = \mathrm{v}_1 E_0 - (\mu + \mathrm{v}_0 + \mathrm{v}_2 + \mathrm{v}_3) I_0;$$
$$V_1 = x_4 \frac{t^{\alpha_4}}{\Gamma(\alpha_4+1)}; where\ x_4 = \Delta k + \mathrm{v}_3 I_0 - \mu V_0 - \mathrm{v}_4 V_0;$$
$$R_1 = x_5 \frac{t^{\alpha_5}}{\Gamma(\alpha_5+1)}; where\ x_5 = \mathrm{v}_2 I_0 - \mu R_0;$$

(2) If n=1, then the second iteration of S_2, E_2, I_2, V_2 *and* R_2 is given by:

$S_2 = S^{-1}\left[u^{\alpha_1} S\left\{\Delta - \Delta k - \mu S_1 - \frac{\beta P_1}{\Psi(I_1)} + \mathrm{v}_4 V_1\right\}\right]$; and since $P_i = \sum_{j=0}^{i} I_j S_{i-j}$; $\Rightarrow P_1 = S_0 I_1 + S_1 I_0;$

$$S_2 = S^{-1}\left[u^{\alpha_1} S\left\{-\mu S_1 - \frac{\beta P_1}{\Psi(I_1)}(S_0 I_1 + S_1 I_0) + \mathrm{v}_4 V_1\right\}\right]$$
$$= -\left(\mu x_1 + \frac{\beta I_0 x_1}{\Psi(I_1)}\right)\frac{t^{2\alpha_1}}{\Gamma(2\alpha_1+1)} - \frac{\beta S_0 x_3 t^{\alpha_1+\alpha_3}}{\Psi(I_1)\Gamma(\alpha_1+\alpha_3+1)} + \mathrm{v}_4 x_4 \frac{t^{\alpha_1+\alpha_4}}{\Gamma(\alpha_1+\alpha_4+1)}$$
$$E_2 = S^{-1}\left[u^{\alpha_2} \mathrm{S}\left\{\frac{\beta P_1}{\Psi(I_1)} - \mathrm{v}_1 E_1 - \mu E_1\right\}\right];$$
$$= \frac{\beta S_0 x_3}{\Psi(I_1)} \frac{t^{\alpha_1+\alpha_3}}{\Gamma(\alpha_1+\alpha_3+1)} + \frac{\beta I_0 x_1}{\Psi(I_1)} \frac{t^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} - (\mathrm{v}_1 + \mu) x_2 \frac{t^{2\alpha_2}}{\Gamma(2\alpha_2+1)}$$
$$\mathrm{I}_2 = S^{-1}[u^{\alpha_3} S\{\mathrm{v}_1 E_1 - (\mu + \mathrm{v}_0 + \mathrm{v}_2 + \mathrm{v}_3)\}]$$
$$= \mathrm{v}_1 x_2 \frac{t^{\alpha_2+\alpha_3}}{\Gamma(\alpha_2+\alpha_3+1)} - x_3(\mu + \mathrm{v}_0 + \mathrm{v}_2 + \mathrm{v}_3)\frac{\mathrm{t}^{2\alpha_3}}{\Gamma(2\alpha_3+1)}$$
$$V_2 = S^{-1}[u^{\alpha_4} S\{\Delta k + \mathrm{v}_3 I_1 - \mu V_1 - \mathrm{v}_4 V_1\}]$$
$$= \mathrm{v}_3 x_3 \frac{t^{\alpha_3+\alpha_4}}{\Gamma(\alpha_3+\alpha_4+1)} - (\mu + \mathrm{v}_4) x_4 \frac{t^{2\alpha_4}}{\Gamma(2\alpha_4+1)}$$
$$R_2 = S^{-1}[u^{\alpha_5} S\{\mathrm{v}_2 I_1 - \mu R_1\}]$$
$$= \mathrm{v}_2 x_3 \frac{t^{\alpha_3+\alpha_5}}{\Gamma(\alpha_3+\alpha_5+1)} - \mu x_5 \frac{t^{2\alpha_5}}{\Gamma(2\alpha_5+1)}$$

(3) If n=2, the third iteration S_3, E_3, I_3, V_3, R_3 is given by :

$$S_3 = S^{-1}[u^{\alpha_1} S\{\Delta - \Delta k - \mu S_2 - \frac{\beta P_2}{\Psi(I_2)} + \mathrm{v}_4 V_2\}]$$

and since $P_i = \sum_{j=0}^{i} I_j S_{i-j} \Rightarrow P_2 = I_0 S_2 + I_1 S_1 + I_2 S_0$

$$S_3 = S^{-1}\Bigg[u^{\alpha_1} S\Bigg\{\Delta - \Delta k - \mu\left(\frac{\beta I_0}{\Psi(I_1)} - \mu\right) x_1 \frac{t^{2\alpha_1}}{\Gamma(2\alpha_1+1)} - \frac{\beta S_0}{\Psi(I_1)} x_3 \frac{t^{\alpha_1+\alpha_3}}{\Gamma(\alpha_1+\alpha_3+1)} + \mathrm{v}_4 x_4 \frac{t^{\alpha_1+\alpha_4}}{\Gamma(\alpha_1+\alpha_4+1)} - \frac{\beta}{\Psi(I_2)}(I_0 S_2 + I_1 S_1 + I_2 S_0) + \mathrm{v}_4\left(\mathrm{v}_3 x_3 \frac{t^{\alpha_3+\alpha_4}}{\Gamma(\alpha_3+\alpha_4+1)} - (\mu + \mathrm{v}_4) x_4 \frac{t^{2\alpha_4}}{\Gamma(2\alpha_4+1)}\right)\Bigg\}\Bigg]$$

$$= -\left(\mu\frac{\beta I_0}{\Psi(I_1)} - \mu^2 + \frac{\beta^2 I_0^2}{\Psi(I_1)\Psi(I_2)}\right)x_1\frac{t^{4\alpha_1}}{\Gamma(4\alpha_1+1)} + \left(\frac{\beta^2 I_0^2}{\Psi(I_1)\Psi(I_2)} - \frac{\beta S_0}{\Psi(I_1)}\right)x_3\frac{t^{3\alpha_1+\alpha_3}}{\Gamma(3\alpha_1+\alpha_3+1)} + \left(\mathrm{v}_4 - \frac{\mathrm{v}_4\beta I_0}{\Psi(I_2)}\right)x_4\frac{t^{3\alpha_1+\alpha_4}}{\Gamma(3\alpha_1+\alpha_4+1)} \quad (22)$$

$$E_3 = S^{-1}\left[u^{\alpha_2}S\left\{\frac{\beta(I_0S_2+I_1S_1+I_2S_0)}{\Psi(I_2)} - (\mathrm{v}_1+\mu)\left[\frac{\beta S_0}{\Psi(I_1)}x_3\frac{t^{\alpha_1+\alpha_3}}{\Gamma(\alpha_1+\alpha_3+1)} + \frac{\beta I_0}{\Psi(I_1)}x_1\frac{t^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} - (\mathrm{v}_1+\mu)x_2\frac{t^{2\alpha_2}}{\Gamma(2\alpha_2+1)}\right]\right\}\right]$$

$$E_3 = \left(\frac{\beta^2 I_0^2}{\Psi(I_1)\Psi(I_2)} - \frac{\mu\beta I_0}{\Psi(I_2)}\right)x_1\frac{t^{2\alpha_1+\alpha_2}}{\Gamma(2\alpha_1+\alpha_2+1)} - \left(\frac{\beta^2 I_0 S_0}{\Psi(I_1)\Psi(I_2)} + (\mathrm{v}_1+\mu)\frac{\beta S_0}{\Psi(I_1)}\right)x_3\frac{t^{\alpha_1+\alpha_2+\alpha_3}}{\Gamma(\alpha_1+\alpha_2+\alpha_3+1)} + \frac{\beta I_0\mathrm{v}_4}{\Psi(I_2)}x_4\frac{t^{\alpha_1+\alpha_2+\alpha_4}}{\Gamma(\alpha_1+\alpha_2+\alpha_4+1)} + \frac{\beta}{\Psi(I_2)}x_3x_4\frac{t^{\alpha_2+\alpha_3+\alpha_4}}{\Gamma(\alpha_2+\alpha_3+\alpha_4+1)} + \frac{\beta S_0}{\Psi(I_2)}x_3\frac{t^{\alpha_2+\alpha_3}}{\Gamma(\alpha_2+\alpha_3+1)} - (\mathrm{v}_1+\mu)\frac{\beta I_0}{\Psi(I_1)}x_1\frac{t^{\alpha_1+2\alpha_2}}{\Gamma(\alpha_1+2\alpha_2+1)} + (\mathrm{v}_1+\mu)^2x_2\frac{t^{3\alpha_2}}{\Gamma(3\alpha_2+1)}\ . \quad (16)$$

$$I_3 = S^{-1}[u^{\alpha_3}S\{\mathrm{v}_1E_2 - (\mu+\mathrm{v}_0+\mathrm{v}_2+\mathrm{v}_3)I_2\}]$$

$$I_3 = \frac{\mathrm{v}_1\beta S_0x_3}{\Psi(I_1)}\frac{t^{\alpha_1+2\alpha_3}}{\Gamma(\alpha_1+2\alpha_3+1)} + \frac{\mathrm{v}_1\beta I_0x_1}{\Psi(I_1)}\frac{t^{\alpha_1+\alpha_2+\alpha_3}}{\Gamma(\alpha_1+\alpha_2+\alpha_3+1)} - (\mathrm{v}_1^2+\mathrm{v}_1\mu)x_2\frac{t^{2\alpha_2+\alpha_3}}{\Gamma(2\alpha_2+\alpha_3+1)} - (\mu\mathrm{v}_1+\mathrm{v}_0\mathrm{v}_1+\mathrm{v}_1\mathrm{v}_2+\mathrm{v}_1\mathrm{v}_3)x_2\frac{t^{\alpha_2+2\alpha_3}}{\Gamma(\alpha_2+2\alpha_3+1)} + \mathrm{v}_1(\mu+\mathrm{v}_0+\mathrm{v}_2+\mathrm{v}_3)^2x_3\frac{t^{3\alpha_3}}{\Gamma(3\alpha_3+1)} \quad (17)$$

$$V_3 = S^{-1}[u^{\alpha_4}S\{\Delta k + \mathrm{v}_3I_2 - (\mu+\mathrm{v}_4)V_2\}]$$

$$V_3 = \mathrm{v}_1\mathrm{v}_3x_2\frac{t^{\alpha_2+\alpha_3+\alpha_4}}{\Gamma(\alpha_2+\alpha_3+\alpha_4+1)} - \mathrm{v}_3x_3(\mu+\mathrm{v}_0+\mathrm{v}_2+\mathrm{v}_3)\frac{t^{2\alpha_3+\alpha_4}}{\Gamma(2\alpha_3+\alpha_4+1)} - \mathrm{v}_3(\mu+\mathrm{v}_4)x_3\frac{t^{\alpha_3+2\alpha_4}}{\Gamma(\alpha_3+2\alpha_4+1)} + (\mu+\mathrm{v}_4)^2x_4\frac{t^{3\alpha_4}}{\Gamma(3\alpha_4+1)} \quad (18)$$

Similarly for n = 3 and 4.

Now the sum first five terms of I(t) is $\mathrm{I(t)} = \mathrm{I}_0 + \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3 + \mathrm{I}_4$ which represents the infected population

$$\mathrm{I(t)} = \mathrm{I}_0 + x_3\frac{t^{\alpha_3}}{\Gamma(\alpha_3+1)} + \mathrm{v}_1x_2\frac{t^{\alpha_2+\alpha_3}}{\Gamma(\alpha_2+\alpha_3+1)} - x_3(\mu+\mathrm{v}_0+\mathrm{v}_2+\mathrm{v}_3)\frac{t^{2\alpha_3}}{\Gamma(2\alpha_3+1)} + \frac{\mathrm{v}_1\beta S_0x_3}{\Psi(I_1)}\frac{t^{\alpha_1+2\alpha_3}}{\Gamma(\alpha_1+2\alpha_3+1)} + \frac{\mathrm{v}_1\beta I_0x_1}{\Psi(I_1)}\frac{t^{\alpha_1+\alpha_2+\alpha_3}}{\Gamma(\alpha_1+\alpha_2+\alpha_3+1)} - (\mathrm{v}_1^2+\mathrm{v}_1\mu)x_2\frac{t^{2\alpha_2+\alpha_3}}{\Gamma(2\alpha_2+\alpha_3+1)} - (\mu\mathrm{v}_1+\mathrm{v}_0\mathrm{v}_1+\mathrm{v}_1\mathrm{v}_2+\mathrm{v}_1\mathrm{v}_3)x_2\frac{t^{\alpha_2+2\alpha_3}}{\Gamma(\alpha_2+2\alpha_3+1)} + \mathrm{v}_1(\mu+\mathrm{v}_0+\mathrm{v}_2+\mathrm{v}_3)^2x_3\frac{t^{3\alpha_3}}{\Gamma(3\alpha_3+1)} + \left(\frac{\mathrm{v}_1\beta^2I_0^2}{\Psi(I_1)\Psi(I_2)} - \frac{\mathrm{v}_1\mu\beta I_0}{\Psi(I_2)}\right)x_1\frac{t^{2\alpha_1+\alpha_2+\alpha_3}}{\Gamma(2\alpha_1+\alpha_2+\alpha_3+1)} - \left[\left(\frac{\mathrm{v}_1\beta^2I_0S_0}{\Psi(I_1)\Psi(I_2)} + (\mathrm{v}_1+\mu)\frac{\mathrm{v}_1\beta S_0}{\Psi(I_1)}\right)x_3 + (\mu+\mathrm{v}_0+\mathrm{v}_2+\mathrm{v}_3)\frac{\mathrm{v}_1\beta I_0x_1}{\Psi(I_1)}\right]\frac{t^{\alpha_1+\alpha_2+2\alpha_3}}{\Gamma(\alpha_1+\alpha_2+2\alpha_3+1)} + \frac{\mathrm{v}_1\beta I_0\mathrm{v}_4}{\Psi(I_2)}x_4\frac{t^{\alpha_1+\alpha_2+\alpha_3+\alpha_4}}{\Gamma(\alpha_1+\alpha_2+\alpha_3+\alpha_4+1)} + \frac{\mathrm{v}_1\beta}{\Psi(I_2)}x_3x_4\frac{t^{\alpha_2+2\alpha_3+\alpha_4}}{\Gamma(\alpha_2+2\alpha_3+\alpha_4+1)} + \frac{\mathrm{v}_1\beta S_0}{\Psi(I_2)}x_3\frac{t^{\alpha_2+2\alpha_3}}{\Gamma(\alpha_2+2\alpha_3+1)} - (\mathrm{v}_1+\mu)\frac{\mathrm{v}_1\beta I_0}{\Psi(I_1)}x_1\frac{t^{\alpha_1+2\alpha_2+\alpha_3}}{\Gamma(\alpha_1+2\alpha_2+\alpha_3+1)} - (\mu+\mathrm{v}_0+\mathrm{v}_2+\mathrm{v}_3)\frac{\mathrm{v}_1\beta S_0x_3}{\Psi(I_1)}\frac{t^{\alpha_1+3\alpha_3}}{\Gamma(\alpha_1+3\alpha_3+1)} + (\mu\mathrm{v}_1^2+\mathrm{v}_1\mu^2+\mathrm{v}_0\mathrm{v}_1^2+\mathrm{v}_0\mathrm{v}_1\mu+\mathrm{v}_1^2\mathrm{v}_2+\mathrm{v}_1\mathrm{v}_2\mu+\mathrm{v}_1^2\mathrm{v}_3+\mathrm{v}_1\mathrm{v}_3\mu)x_2\frac{t^{2\alpha_2+2\alpha_3}}{\Gamma(2\alpha_2+2\alpha_3+1)} + (\mu^2\mathrm{v}_1+\mathrm{v}_0\mathrm{v}_1^2+\mathrm{v}_1\mathrm{v}_3^2+2\mu\mathrm{v}_0\mathrm{v}_1+2\mu\mathrm{v}_1\mathrm{v}_2+2\mu\mathrm{v}_1\mathrm{v}_3+2\mathrm{v}_0\mathrm{v}_1\mathrm{v}_2+3\mathrm{v}_1\mathrm{v}_2\mathrm{v}_3)x_2\frac{t^{\alpha_2+3\alpha_3}}{\Gamma(\alpha_2+3\alpha_3+1)} + \mathrm{v}_1(\mathrm{v}_1+\mu)^2x_2\frac{t^{3\alpha_2+\alpha_3}}{\Gamma(3\alpha_2+\alpha_3+1)} - (\mu+\mathrm{v}_0+\mathrm{v}_2+\mathrm{v}_3)^2x_3\frac{t^{4\alpha_3}}{\Gamma(4\alpha_3+1)} \quad (19)$$

Also the exposed population is given by :-

$$\mathrm{E(t)} = \mathrm{E}_0 + \mathrm{E}_1 + \mathrm{E}_2 + \mathrm{E}_3 + \mathrm{E}_4 + \mathrm{E}_5$$

$$= \mathrm{E}_0 + x_2 \frac{t^{\alpha_2}}{\Gamma(\alpha_2+1)} + \frac{\beta S_0 x_3}{\Psi(I_1)} \frac{t^{\alpha_1+\alpha_3}}{\Gamma(\alpha_1+\alpha_3+1)} + \frac{\beta I_0 x_1}{\Psi(I_1)} \frac{t^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} - (\mathrm{v}_1+\mu) x_2 \frac{t^{2\alpha_2}}{\Gamma(2\alpha_2+1)} +$$
$$\left(\frac{\beta^2 I_0^2}{\Psi(I_1)\Psi(I_2)} - \frac{\mu\beta I_0}{\Psi(I_2)}\right) x_1 \frac{t^{2\alpha_1+\alpha_2}}{\Gamma(2\alpha_1+\alpha_2+1)} - \left(\frac{\beta^2 I_0 S_0}{\Psi(I_1)\Psi(I_2)} + (\mathrm{v}_1+\mu)\frac{\beta S_0}{\Psi(I_1)}\right) x_3 \frac{t^{\alpha_1+\alpha_2+\alpha_3}}{\Gamma(\alpha_1+\alpha_2+\alpha_3+1)} +$$
$$\frac{\beta I_0 \mathrm{v}_4}{\Psi(I_2)} x_4 \frac{t^{\alpha_1+\alpha_2+\alpha_4}}{\Gamma(\alpha_1+\alpha_2+\alpha_4+1)} + \frac{\beta}{\Psi(I_2)} x_3 x_4 \frac{t^{\alpha_2+\alpha_3+\alpha_4}}{\Gamma(\alpha_2+\alpha_3+\alpha_4+1)} + \frac{\beta S_0}{\Psi(I_2)} x_3 \frac{t^{\alpha_2+\alpha_3}}{\Gamma(\alpha_2+\alpha_3+1)} - (\mathrm{v}_1 +$$
$$\mu)\frac{\beta I_0}{\Psi(I_1)} x_1 \frac{t^{\alpha_1+2\alpha_2}}{\Gamma(\alpha_1+2\alpha_2+1)} + (\mathrm{v}_1+\mu)^2 x_2 \frac{t^{3\alpha_2}}{\Gamma(3\alpha_2+1)} + \frac{\mathrm{v}_1\beta^2 {S_0}^2}{\Psi(I_1)\Psi(I_3)} x_3 \frac{t^{\alpha_1+\alpha_2+2\alpha_3}}{\Gamma(\alpha_1+\alpha_2+2\alpha_3+1)} +$$
$$\frac{\mathrm{v}_1\beta^2 S_0 I_0 x_1}{\Psi(I_1)\Psi(I_3)} \frac{t^{\alpha_1+2\alpha_2+\alpha_3}}{\Gamma(\alpha_1+2\alpha_2+\alpha_3+1)} - \frac{\beta S_0}{\Psi(I_3)}(\mathrm{v}_1^2 + \mathrm{v}_1\mu) x_2 \frac{t^{3\alpha_2+\alpha_3}}{\Gamma(3\alpha_2+\alpha_3+1)} - \frac{\beta S_0}{\Psi(I_3)}(\mu\mathrm{v}_1 + \mathrm{v}_0\mathrm{v}_1 + \mathrm{v}_1\mathrm{v}_2 +$$
$$\mathrm{v}_1\mathrm{v}_3) x_2 \frac{t^{2\alpha_2+2\alpha_3}}{\Gamma(2\alpha_2+2\alpha_3+1)} + \frac{\beta S_0}{\Psi(I_3)} \mathrm{v}_1(\mu + \mathrm{v}_0 + \mathrm{v}_2 + \mathrm{v}_3)^2 x_3 \frac{t^{\alpha_2+3\alpha_3}}{\Gamma(\alpha_2+3\alpha_3+1)} +$$
$$\frac{\beta}{\Psi(I_3)} \mathrm{v}_1 x_1 x_2 \frac{t^{\alpha_1+2\alpha_2+\alpha_3}}{\Gamma(\alpha_1+\alpha_2+\alpha_3+1)} - \frac{\beta}{\Psi(I_3)} x_1 x_3 (\mu + \mathrm{v}_0 + \mathrm{v}_2 + \mathrm{v}_3) \frac{t^{\alpha_1+\alpha_2+2\alpha_3}}{\Gamma(\alpha_1+\alpha_2+2\alpha_3+1)} - \left(\frac{\beta\mu}{\Psi(I_3)} +\right.$$
$$\left.\frac{\beta^2 I_0}{\Psi(I_1)}\right) x_1 x_3 \frac{t^{2\alpha_1+\alpha_2+\alpha_3}}{\Gamma(2\alpha_1+\alpha_3+1)} - \frac{\beta^2 S_0 {x_3}^2 t^{\alpha_1+\alpha_2+2\alpha_3}}{\Psi(I_1)\Psi(I_3)\Gamma(\alpha_1+2\alpha_3+1)} + \frac{\beta}{\Psi(I_3)} \mathrm{v}_4 x_3 x_4 \frac{t^{\alpha_1+\alpha_2+\alpha_3+\alpha_4}}{\Gamma(\alpha_1+\alpha_3+\alpha_4+1)} - \left(\frac{\mu\beta^2 {I_0}^2}{\Psi(I_1)\Psi(I_3)} -\right.$$
$$\left.\frac{\beta I_0}{\Psi(I_3)}\mu^2 + \frac{\beta^3 I_0^3}{\Psi(I_1)\Psi(I_2)\,\Psi(I_3)}\right) x_1 \frac{t^{4\alpha_1+\alpha_2}}{\Gamma(4\alpha_1+\alpha_2+1)} + \left(\frac{\beta^3 I_0^3}{\Psi(I_1)\Psi(I_2)\Psi(I_3)} - \frac{\beta^2 I_0 S_0}{\Psi(I_1)\Psi(I_3)}\right) x_3 \frac{t^{3\alpha_1+\alpha_2+\alpha_3}}{\Gamma(3\alpha_1+\alpha_2+\alpha_3+1)} +$$
$$\left(\frac{\beta I_0}{\Psi(I_3)}\mathrm{v}_4 - \frac{\mathrm{v}_4\beta^2 {I_0}^2}{\Psi(I_2)\Psi(I_3)^2}\right) x_4 \frac{t^{3\alpha_1+\alpha_2+\alpha_4}}{\Gamma(3\alpha_1+\alpha_2+\alpha_4+1)} - \left[(\mathrm{v}_1+\mu)\frac{\beta^2 I_0^2}{\Psi(I_1)\Psi(I_2)} - (\mathrm{v}_1 +\right.$$
$$\left.\mu)\frac{\mu\beta I_0}{\Psi(I_2)}\right] x_1 \frac{t^{2\alpha_1+2\alpha_2}}{\Gamma(2\alpha_1+2\alpha_2+1)} + \left[(\mathrm{v}_1+\mu)\frac{\beta^2 I_0 S_0}{\Psi(I_1)\Psi(I_2)} + (\mathrm{v}_1+\mu)^2 \frac{\beta S_0}{\Psi(I_1)}\right] x_3 \frac{t^{\alpha_1+2\alpha_2+\alpha_3}}{\Gamma(\alpha_1+2\alpha_2+\alpha_3+1)} -$$
$$(\mathrm{v}_1+\mu)\frac{\beta I_0 \mathrm{v}_4}{\Psi(I_2)} x_4 \frac{t^{\alpha_1+2\alpha_2+\alpha_4}}{\Gamma(\alpha_1+2\alpha_2+\alpha_4+1)} - (\mathrm{v}_1+\mu)\frac{\beta}{\Psi(I_2)} x_3 x_4 \frac{t^{2\alpha_2+\alpha_3+\alpha_4}}{\Gamma(\alpha_2+\alpha_3+\alpha_4+1)} - (\mathrm{v}_1 +$$
$$\mu)\frac{\beta S_0}{\Psi(I_2)} x_3 \frac{t^{2\alpha_2+\alpha_3}}{\Gamma(2\alpha_2+\alpha_3+1)} + (\mathrm{v}_1+\mu)^2 \qquad \frac{\beta I_0}{\Psi(I_1)} x_1 \frac{t^{\alpha_1+3\alpha_2}}{\Gamma(\alpha_1+3\alpha_2+1)} - (\mathrm{v}_1+\mu)^3 x_2 \frac{t^{4\alpha_2}}{\Gamma(4\alpha_2+1)} +$$
$$\left[\left(\frac{\mathrm{v}_1 S_0 \beta^3 I_0^2}{\Psi(I_1)\Psi(I_2)\Psi(I_4)} - \frac{\mathrm{v}_1\mu S_0\beta^2 I_0}{\Psi(I_2)\Psi(I_4)}\right) x_1 + \frac{\mathrm{v}_1\beta^2 I_0 {x_1}^2}{\Psi(I_1)\Psi(I_4)} - \frac{\beta^2 I_0 P_3}{\Psi(I_3)\Psi(I_4)} \mathrm{v}_1 x_1 x_2\right] \frac{t^{2\alpha_1+2\alpha_2+\alpha_3}}{\Gamma(2\alpha_1+2\alpha_2+\alpha_3+1)} -$$
$$\left[\left(\frac{\mathrm{v}_1\beta^3 I_0 {S_0}^2}{\Psi(I_1)\Psi(I_2)\Psi(I_4)} + (\mathrm{v}_1+\mu)\frac{\mathrm{v}_1\beta^2 {S_0}^2}{\Psi(I_1)\Psi(I_4)}\right) x_3 + (\mu + \mathrm{v}_0 + \mathrm{v}_2 + \mathrm{v}_3)\frac{\mathrm{v}_1\beta^2 S_0 I_0 x_1}{\Psi(I_1)\Psi(I_4)}\right] \frac{t^{\alpha_1+2\alpha_2+2\alpha_3}}{\Gamma(\alpha_1+2\alpha_2+2\alpha_3+1)} +$$
$$\left[\frac{\mathrm{v}_1\beta^2 S_0 I_0 \mathrm{v}_4}{\Psi(I_2)\Psi(I_4)} x_4 + \frac{\beta I_0}{\Psi(I_4)} \mathrm{v}_1\mathrm{v}_3\mathrm{v}_4 x_2\right] \frac{t^{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}}{\Gamma(\alpha_1+2\alpha_2+\alpha_3+\alpha_4+1)} + \frac{\mathrm{v}_1 S_0 \beta^2}{\Psi(I_2)\Psi(I_4)} x_3 x_4 \frac{t^{2\alpha_2+2\alpha_3+\alpha_4}}{\Gamma(2\alpha_2+2\alpha_3+\alpha_4+1)} +$$
$$\frac{\mathrm{v}_1\beta^2 {S_0}^2}{\Psi(I_2)\Psi(I_4)} x_3 \frac{t^{2\alpha_2+2\alpha_3}}{\Gamma(2\alpha_2+2\alpha_3+1)} - \left[(\mathrm{v}_1+\mu)\frac{\mathrm{v}_1\beta^2 S_0 I_0}{\Psi(I_1)\Psi(I_4)} x_1 + \frac{\beta}{\Psi(I_4)}(\mathrm{v}_1^2 + \mathrm{v}_1\mu) x_1 x_2\right] \frac{t^{\alpha_1+3\alpha_2+\alpha_3}}{\Gamma(\alpha_1+3\alpha_2+\alpha_3+1)} -$$
$$\left[(\mu + \mathrm{v}_0 + \mathrm{v}_2 + \mathrm{v}_3)\frac{\mathrm{v}_1\beta^2 {S_0}^2 x_3}{\Psi(I_1)\Psi(I_4)} - \frac{\beta}{\Psi(I_4)}\mathrm{v}_1(\mu + \mathrm{v}_0 + \mathrm{v}_2 + \mathrm{v}_3)^2 x_1 x_3 + \frac{\beta^2 S_0 I_0 P_3}{\Psi(I_3)\Psi(I_4)}\mathrm{v}_1(\mu + \mathrm{v}_0 +\right.$$
$$\left.\mathrm{v}_2 + \mathrm{v}_3)^2 x_1 x_3\right] \qquad \frac{t^{\alpha_1+\alpha_2+3\alpha_3}}{\Gamma(\alpha_1+\alpha_2+3\alpha_3+1)} + \frac{\beta S_0}{\Psi(I_4)}(\mu\mathrm{v}_1^2 + \mathrm{v}_1\mu^2 + \mathrm{v}_0\mathrm{v}_1^2 + \mathrm{v}_0\mathrm{v}_1\mu + \mathrm{v}_1^2\mathrm{v}_2 + \mathrm{v}_1\mathrm{v}_2\mu +$$
$$\mathrm{v}_1^2\mathrm{v}_3 + \mathrm{v}_3\mu) x_2 \frac{t^{3\alpha_2+2\alpha_3}}{\Gamma(3\alpha_2+2\alpha_3+1)} + \frac{\beta S_0}{\Psi(I_4)}(\mu^2\mathrm{v}_1 + \mathrm{v}_0\mathrm{v}_1^2 + \mathrm{v}_1\mathrm{v}_3^2 + 2\mu\mathrm{v}_0\mathrm{v}_1 + 2\mu\mathrm{v}_1\mathrm{v}_2 + 2\mu\mathrm{v}_1\mathrm{v}_3 +$$
$$2\mathrm{v}_0\mathrm{v}_1\mathrm{v}_2 + 3\mathrm{v}_1\mathrm{v}_2\mathrm{v}_3) x_2 \frac{t^{2\alpha_2+3\alpha_3}}{\Gamma(2\alpha_2+3\alpha_3+1)} + \frac{\beta S_0}{\Psi(I_4)}\mathrm{v}_1(\mathrm{v}_1+\mu)^2 x_2 \frac{t^{4\alpha_2+\alpha_3}}{\Gamma(4\alpha_2+\alpha_3+1)} - \frac{\beta S_0}{\Psi(I_4)}(\mu + \mathrm{v}_0 +$$
$$\mathrm{v}_2 + \mathrm{v}_3)^2 x_3 \frac{t^{\alpha_2+4\alpha_3}}{\Gamma(\alpha_2+4\alpha_3+1)} - \frac{\beta}{\Psi(I_4)}(\mu\mathrm{v}_1 + \mathrm{v}_0\mathrm{v}_1 + \mathrm{v}_1\mathrm{v}_2 + \mathrm{v}_1\mathrm{v}_3)\mathrm{x}_1\mathrm{x}_2 \frac{\mathrm{t}^{\alpha_1+2\alpha_2+2\alpha_3}}{\Gamma(\alpha_1+2\alpha_2+2\alpha_3+1)} + \frac{\beta}{\Psi(I_4)} \quad =$$
$$\frac{\mathrm{v}_1\beta S_0 x_2\ {x_3}^2 t^{\alpha_1+2\alpha_2+4\alpha_3}}{\Psi(I_1)\Gamma(\alpha_1+2\alpha_2+4\alpha_3+1)} - \frac{\beta}{\Psi(I_4)}(\mu + \mathrm{v}_0 + \mathrm{v}_2 + \mathrm{v}_3)\mathrm{v}_1\mathrm{v}_4 x_2 x_3 x_4 - \left[\frac{\beta}{\Psi(I_4)}\left(-\mu^2 +\right.\right.$$
$$\left.\frac{\beta^2 I_0^2}{\Psi(I_1)\Psi(I_2)}\right) x_1 x_3 + \frac{\beta}{\Psi(I_4)}\left(\mu\frac{\beta I_0}{\Psi(I_1)} - \mu^2 + \frac{\beta^2 I_0^2}{\Psi(I_1)\Psi(I_2)}\right) x_1 x_3 + [$$

$$\left(\frac{\mu\beta^3 I_0^3}{\Psi(I_1)\Psi(I_2)\Psi(I_4)} - \frac{\mu\beta^2 I_0 S_0}{\Psi(I_1)\Psi(I_4)}\right)$$
$$+ \frac{\beta^2 {I_0}^2 P_3}{\Psi(I_3)\Psi(I_4)}\left(\frac{\beta^2 I_0^2}{\Psi(I_1)\Psi(I_2)} - \frac{\beta S_0}{\Psi(I_1)}\right)]\, x_3]\, \frac{t^{4\alpha_1+\alpha_2+\alpha_3}}{\Gamma(4\alpha_1+\alpha_2+\alpha_3+1)}$$
$$+ \frac{\beta}{\Psi(I_4)}[\left(\mu I_0 \mathrm{v}_4 - \mu\frac{\mathrm{v}_4\beta {I_0}^2}{\Psi(I_2)}\right) - \frac{\beta {I_0}^2 P_3}{\Psi(I_3)}\left(\mathrm{v}_4 - \frac{\mathrm{v}_4\beta I_0}{\Psi(I_2)}\right)]x_4\, \frac{t^{4\alpha_1+\alpha_2+\alpha_4}}{\Gamma(4\alpha_1+\alpha_2+\alpha_4+1)}$$
$$+ 2\frac{\beta^2 I_0 P_3}{\Psi(I_3)\Psi(I_4)}\left(\mu x_1 + \frac{\beta I_0 x_1}{\Psi(I_1)}\right)x_3\, \frac{t^{3\alpha_1+\alpha_2+\alpha_3}}{\Gamma(3\alpha_1+\alpha_2+\alpha_3+1)}$$
$$+ \frac{\beta^2 I_0 P_3}{\Psi(I_3)\Psi(I_4)}x_1 x_3(\mu + \mathrm{v}_0 + \mathrm{v}_2 + \mathrm{v}_3)]\, \frac{t^{2\alpha_1+\alpha_2+2\alpha_3}}{\Gamma(2\alpha_1+\alpha_2+2\alpha_3+1)}$$
$$- \frac{\beta^2 I_0 P_3}{\Psi(I_3)\Psi(I_4)}\mathrm{v}_4 x_3 x_4\, \frac{t^{2\alpha_1+\alpha_2+\alpha_3+\alpha_4}}{\Gamma(2\alpha_1+\alpha_2+\alpha_3+\alpha_4+1)}$$
$$- \frac{\mathrm{v}_1\beta^3 I_0 {S_0}^2 x_1 x_3}{\Psi(I_1)\Psi(I_3)\Psi(I_4)}\frac{t^{3\alpha_1+\alpha_2+2\alpha_3}}{\Gamma(3\alpha_1+\alpha_2+2\alpha_3+1)} - \frac{\mathrm{v}_1\beta^3 P_3 S_0 {I_0}^2 {x_1}^2}{\Psi(I_1)\Psi(I_3)\Psi(I_4)}\frac{t^{3\alpha_1+2\alpha_2+\alpha_3}}{\Gamma(3\alpha_1+2\alpha_2+\alpha_3+1)}$$
$$+ \frac{\beta^2 S_0 I_0 P_3}{\Psi(I_3)\Psi(I_4)}(\mathrm{v}_1^2 + \mathrm{v}_1\mu)x_1 x_2\, \frac{t^{2\alpha_1+3\alpha_2+\alpha_3}}{\Gamma(2\alpha_1+3\alpha_2+\alpha_3+1)}$$
$$+ \frac{\beta^2 S_0 I_0 P_3}{\Psi(I_3)\Psi(I_4)}(\mu\mathrm{v}_1 + \mathrm{v}_0\mathrm{v}_1 + \mathrm{v}_1\mathrm{v}_2 + \mathrm{v}_1\mathrm{v}_3)x_1 x_2\, \frac{t^{2\alpha_1+2\alpha_2+2\alpha_3}}{\Gamma(2\alpha_1+2\alpha_2+2\alpha_3+1)}$$
$$- \frac{\beta^2 S_0 I_0 P_3}{\Psi(I_3)\Psi(I_4)}\mathrm{v}_1(\mu + \mathrm{v}_0 + \mathrm{v}_2 + \mathrm{v}_3)^2 x_1 x_3\, \frac{t^{\alpha_1+\alpha_2+3\alpha_3}}{\Gamma(\alpha_1+\alpha_2+3\alpha_3+1)}$$
$$- \frac{\beta I_0}{\Psi(I_4)}\mathrm{v}_3\mathrm{v}_4 x_3(\mu + \mathrm{v}_0 + \mathrm{v}_2 + \mathrm{v}_3)\, \frac{t^{\alpha_1+\alpha_2+2\alpha_3+\alpha_4}}{\Gamma(\alpha_1+\alpha_2+2\alpha_3+\alpha_4+1)}$$
$$- \frac{\beta I_0}{\Psi(I_4)}\mathrm{v}_3\mathrm{v}_4(\mu + \mathrm{v}_4)x_3\, \frac{t^{\alpha_1+\alpha_2+\alpha_3+2\alpha_4}}{\Gamma(\alpha_1+\alpha_2+\alpha_3+2\alpha_4+1)}$$
$$- \frac{\beta}{\Psi(I_4)}(\mu + \mathrm{v}_0 + \mathrm{v}_2$$
$$+ \mathrm{v}_3)\mathrm{v}_1\mathrm{v}_4 x_2 x_3 x_4\, \frac{t^{\alpha_1+2\alpha_2+3\alpha_3+\alpha_4}}{\Gamma(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4+1)} \qquad (20)$$

5.<u>Numerical Simulation</u> :

S(t) = 10+11.71t $-0.17t^2 - 12.00088712t^4 - 7.952860234t^5 - 65.57t^7 + 922.3285701t^8 + 608.5332174t^9$

Now if we take $E_0 = 50;\ I_0 = 10;\ \mu = 0.025;\ \mathrm{v}_0 = 0.001;\ \mathrm{v}_1 0.009; \mathrm{v}_2 = 0.07;$ $\mathrm{v}_3 = 0.04;\ \mathrm{v}_4 = 0.2;\ \beta = 0.002;$ k = 0; $S_0 = 10;\ V_0 10;\ P_0 = S_0^2 = 100;$ $R_0 = 5\ ; \Psi(I_0) = \Psi(I_1) = \Psi(I_2) = \Psi(I_3) = \Psi(I_4) = 1, \alpha_1, \alpha_2, .\alpha_3,$

E(t) = 50 + $0.65t$ $+4.012t^2$ $-12.001t^3 - 20.13t^4 + 1.8t^5 - 1.000000003t^6-5t^7-181.3418t^8-1724.330424t^9 - 2.101t^{10}$

I(t) = 10 + $1.36t^2 - 2.001t^2 - 2.2t^3 + 3.7\, t^4$

V(t) =10 $-0.4t + 0.001t^2 - 2.0127\, t^3$

R(t) = 5 -0.7 t +2.017 t^2

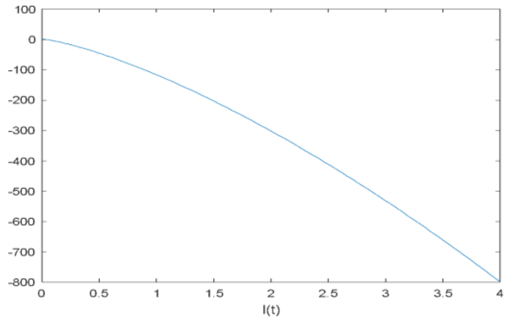

Figure (2): I(t)

Now if we choose $I_0 = 10;\ x_1 = 8.930;\ x_2 = -33.1;\ x_3 = 7.64; x_4 = -0.399$; thus :
$E(t) = -0.000051t^6 + 3.12\ t^5$+6.12001 $t^4 - 0.00064t^3 + 45.50223t^2 - 33.1t + 1000;$

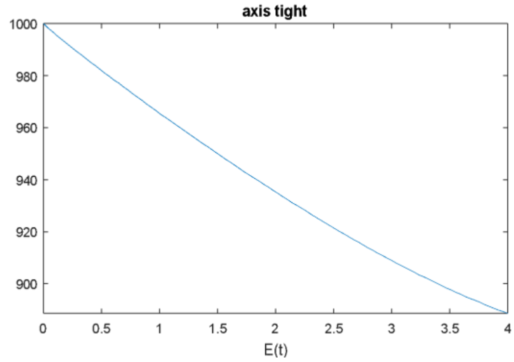

Figure3: E(t).

Table 1

t	0	0.1	0.2	0.3	0.4	1
S(t)	10	20.031	12.2	12.1	11.697	13.599
E(t)	50	89.96999	207.1828825	393.3804	632.7155	131.6715
I(t)	10	10.11416	10.18028	10.19848	10.17776	10.859
V(t)	10	9.957997	9.9039384	9.825747	9.711347	7.5883
R(t)	5	4.95017	4.94068	4.97153	5.04272	6.317

From the previous figures and table we notice :

(1) The susceptible population S(t) increases rapidly at the beginning because they do not know the danger then it stabilities at acceptable number.

(2) The exposed E(t) also increases because either they don't know the infected or the infected people don't declare their disease after that according to the limitation ordered by medical organization the number of E(t) decreases.

Similar interpretation for I(t), V(t) and R(t).

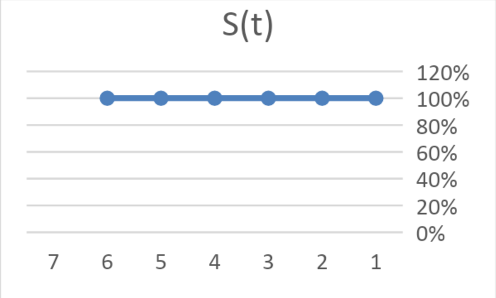

Figure4: S(t)

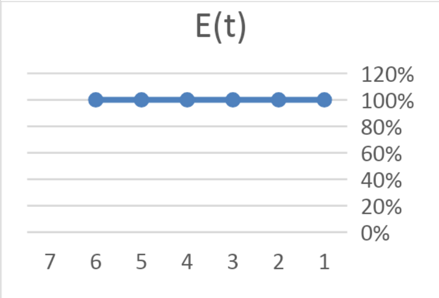

Figure5: E(t)

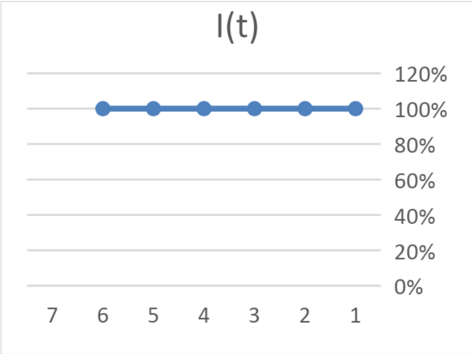

Figure 6: I(t)

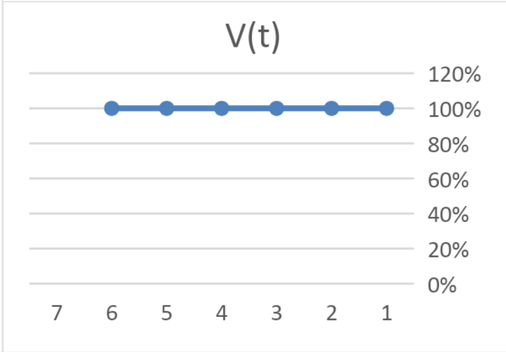

Figure7: V(t)

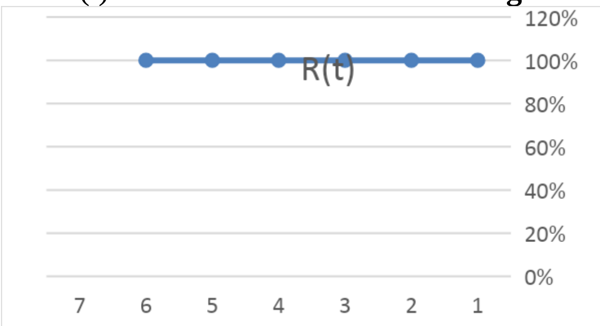

Figure 8: R(t)

Conclusion

The dynamical behaviors of model (9) is discussed in this paper. Model (9) contains five fractional differential equations with different orders. Due to the nonlinearity of the system, an approximate solution is evaluated by using the Sumudu transformation after applying Adomain decomposition method. The satisfactory property of the given approximate solution is illustrated through graphs and tables. It is clearly that the finding approximate solution to the problem is better than the qualitative study because we get quantitative results as well as one can estimate the size of populations after successive times. Finally, we can note that for real data one can determine the values of the orders of the fractional derivatives that give the best estimation.

Future work

1- One can consider the approximate solution as probability density function and then study the statistical properties of the model.
2- One can insert the delay in the model.
3- One can study the system as a stochastic fractional delay system.

Conflict of interest

The authors have no conflicts of interest.

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