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On A Certain Class of Meromorphic Multivalent Functions Defined by Fractional Calculus Operator

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Abstract
In this work, we study a new class of meromorphicmultivalent functions, defined by fractional differ-integral operator.We obtain some geometricproperties, such ascoefficient inequality, growth and distortion bounds, convolution properties, integral representation, radii of starlikeness, convexity, extreme pointsproperties, weighted mean and arithmetic meanproperties.

Keywords: Meromorphic multivalent function, fractional calculus operator.

حول صنف حديث لدوال متعددة التكافؤ المرمورفيه معرفه بواسطة التفاضل ألكسري

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ألخلاصه
في هذا العمل الحالي, درسنا صنف حديث من الدوال متعددة التكافؤ المرمورفيه, المعرفة بواسطة المؤثر التفاضلي ألكسري. حصلنا بعض الخصائص الهندسية, مثل, مترا حجه المعاملات, وقيود النمو و التشوه, خاصية الالتواء, خاصية, التكامل, إنصاف أقطار ألنجميه ولتحدبيه, الوسط الحسابي الموزون, و الوسط الحسابي الرياضي لتلك الدوال.

1. Introduction

Let Σ_p stand for the class of functionsof the format:

$$f(w) = \frac{1}{w^p} - \sum_{i=p}^{\infty} a_i w^i \quad , \quad (a_i \geq 0, p \in \mathbb{N} = \{1,2,\ldots\}), \tag{1.1}$$

which are holomorphic and multivalent in the punctured unit disk $U^* = \{w: w\epsilon\mathbb{C} and\ 0 < |w| < 1\} = U \setminus \{0\}$.

A function $f \in \Sigma_p$ is ameromorphic multivalent starlikefunction of order ϕ $(0 \leq \phi < p)$, if

$$Re\left\{-\frac{wf^{`}(w)}{f(w)}\right\} > \varphi\, , (0 \leq \phi < p), \mathrm{w} \in U^*, \tag{1.2}$$

a function$f \in \Sigma_p$ is a meromorphic multivalent convex function of order φ $(0 \leq \varphi < p)$, if

$$Re\left\{-\left(1 + \frac{wf^{``}(w)}{f(w)}\right)\right\} > \phi\, , (0 \leq \phi < p), \mathrm{w} \in U^*. \tag{1.3}$$

For functions $f \in \Sigma_p$ presented in (1.1) and $g(w) \in \Sigma_p$ presented by

$$g(w) = \frac{1}{\mathrm{w^p}} - \sum_{i=p}^{\infty} b_i w^i \quad , \quad (b_i \geq 0, p \in \mathbb{N} = \{1,2,\ldots\}). \tag{1.4}$$

We define by the convolution of $f(w) and\ g(w)$

$$(f * g)(w) = \frac{1}{\mathrm{w^p}} - \sum_{i=p}^{\infty} a_i b_i w^i = (g * f)(w). \tag{1.5}$$

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In this article, we investigate a new class of meromorphic multivalent functions by making use of the fractional differ-integral operator,defined asfollows:

Definition (1 .1)[1]: Let function *f(w)*$\in \Sigma_p$defined in (1.1). Then

$$\mathcal{F}_{0,w}^{\kappa,\mu,\nu,\eta} f(w)=\begin{cases} \frac{\Gamma(\mu+\nu+\eta-\kappa)\Gamma(\eta)}{\Gamma(\mu+\eta)\Gamma(\nu+\eta)} w^{-p-\eta+1} J_{0,w}^{\kappa,\mu,\nu,\eta}[w^{\mu+p} f(w)](0 \leq \kappa < 1) \\ \frac{\Gamma(\mu+\nu+\eta-\lambda)\Gamma(\eta)}{\Gamma(\mu+\eta)\Gamma(\nu+\eta)} w^{-p-\eta+1} J_{0,w}^{-\kappa,\mu,\nu,\eta}[w^{\mu+p} f(w)](\infty < \kappa < 0), \end{cases} \quad (1.6)$$

where$J_{0,w}^{\kappa,\mu,\nu,\eta} f(w)$ is the circulatedfractional derivative operator of order κ defined by

$$J_{0,w}^{\kappa,\mu,\nu,\eta} f(w) = \frac{1}{\Gamma(1-\kappa)} \frac{d}{dw} \left\{ w^{\kappa-\mu} \int_0^w t^{\eta-1} (w-t)^{-\kappa} f_1(\mu-\kappa, 1-\nu, 1-\kappa, 1-\frac{t}{w} f(t)dt \right\} (1.7)$$

$(0 \leq \kappa < 1, \mu, \eta \in R, \nu \in R^+ and r > (\max\{0,\mu\} - \eta))$,

where *f(w),* is an analytic function in a simply-connected region of the w-plane containing the origin and the multiplicity of $(w-t),^{-\kappa}$is removed by requiring *log*(w-t) to be real when $(w-t) > 0$, provided that

$$f(w) = o(|w|^r)(w \to 0) \quad (1.8)$$

and $I_{0,w}^{\kappa,\mu,\nu,\eta}$ is the generalized fractional integral operator of order - $\kappa(\infty < \kappa < 0)$ defined by

$$I_{0,w}^{\kappa,\mu,\nu,\eta} f(w) = \frac{w^{-(\kappa+\mu)}}{\Gamma(\kappa)} \int_0^w t^{\eta-1} (w-t)^{\kappa-1} \left(f_1(\mu+\kappa, -\nu, \kappa, 1-\frac{t}{w} \right) f(t)dt \quad (1.9)$$

$(\kappa > 0, \mu, \eta \in R, \nu \in R^+ and r > (\max\{0,\mu\} - \eta))$,

where *f*(w), is constrained, the multiplicity of $(w-t)^{\kappa-1}$is removed as above and r is presented by order estimate (1.8).

$$J_{0,w}^{\kappa,\mu,\nu,1} f(w) = I_{0,w}^{\kappa,\mu,\nu} f(w), \quad (1.10)$$

and

$$I_{0,w}^{\kappa,\mu,\nu,1} f(w) = I_{0,w}^{\kappa,\mu,\nu} f(w), \quad (1,11)$$

where $J_{0,w}^{\kappa,\mu,\nu,1} f(w) and I_{0,w}^{\kappa,\mu,\nu} f(w)$, are the Owa-Saigo-Srivastvav generalized fractional derivative and integral operators (see [2,3,4]).

Also

$$J_{0,w}^{\kappa,\kappa,\nu,1} f(w) = D_w^{\kappa} f(w), \qquad (0 \leq \kappa < 1), \quad (1,12)$$

and

$$I_{0,w}^{\kappa,-\kappa,\nu,1} f(w) = D_w^{-\kappa} f(w), \qquad (0 \leq \kappa < 1) \quad (1.13)$$

whereD_w^{κ} and $D_w^{-\kappa}$are the familiar Owa-Srivastava fractional derivative and integral of order λ, respectively (see [5,6]). Furthermore, in Gamma function, we have

$$J_{0,w}^{\kappa,\mu,\nu,\eta} w^i = \frac{\Gamma(i+\eta)\Gamma(i+\eta-\mu+\nu)}{\Gamma(i+\eta-\mu)\Gamma(i+\eta-\lambda+\nu)} w^{i+\eta-\mu-1} \quad (1.14)$$

$(0 \leq \kappa < 1, \mu, \eta \in R, \nu \in R^+ andi > (\max\{0,\mu\} - \eta))$,

and

$$I_{0,w}^{\kappa,\mu,\nu,\eta} w^i = \frac{\Gamma(i+\eta)\Gamma(i+\eta-\mu+\nu)}{\Gamma(i+\eta-\mu)\Gamma(i+\eta+\kappa+\nu)} w^{i+\eta-\mu-1} \quad (1.15)$$

$(\kappa > 0, \mu, \eta \in R, \nu \in R^+ andi > (\max\{0,\mu\} - \eta))$.

Now , using (1.1), (1.14), (1.15), in (1.6), we get

$$\mathcal{F}_{0,w}^{\lambda,\mu,\nu,\eta} f(w) = \frac{1}{w^p} - \sum_{i=p}^{\infty} \Gamma_{p,i}^{\kappa,\mu,\nu,\eta} a_i w^i \quad (1.16)$$

provided that $-\infty < \kappa < 1, \mu+\nu+\eta > \kappa, \mu > -\eta, \nu > -\eta, \eta > 0, p \in N, f \in \Sigma_p$and

$$\Gamma_{p,i}^{\kappa,\mu,\nu,\eta} = \frac{(\mu+\eta)_{p+i}(\nu+\eta)_{p+i}}{(\mu+\nu+\eta-\kappa)_{p+i}(\eta)_{p+i}} . \quad (1.17)$$

$$\left(\mathcal{F}_{0,w}^{\kappa,\mu,\nu,\eta} f(w)\right)^{`} = -pw^{-p-1} - i \sum_{i=p}^{\infty} \Gamma_{p,i}^{\kappa,\mu,\nu,\eta} a_i w^{i-1}$$

$$\left(\mathcal{F}_{0,w}^{\kappa,\mu,\nu,\eta} f(w)\right)^{''} = -p(p-1)w^{-p-2} - i(i-1)\sum_{i=p}^{\infty} \Gamma_{p,i}^{\kappa,\mu,\nu,\eta} \; a_i w^{i-2}$$

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$$\left(\mathcal{F}_{0,w}^{\kappa,\mu,\nu,\eta} f(w)\right)^{q} = \frac{(p+q-1)!}{(p-1)!}(-1)^q w^{-p-q} + \sum_{i=p}^{\infty} \Gamma_{p,w}^{\kappa,\mu,\nu,\eta} \frac{i!}{(i-q)!} a_i w^{i-q}$$

It may be worth noting that, be choosing $\mu = \kappa$ and $\eta = p = 1$, the operator$\mathcal{F}_{0,w}^{\kappa,\mu,\nu,\eta} f(w)$, educes to the well-known Ruscheweyh derivative $D_w^{\kappa} f(w)$, for meromorphic univalentfunctions[7]. Following our new subclass of meromorphic multivalent functions.

Definition (1.2): Let $\Sigma_p(\gamma,\alpha,\beta,\mu,\nu,\eta)$,be denoted the new class of functions $f \in \Sigma_p$ which satisfies the condition :

$$\left| \frac{\frac{w\left(\mathcal{F}_{0,w}^{\kappa,\mu,\nu,\eta} f(w)\right)^{q+1}}{\left(\mathcal{F}_{0,w}^{\kappa,\mu,\nu,\eta} f(w)\right)^{q}} + (p+1)}{\frac{\gamma w\left(\mathcal{F}_{0,w}^{\kappa,\mu,\nu,\eta} f(w)\right)^{q+1}}{\left(\mathcal{F}_{0,w}^{\kappa,\mu,\nu,\eta} f(w)\right)^{q}} - \alpha} \right| < \beta \quad , \tag{1.18}$$

where$0 \le \gamma \le 1, 0 < \alpha \le 1, 0 < \beta \le 1, \infty < \kappa < 1, \mu+\nu+\eta > \kappa, \mu > -\eta, \nu > -\eta and \eta > 0, p \in N and w \in U^*$,q$\in N_0 = N \cup \{0\}$,

Similar studieswere executed by several different authors, including Aouf et al. [8], Atshan et al.[9] , Panigrahiand Jena [10], and Juma and Dhayea [11],but usinganother class.

2.Coefficient Inequality

In the following theorem, we obtain the requisite and sufficient condition for the function f to be in the class $\Sigma_p(\gamma,\alpha,\beta,\mu,\nu,\eta)$.

Theorem (2.1): Let $f \in \Sigma_p$. Then f is in the class $\Sigma_p(\gamma,\alpha,\beta,\mu,\nu,\eta)$ if and only if

$\sum_{i=p}^{\infty} \Gamma_{p,i}^{\kappa,\mu,\nu,\eta} \frac{i!}{(i-q-1)!}\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{i-q}\right) a_i \le \frac{(p+q-1)!}{(p-1)!}(\beta(\gamma(p+q)+\alpha)-(1-q))$, (2.1)

where $\Gamma_{p,i}^{\kappa,\mu,\nu,\eta}$is given in (1.17), $0 \le \gamma \le 1$,$< \alpha \le 1, 0 \le \beta < 1$ and$w \in U^*$.

The outcome is sharp for the function

$$f(w) = \frac{1}{w^p} - \frac{\frac{(p+q-1)!}{(p-1)!}(\beta(\gamma(p+q)+\alpha)-(1-q))}{\Gamma_{p,i}^{\lambda,\mu,\nu,\eta}\frac{i!}{(i-q-1)!}\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{i-q}\right)} w^i, \qquad i \ge p \quad (2.2)$$

Proof: Presume that the inequality (2.1) holds and|z|=1. Then, we have

$$\left| w\left(\mathcal{F}_{0,w}^{\kappa,\mu,\nu,\eta} f(w)\right)^{q+1} + (p+1)\left(\mathcal{F}_{0,w}^{\kappa,\mu,\nu,\eta} f(w)\right)^{q} \right| - \beta\left| \gamma w\left(\mathcal{F}_{0,w}^{\kappa,\mu,\nu,\eta} f(w)\right)^{q+1} - \alpha\left(\mathcal{F}_{0,w}^{\kappa,\mu,\nu,\eta} f(w)\right)^{q} \right|$$

$=\Big|\frac{(p+q-1)!}{(p-1)!}(-1)^{q+1}w^{-p-q} + \sum_{i=p}^{\infty} \mathcal{F}_{0,w}^{\kappa,\mu,\nu,\eta} \frac{i!}{(i-q-1)!} a_i w^{i-q} + \frac{p(p+q-1)!}{(p-1)!}(-1)^q w^{-p-q}(1-q) + \sum_{i=p}^{\infty} \mathcal{F}_{0,w}^{\kappa,\mu,\nu,\eta} \frac{pi!}{(i-q)!} a_i w^{i-q} + \frac{(p+q-1)!}{(p-1)!}(-1)^q w^{-p-q} + \sum_{i=p}^{\infty} \mathcal{F}_{0,w}^{\kappa,\mu,\nu,\eta} \frac{i!}{(i-q)!} a_i w^{i-q}\Big| - \beta\left|\sum_{i=p}^{\infty} \mathcal{F}_{0,w}^{\kappa,\mu,\nu,\eta} \frac{i!}{(i-q-1)!} a_i w^{i-q}\left(1+\frac{(p+1)}{(p-1)}\right)(-1)^q w^{-p-q}(1-q)\right| - \beta\left|\sum_{i=p}^{\infty} \mathcal{F}_{0,w}^{\kappa,\mu,\nu,\eta} \frac{i!}{(i-q-1)!} a_i w^{i-q}\left(\gamma - \frac{\alpha}{i-q}\right) + \frac{(p+q-1)!}{(p-1)!}(-1)^q w^{-p-q}(-(\gamma(p+q)+\alpha)\right|$

$$= \left| \sum_{i=p}^{\infty} \Gamma_{p,i}^{\kappa,\mu,\nu,\eta} \frac{i!}{(i-q-1)!} a_i w^{n-q}\left(1+\frac{p+1}{i-q}\right) + \frac{(p+q-1)!}{(p-1)!}(-1)^q w^{-p-q}(1-q) \right|$$

$$-\beta\left| \sum_{i=p}^{\infty} \Gamma_{p,i}^{\kappa,\mu,\nu,\eta} \frac{i!}{(i-q-1)!} a_i w^{i-q}\left(\gamma - \frac{\alpha}{i-q}\right) + \frac{(p+q-1)!}{(p-1)!}(-1)^q w^{-p-q}(-(\gamma(p+q)+\alpha) \right|$$

$$\leq \sum_{i=p}^{\infty} \Gamma_{p,i}^{\kappa,\mu,\nu,\eta} \frac{i!}{(i-q-1)!}\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{i-q}\right)a_i - \frac{(p+q-1)!}{(p-1)!}\left(\beta(\gamma(p+q)+\alpha)-(1-q)\right)$$
$$\leq 0\ ,$$

Hence,the hypothesis is .

Principle, $f \in \Sigma_p(\gamma,\alpha,\beta,\mu,\nu,\eta)$.

Conversely, suppose that $f \in \Sigma_p(\gamma,\alpha,\beta,\mu,\nu,\eta)$,then from (1.18) we get

$$\left|\frac{\dfrac{w\left(\mathcal{F}_{0,w}^{\kappa,\mu,\nu,\eta} f(w)\right)^{q+1}}{\left(\mathcal{F}_{0,w}^{\kappa,\mu,\nu,\eta} f(w)\right)^{q}}+(p+1)}{\dfrac{\gamma w\left(\mathcal{F}_{0,w}^{\kappa,\mu,\nu,\eta} f(w)\right)^{q+1}}{\left(\mathcal{F}_{0,w}^{\kappa,\mu,\nu,\eta} f(w)\right)^{q}}-\alpha}\right|$$
$$=\left|\frac{\sum_{i=p}^{\infty}\Gamma_{p,w}^{\kappa,\mu,\nu,\eta}\frac{i!}{(i-q-1)!}\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{i-q}\right)a_i w^{i-1}}{\sum_{i=p}^{\infty}\Gamma_{p,i}^{\kappa,\mu,\nu,\eta}\frac{i!}{(i-q-1)!}a_i w^{i-q}\left(\gamma-\frac{\alpha}{i-q}\right)+\frac{(p+q-1)!}{(p-1)!}(-1)^q w^{-p-q}(-(\gamma(p+q)+\alpha)}\right|$$
$$<\beta,$$

since Re(w)≤|w| for all w (w∈ U^*), we get

$$Re\left\{\frac{\sum_{i=p}^{\infty}\Gamma_{p,w}^{\kappa,\mu,\nu,\eta}\frac{i!}{(i-q-1)!}\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{i-q}\right)a_i w^{i-1}}{\sum_{i=p}^{\infty}\Gamma_{p,i}^{\kappa,\mu,\nu,\eta}\frac{i!}{(i-q-1)!}a_i w^{i-q}\left(\gamma-\frac{\alpha}{i-q}\right)+\frac{(p+q-1)!}{(p-1)!}(-1)^q w^{-p-q}(-(\gamma(p+q)+\alpha)}\right\}$$
$$<\beta.$$

We choose the value ofw on the real axis so

$\frac{w\left(\mathcal{F}_{0,w}^{\kappa,\mu,\nu,\eta} f(w)\right)^{q+1}}{\left(\mathcal{F}_{0,w}^{\kappa,\mu,\nu,\eta} f(w)\right)^{q}}$ is real.

Letting w→ 1^-,tgrough real values, we gain the inequality (2.1).

Finally, sharpness follows if we take

$$f(w)=\frac{1}{w^p}-\frac{\frac{(p+q-1)!}{(p-1)!}\left(\beta(\gamma(p+q)+\alpha)-(1-q)\right)}{\Gamma_{p,i}^{\kappa,\mu,\nu,\eta}\frac{i!}{(i-q-1)!}\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{i-q}\right)}w^i,\ \ i\geq p.$$

Corollary:Let $f(w) \in \Sigma_p(\gamma,\alpha,\beta,\mu,\nu,\eta)$,then

$$a_i \leq \frac{\frac{(p+q-1)!}{(p-1)!}\left(\beta(\gamma(p+q)+\alpha)-(1-q)\right)}{\Gamma_{p,i}^{\kappa,\mu,\nu,\eta}\frac{i!}{(i-q-1)!}\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{i-q}\right)}\, i\geq p. \tag{2.3}$$

3.Growth and Distortion Bounds

Next, we obtain the growth and distortion bounds for the linear operator $\mathcal{F}_{0,w}^{\kappa,\mu,\nu,\eta}$.

<u>Theorem (3.1)</u>: If$f \in \Sigma_p(\gamma,\alpha,\beta,\mu,\nu,\eta)$,then

$$\frac{1}{r^p}-\frac{\frac{(p+q-1)!}{(p-1)!}\left(\beta(\gamma(p+q)+\alpha)-(1-q)\right)}{\frac{p!}{(p-q-1)!}\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{p-q}\right)}r^p \leq \left|\mathcal{F}_{0,w}^{\kappa,\mu,\nu,\eta} f(w)\right| \leq \frac{1}{r^p}+\frac{\frac{(p+q-1)!}{(p-1)!}\left(\beta(\gamma(p+q)+\alpha)-(1-q)\right)}{\frac{p!}{(p-q-1)!}\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{p-q}\right)}r^p \tag{3.1}$$
$$(0<|\mathrm{w}|=\mathrm{r}<1).$$

The outcome is sharp for the function

$$f(w)=\frac{1}{w^p}-\frac{\frac{(p+q-1)!}{(p-1)!}\left(\beta(\gamma(p+q)+\alpha)-(1-q)\right)}{\Gamma_{p,p}^{\kappa,\mu,\nu,\eta}\frac{p!}{(p-q-1)!}\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{p-q}\right)}w^p, \tag{3.2}$$

Proof : Let $f \in \Sigma_p(\gamma,\alpha,\beta,\mu,\nu,\eta)$.Then by theorem (2.1), we get

$$\sum_{i=p}^{\infty}\Gamma_{p,p}^{\kappa,\mu,\nu,\eta}a_i\frac{p!}{(p-q-1)!}\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{p-q}\right) \leq \sum_{i=p}^{\infty}\Gamma_{p,i}^{\kappa,\mu,\nu,\eta}a_i\frac{i!}{(i-q-1)!}\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{i-q}\right) \leq \frac{(p+q-1)!}{(p-1)!}(\beta(\gamma(p+q)+\alpha)-(1-q))$$

or

$$\sum_{i=p}^{\infty} a_i \leq \frac{\frac{(p+q-1)!}{(p-1)!}(\beta(\gamma(p+q)+\alpha)-(1-q))}{\Gamma_{p,p}^{\kappa,\mu,\nu,\eta}\frac{p!}{(p-q-1)!}\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{p-q}\right)}. \quad (3.3)$$

Hence,

$$\left|\mathcal{F}_{0,w}^{\kappa,\mu,\nu,\eta}f(z)\right| \leq \frac{1}{|w|^p}+\sum_{i=p}^{\infty}\Gamma_{p,i}^{\kappa,\mu,\nu,\eta}a_i|w|^p \leq \frac{1}{|w|^p}+\Gamma_{p,p}^{\kappa,\mu,\nu,\eta}a_i|w|^p\sum_{i=p}^{\infty}a_i = \frac{1}{r^p}+$$

$$\Gamma_{p,p}^{\kappa,\mu,\nu,\eta}r^p\sum_{i=p}^{\infty}a_i \leq \frac{1}{r^p}+\frac{\frac{(p+q-1)!}{(p-1)!}(\beta(\gamma(p+q)+\alpha)-(1-q))}{\Gamma_{p,p}^{\kappa,\mu,\nu,\eta}\frac{p!}{(p-q-1)!}\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{p-q}\right)}r^p \quad (3.4)$$

Similarly,

$$\left|\mathcal{F}_{0,w}^{\kappa,\mu,\nu,\eta}f(w)\right| \geq \frac{1}{|w|^p}-\sum_{i=p}^{\infty}\Gamma_{p,i}^{\kappa,\mu,\nu,\eta}a_i|w|^p \geq \frac{1}{|w|^p}-\Gamma_{p,p}^{\kappa,\mu,\nu,\eta}a_i|w|^p\sum_{i=p}^{\infty}a_i$$

$$=\frac{1}{r^p}-\Gamma_{p,p}^{\kappa,\mu,\nu,\eta}r^p\sum_{i=p}^{\infty}a_i \geq \frac{1}{r^p}-\frac{\frac{(p+q-1)!}{(p-1)!}(\beta(\gamma(p+q)+\alpha)-(1-q))}{\Gamma_{p,p}^{\kappa,\mu,\nu,\eta}\frac{p!}{(p-q-1)!}\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{p-q}\right)}r^p \quad (3.5)$$

From (3.2) and (3.5), we get to (3.1) and the proof is entier.

Theorem (3 .2) : If $f \in \Sigma_p(\gamma,\alpha,\beta,\mu,\nu,\eta)$.Then

$$\frac{p}{r^{p+1}}-\frac{\frac{(p+q-1)!}{(p-1)!}p(\beta(\gamma(p+q)+\alpha)-(1-q))}{\frac{p!}{(p-q-1)!}\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{p-q}\right)}r^{p-1} \leq \left|\left(\mathcal{F}_{p,w}^{\kappa,\mu,\nu,\eta}f(w)\right)'\right|$$

$$\leq \frac{p}{r^{p+1}}+\frac{\frac{(p+q-1)!}{(p-1)!}p(\beta(\gamma(p+q)+\alpha)-(1-q))}{\frac{p!}{(p-q-1)!}\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{p-q}\right)}r^{p-1} \quad (3.6)$$

$$(0<|w|=r<1).$$

The outcome is sharp for the function *f*presented in (3.2).

Proof: is comparable to that of theorem (3.1).

4. Convolution Properties

Theorem (4.1): Let the function $f_j(j=1.2)$ defind by

$$f_j(w)=\frac{1}{w^p}-\sum_{i=p}^{\infty} a_{i,j}w^i, \qquad (a_{i,j}\geq 0, j=1,2), \quad (4.1)$$

Which is in the class $\Sigma_p(\gamma,\alpha,\beta,\mu,\nu,\eta)$.Then $f_1 * f_2 \in \Sigma_p(\gamma,\alpha,\delta,\mu,\nu,\eta)$.where

$$\delta \leq \frac{\frac{i!}{(i-q-1)!}\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{p-q}\right)^2\Gamma_{p,i}^{\kappa,\mu,\nu,\eta}}{\frac{(p+q-1)!}{(p-1)!}(\beta^2(i+p)(\gamma(p+q)+\alpha)-(1-q))} - \frac{\frac{(p+q-1)!}{(p-1)!}(\beta^2(i+p)(\gamma(p+q)+\alpha)-(1-q))\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{p-q}\right)}{\frac{(p+q-1)!}{(p-1)!}(\beta^2(i+p)(\gamma(p+q)+\alpha)-(1-q))}$$

Proof : we must find the largestδ such that

$$\sum_{i=p}^{\infty}\frac{\Gamma_{p,i}^{\kappa,\mu,\nu,\eta}\frac{i!}{(i-q-1)!}\left(1+\delta\gamma+\frac{p+1+\delta\gamma}{i-q}\right)}{\frac{(p+q-1)!}{(p-1)!}(\delta(\gamma(p+q)+\alpha)-(1-q))}a_{i,1}a_{i,2} \leq 1$$

Since$f_j \in \Sigma_p(\gamma,\alpha,\beta,\mu,\nu,\eta)$. $(j=1.2)$ then

$$\sum_{i=p}^{\infty}\frac{\Gamma_{p,i}^{\kappa,\mu,\nu,\eta}\frac{i!}{(i-q-1)!}\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{i-q}\right)}{\frac{(p+q-1)!}{(p-1)!}(\beta(\gamma(p+q)+\alpha)-(1-q))}a_{i,j}\leq 1,(j=1,2). \quad (4.2)$$

By Cauchy –Schwarz inequality,we have

$$\sum_{i=p}^{\infty}\frac{\Gamma_{p,i}^{\kappa,\mu,\nu,\eta}\frac{i!}{(i-q-1)!}\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{i-q}\right)}{\frac{(p+q-1)!}{(p-1)!}(\beta(\gamma(p+q)+\alpha)-(1-q))}\sqrt{a_{i,1}a_{i,2}}\leq 1,(j=1,2). \quad (4.3)$$

So,we only show that

$$\frac{\Gamma_{p,i}^{\kappa,\mu,\nu,\eta}\frac{i!}{(i-q-1)!}\left(1+\delta\gamma+\frac{p+1+\delta\gamma}{i-q}\right)}{\frac{(p+q-1)!}{(p-1)!}(\delta(\gamma(p+q)+\alpha)-(1-q))}a_{i,1}a_{i,2}\leq\frac{\Gamma_{p,i}^{\kappa,\mu,\nu,\eta}\frac{i!}{(i-q-1)!}\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{i-q}\right)}{\frac{(p+q-1)!}{(p-1)!}(\beta(\gamma(p+q)+\alpha)-(1-q))}\sqrt{a_{i,1}a_{i,2}}\,.$$

This equivalent to

$$\sqrt{a_{i,1}a_{i,2}}\leq\frac{\frac{\delta!}{(\delta-q-1)!}\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{i-q}\right)}{\frac{\beta!}{(\beta-q-1)!}\left(1+\delta\gamma+\frac{p+1+\delta\gamma}{i-q}\right)}\,.$$

From (4.3), we have

$$\sqrt{a_{i,1}a_{i,2}}\leq\frac{\frac{(p+q-1)!}{(p-1)!}(\beta(\gamma(p+q)+\alpha)-(1-q))}{\Gamma_{p,i}^{\kappa,\mu,\nu,\eta}\frac{i!}{(i-q-1)!}\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{i-q}\right)},$$

which is sufficient to prove that

$$\frac{\frac{(p+q-1)!}{(p-1)!}(\beta(\gamma(p+q)+\alpha)-(1-q))}{\Gamma_{p,i}^{\kappa,\mu,\nu,\eta}\frac{i!}{(i-q-1)!}\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{i-q}\right)}\leq\frac{\frac{\delta!}{(\delta-q-1)!}\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{i-q}\right)}{\frac{\beta!}{(\beta-q-1)!}\left(1+\delta\gamma+\frac{p+1+\delta\gamma}{i-q}\right)}$$

That implies to

$$\delta\leq\frac{\frac{i!}{(i-q-1)!}\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{p-q}\right)^2\Gamma_{p,i}^{\kappa,\mu,\nu,\eta}}{\frac{(p+q-1)!}{(p-1)!}(\beta^{2}(i+p)(\gamma(p+q)+\alpha)-(1-q))}-\frac{\frac{(p+q-1)!}{(p-1)!}(\beta^{2}(i+p)(\gamma(p+q)+\alpha)-(1-q))\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{p-q}\right)}{\frac{(p+q-1)!}{(p-1)!}(\beta^{2}(i+p)(\gamma(p+q)+\alpha)-(1-q))}$$

Theorem (4.2):Let the function $f_j(j=1.2)$ that is defined in (4.1) be in class$\Sigma_p(\gamma,\alpha,\beta,\mu,\nu,\eta)$. Then the function k defined by

$$k(w)=\frac{1}{w^p}-\left({a^2}_{i,1}+{a^2}_{i,1}\right)w^2. \quad (4.4)$$

belongsto the class$\Sigma_p(\gamma,\alpha,\epsilon,\mu,\nu,\eta)$.where

$$\epsilon\leq\frac{\frac{i!}{(i-q-1)!}\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{p-q}\right)^2\Gamma_{p,i}^{\kappa,\mu,\nu,\eta}}{\frac{(p+q-1)!}{(p-1)!}2\beta(i+p)(\beta(\gamma(p+q)+\alpha)-(1-q))}-\frac{\frac{i!}{(i-q-1)!}2\beta^{2}(i+p)\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{p-q}\right)(\beta(i+p)(\gamma(p+q)+\alpha)-(1-q))}{\frac{(p+q-1)!}{(p-1)!}(2\beta^{2}(i+p)(\gamma(p+q)+\alpha)-(1-q))}$$

Proof : We must find the largest ϵ such that

$$\sum_{i=p}^{\infty}\frac{\frac{i!}{(i-q-1)!}\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{p-q}\right)\Gamma_{p,i}^{\kappa,\mu,\nu,\eta}}{\frac{(p+q-1)!}{(p-1)!}\epsilon\left((\gamma(p+q)+\alpha)-(1-q)\right)}\left({a^2}_{i,1}+{a^2}_{i,1}\right)\leq 1\quad,$$

since $f_j\in\Sigma_p(\gamma,\alpha,\beta,\mu,\nu,\eta)$, $(j=1.2)$, we get

$$\sum_{i=p}^{\infty}\left(\frac{\frac{i!}{(i-q-1)!}\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{p-q}\right)\Gamma_{p,i}^{\kappa,\mu,\nu,\eta}}{\frac{(p+q-1)!}{(p-1)!}(\beta(\gamma(p+q)+\alpha)-(1-q))}\right)^2{a^2}_{i,1}\leq\left(\sum_{i=p}^{\infty}\frac{\frac{i!}{(i-q-1)!}\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{p-q}\right)\Gamma_{p,i}^{\kappa,\mu,\nu,\eta}}{\frac{(p+q-1)!}{(p-1)!}(\beta(\gamma(p+q)+\alpha)-(1-q))}a_{i,1}\right)^2\leq$$

$1,$ (4.5)

and

$$\sum_{i=p}^{\infty}\left(\frac{\frac{i!}{(i-q-1)!}\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{p-q}\right)\ \Gamma_{p,i}^{\lambda,\mu,\nu,\eta}}{\frac{(p+q-1)!}{(p-1)!}(\beta(\gamma(p+q)+\alpha)-(1-q))}\right)^2 a^2{}_{i,2}\leq\left(\sum_{i=p}^{\infty}\ \frac{\frac{i!}{(i-q-1)!}\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{p-q}\right)\ \Gamma_{p,i}^{\kappa,\mu,\nu,\eta}}{\frac{(p+q-1)!}{(p-1)!}(\beta(\gamma(p+q)+\alpha)-(1-q))}a\ \ _{i,2}\right)^2\leq 1. \quad (4.6)$$

Annexation of the inequalities (4.5) and (4.6) gives

$$\sum_{i=p}^{\infty}\frac{1}{2}\left(\frac{\frac{i!}{(i-q-1)!}\left(1+\epsilon\gamma+\frac{p+1+\epsilon\gamma}{p-q}\right)\ \Gamma_{p,i}^{\kappa,\mu,\nu,\eta}}{\frac{(p+q-1)!}{(p-1)!}(\epsilon(\gamma(p+q)+\alpha)-(1-q))}\right)^2 (\ a^2{}_{i,1}+a^2{}_{i,2})\leq 1. \quad (4.7)$$

But $k\in\Sigma_p(\gamma,\alpha,\beta,\mu,\nu,\eta)$,if and if

$$\sum_{i=p}^{\infty}\frac{\frac{i!}{(i-q-1)!}\left(1+\epsilon\gamma+\frac{p+1+\epsilon\gamma}{p-q}\right)\ \Gamma_{p,i}^{\kappa,\mu,\nu,\eta}}{\frac{(p+q-1)!}{(p-1)!}(\epsilon(\gamma(p+q)+\alpha)-(1-q))}\ (\ a^2{}_{i,1}+a^2{}_{i,2})\leq 1. \quad (4.8)$$

The inequality (4.8) will be satisfied

$$\epsilon \leq \frac{\frac{i!}{(i-q-1)!}\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{p-q}\right)^2\Gamma_{p,i}^{\kappa,\mu,\nu,\eta}}{\frac{(p+q-1)!}{(p-1)!}2\beta(i+p)(\beta(\gamma(p+q)+\alpha)-(1-q))} - \frac{\frac{i!}{(i-q-1)!}\ 2\beta\ \ ^2(i+p)\ \left(1\ +\beta\gamma+\frac{p+1+\beta\gamma}{p-q}\right)(\beta\ (i+p)(\gamma(p+q)+\alpha)-(1-q))}{\frac{(p+q-1)!}{(p-1)!}(2\beta\ \ ^2(i+p)(\gamma(p+q)+\alpha)-(1-q))}$$

Theorem (4.3): If $f(w)=\frac{1}{w^p}-\sum_{i=p}^{\infty}a_i\,w^i$ and $g(w)=\frac{1}{w^p}-\sum_{i=p}^{\infty}b_i\,w^i$with $|b_i|\leq 1$ are in the class $\Sigma_p(\gamma,\alpha,\beta,\mu,\nu,\eta)$,then *f(w)*g(w)*$\in\Sigma_p(\gamma,\alpha,\beta,\mu,\nu,\eta)$.

Proof : From theorem (2.1)we get

$$\sum_{i=p}^{\infty}\frac{i!}{(i-q-1)!}\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{p-q}\right)^2\Gamma_{p,i}^{\kappa,\mu,\nu,\eta}\leq\frac{(p+q-1)!}{(p-1)!}(\beta(\gamma(p+q)+\alpha)-(1-q)).$$

Since

$$\sum_{i=p}^{\infty}\frac{\frac{i!}{(i-q-1)!}\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{p-q}\right)\ \ \Gamma_{p,i}^{\kappa,\mu,\nu,\eta}}{\frac{(p+q-1)!}{(p-1)!}\epsilon\big((\gamma(p+q)+\alpha)-(1-q)\big)}\ |a\ \ _ib_i|$$

$$=\sum_{i=p}^{\infty}\frac{\frac{i!}{(i-q-1)!}\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{p-q}\right)\ \ \Gamma_{p,i}^{\kappa,\mu,\nu,\eta}}{\frac{(p+q-1)!}{(p-1)!}\epsilon\big((\gamma(p+q)+\alpha)-(1-q)\big)}a\ \ _i|b_i|$$

$$\leq\sum_{i=p}^{\infty}\frac{\frac{i!}{(i-q-1)!}\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{p-q}\right)\ \ \Gamma_{p,i}^{\kappa,\mu,\nu,\eta}}{\frac{(p+q-1)!}{(p-1)!}\epsilon\big((\gamma(p+q)+\alpha)-(1-q)\big)}a\ \ _i\leq 1\ \ .$$

Thus *f(w)*g(w)*$\in\Sigma_p(\gamma,\alpha,\beta,\mu,\nu,\eta)$.

Hence, the proof is complete.

Corollary (4.1): If $f(w)=\frac{1}{w^p}-\sum_{i=p}^{\infty}a_i\,w^i$ and $g(w)=\frac{1}{w^p}-\sum_{i=p}^{\infty}b_i\,w^i$with $0\leq b_i\leq 1$ are in the class $\Sigma_p(\gamma,\alpha,\beta,\mu,\nu,\eta)$,then *f(w)*g(w)*$\in\Sigma_p(\gamma,\alpha,\beta,\mu,\nu,\eta)$.

5.Integral Representation

Theorem(5.1): Let $f\in\Sigma_p(\gamma,\alpha,\beta,\mu,\nu,\eta)$,then

$\mathcal{F}_{0,w}^{\lambda,\mu,\nu,\eta}f(w)=\int_0^w exp\left[\int_0^w\frac{\alpha\beta\psi(t)+(p+1)}{t(\gamma\beta\psi(t)-1)}dt\right]dt,$

where $|\psi(t)|<1, w\in U^*$.

Proof: By letting

$$\frac{w\left(\mathcal{F}_{0,w}^{\lambda,\mu,\nu,\eta}f(w)\right)^{``}}{\left(\mathcal{F}_{0,w}^{\lambda,\mu,\nu,\eta}f(w)\right)^{`}}=Q(w),(1.18),\text{we get}$$

$$\left|\frac{Q(z)+(p+1)}{\gamma Q(w)-\alpha}\right|<\beta,$$

or equivalently

$$\frac{Q(w)+(p+1)}{\gamma Q(w)-\alpha}=\beta\psi(t),\quad (|\psi(t)|<1,w\in U^*.$$

So

$$\frac{\left(\mathcal{F}_{0,w}^{\kappa,\mu,\nu,\eta}f(w)\right)^{``}}{\left(\mathcal{F}_{0,w}^{\lambda,\mu,\nu,\eta}f(w)\right)^{`}}=\frac{\alpha\beta\psi(t)+(p+1)}{w(\gamma\beta\psi(t)-1)},$$

after integration, we obtain

$$\log\left(\mathcal{F}_{0,w}^{\kappa,\mu,\nu,\eta}f(w)\right)^{`}=\int_0^w\frac{\alpha\beta\psi(t)+(p+1)}{t(\gamma\beta\psi(t)-1)}dt$$

Therefore

$$\left(\mathcal{F}_{0,w}^{\kappa,\mu,\nu,\eta}f(w)\right)^{`}=exp\left[\int_0^w\frac{\alpha\beta\psi(t)+(p+1)}{t(\gamma\beta\psi(t)-1)}dt\right].$$

Again by integration , we have

$$\mathcal{F}_{0,w}^{\lambda,\mu,\nu,\eta}f(z)=\int_0^w exp\left[\int_0^w\frac{\alpha\beta\psi(t)+(p+1)}{t(\gamma\beta\psi(t)-1)}dt\right]dt$$

and this is the required result.

6.Radii of Starlikeness and Convexity

In the following theorem, we introduce the radii of starlikeness and convexity.

Theorem (6.1): If $f\in\Sigma_p(\gamma,\alpha,\beta,\mu,\nu,\eta)$,then *f* is a multivalent meromorphicstarlike of order $\phi(0\leq\phi<p)$in the disk |w|<r_1, where

$$r_1=inf_i\left\{\frac{\frac{i!}{(i-q-1)!}(p-\phi)\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{p-q}\right)\ \Gamma_{p,i}^{\kappa,\mu,\nu,\eta}}{\frac{(p+q-1)!}{(p-1)!}(i-\phi-2p)\left(\beta(\gamma(p+q)+\alpha)-(1-q)\right)}\right\}^{\frac{1}{i+p}}\quad ,i\geq p$$

The consequence is sharp for the function *f* that is presented in (2.2).

Proof: It is enough to show that

$$\left|\frac{w\acute{f}(w)}{f(w)}+p\right|\leq p-\phi for|w|<r_1. \quad (6.1)$$

But

$$\left|\frac{w\acute{f}(w)+pf(w)}{f(w)}\right|\leq\frac{\sum_{i=p}^{\infty}(i+p)a_i|w|^{i+p}}{1-\sum_{i=p}^{\infty}a_i|w|^{i+p}}.$$

Thus, (6.1) will be satisfied if

$$\frac{\sum_{i=p}^{\infty}(i+p)a_i|w|^{i+p}}{1-\sum_{i=p}^{\infty}a_i\,|w|^{i+p}}\leq p-\phi,$$

or if

$$\sum_{i=p}^{\infty}\frac{(i-\phi+2p)}{p-\phi}a_i|w|^{i+p}\leq 1$$

Since $f\in\Sigma_p(\gamma,\alpha,\beta,\mu,\nu,\eta)$, we have

$\sum_{i=p}^{\infty} \frac{\frac{i!}{(i-q-1)!}\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{p-q}\right) \Gamma_{p,i}^{\kappa,\mu,\nu,\eta}}{\frac{(p+q-1)!}{(p-1)!}(\beta(\gamma(p+q)+\alpha)-(1-q))} a_i \leq 1$.

Hence,(6.2) will be true if

$\frac{(i-\phi+2p)}{p-\phi}|w|^{i+p} \leq \frac{\frac{i!}{(i-q-1)!}\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{p-q}\right) \Gamma_{p,i}^{\kappa,\mu,\nu,\eta}}{\frac{(p+q-1)!}{(p-1)!}(\beta(\gamma(p+q)+\alpha)-(1-q))}$,

or equivalently

$$|w| \leq \left\{\frac{\frac{i!}{(i-q-1)!}(p-\phi)\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{p-q}\right) \Gamma_{p,i}^{\kappa,\mu,\nu,\eta}}{\frac{(p+q-1)!}{(p-1)!}(i-\phi-2p)\left(\beta(\gamma(p+q)+\alpha)-(1-q)\right)}\right\}^{\frac{1}{i+p}} \quad ,i \geq p$$

which follows the result.

Theorem (6.2): If $f \in \Sigma_p(\gamma,\alpha,\beta,\mu,\nu,\eta)$,then f is a multivalent meromorphicstarlike of order $\phi(0 \leq \phi < p)$in the disk |w|<r_2, where

$$r_1 = \inf_i \left\{\frac{\frac{i!}{(i-q-1)!}p(p-\phi)\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{p-q}\right) \Gamma_{p,i}^{\kappa,\mu,\nu,\eta}}{\frac{(p+q-1)!}{(p-1)!}i(i-\varphi-2p)\left(\beta(\gamma(p+q)+\alpha)-(1-q)\right)}\right\}^{\frac{1}{i+p}} \quad ,i \geq p$$

The consequence is sharp for the function f that is presented in (2.2).

Proof: It is sufficient to show that

$$\left|\frac{w\acute{\acute{f}}(w)}{\acute{f}(w)}+1+p\right| \leq p-\phi for|w| < r_2 \tag{6.3}$$

But

$$\left|\frac{w\acute{\acute{f}}(w)}{\acute{f}(w)}+1+p\right| = \left|\frac{w\acute{\acute{f}}(w)+(p+1)\acute{f}(w)}{\acute{f}(w)}\right| \frac{\sum_{i=p}^{\infty} i(i+p)a_i|w|^{i+p}}{p-\sum_{i=p}^{\infty} ia_i|w|^{i+p}} \leq p-\phi .$$

Thus, (6.3) will be satisfied if

$$\frac{\sum_{i=p}^{\infty} i(i+p)a_i|w|^{i+p}}{p-\sum_{i=p}^{\infty} ia_i|w|^{i+p}} \leq p-\phi .$$

Or if

$$\sum_{i=p}^{\infty} \frac{i(i-\phi+2p)}{p(p-\phi)} a_i|w|^{i+p} \leq 1, \tag{6.4}$$

since $f \in \Sigma_p(\gamma,\alpha,\beta,\mu,\nu,\eta)$, we have

$\sum_{i=p}^{\infty} \frac{\frac{i!}{(i-q-1)!}\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{p-q}\right) \Gamma_{p,i}^{\kappa,\mu,\nu,\eta}}{\frac{(p+q-1)!}{(p-1)!}(\beta(\gamma(p+q)+\alpha)-(1-q))} a_i \leq 1$.

Hence,(6.4) will be true if

$$\frac{i(i-\phi+2p)}{p(p-\phi)}|w|^{i+p} \leq \frac{\frac{i!}{(i-q-1)!}\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{p-q}\right) \Gamma_{p,i}^{\kappa,\mu,\nu,\eta}}{\frac{(p+q-1)!}{(p-1)!}(\beta(\gamma(p+q)+\alpha)-(1-q))}$$

or equivalently

$$|w| \leq \left\{\frac{\frac{i!}{(i-q-1)!}p(p-\phi)\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{p-q}\right) \Gamma_{p,i}^{\kappa,\mu,\nu,\eta}}{\frac{(p+q-1)!}{(p-1)!}i(i-\phi-2p)\left(\beta(\gamma(p+q)+\alpha)-(1-q)\right)}\right\}^{\frac{1}{i+p}} \quad ,i \geq p$$

which follows the result.

7.Extreme points

We gain here extreme points of the class$\Sigma_p(\gamma,\alpha,\beta,\mu,\nu,\eta)$.

Theorem (7.1): Let $f_{p-1}(w) = w^{-p}$ and

$$f_i(w) = w^{-p} - \frac{\frac{(p+q-1)!}{(p-1)!}\left(\beta(\gamma(p+q)+\alpha)-(1-q)\right)}{\frac{i!}{(i-q-1)!}\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{p-q}\right)\ \Gamma_{p,i}^{\lambda,\mu,\nu,\eta}} w^i, \tag{7.1}$$

where each parameter isbound as in theorem (2.1). then the function f is in the class $\Sigma_p(\gamma,\alpha,\beta,\mu,\nu,\eta)$ if and only if

$$f(w) = \theta_{p-1}w^{-p} + \sum_{i=p}^{\infty}\theta_i f_i(w), \tag{7.2}$$

where $(\theta_{p-1} \geq 0, \theta_i \geq 0, i \geq p)$ and $\theta_{p-1} + \sum_{i=p}^{\infty} \theta_i = 1$

Proof : Suppose f is expressedas (7.2). Then

$$f(w) = \theta_{p-1}w^{-p} + \sum_{i=p}^{\infty} \theta_i \left[w^{-p} - \frac{\frac{(p+q-1)!}{(p-1)!}\left(\beta(\gamma(p+q)+\alpha)-(1-q)\right)}{\frac{i!}{(i-q-1)!}\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{p-q}\right)\ \Gamma_{p,i}^{\kappa,\mu,\nu,\eta}} w^i \right]$$

$$= w^{-p} - \sum_{i=p}^{\infty} \frac{\frac{(p+q-1)!}{(p-1)!}\left(\beta(\gamma(p+q)+\alpha)-(1-q)\right)}{\frac{i!}{(i-q-1)!}\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{p-q}\right)\ \Gamma_{p,i}^{\kappa,\mu,\nu,\eta}} \theta_i w^i .$$

Hence,

$$\sum_{i=p}^{\infty} \frac{\frac{i!}{(i-q-1)!}\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{p-q}\right)\ \Gamma_{p,i}^{\kappa,\mu,\nu,\eta}}{\frac{(p+q-1)!}{(p-1)!}\left(\beta(\gamma(p+q)+\alpha)-(1-q)\right)} \times \frac{\frac{(p+q-1)!}{(p-1)!}\left(\beta(\gamma(p+q)+\alpha)-(1-q)\right)}{\frac{i!}{(i-q-1)!}\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{p-q}\right)\ \Gamma_{p,i}^{\kappa,\mu,\nu,\eta}} \theta_i = \sum_{i=p}^{\infty} \theta_i = 1 - \theta_{p-1} \leq 1.$$

Then$f \in \Sigma_p(\gamma,\alpha,\beta,\mu,\nu,\eta)$.

Conversely, suppose that $f \in \Sigma_p(\gamma,\alpha,\beta,\mu,\nu,\eta)$. We may set

$$\theta_i = \frac{\frac{i!}{(i-q-1)!}\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{p-q}\right)\ \Gamma_{p,i}^{\kappa,\mu,\nu,\eta}}{\frac{(p+q-1)!}{(p-1)!}\left(\beta(\gamma(p+q)+\alpha)-(1-q)\right)} a_i ,$$

where a_i is presented in (2.3). Then

$$f(w) = w^{-p} - \sum_{i=p}^{\infty} a_i w^i = w^{-p} - \sum_{i=p}^{\infty} \frac{\frac{(p+q-1)!}{(p-1)!}\left(\beta(\gamma(p+q)+\alpha)-(1-q)\right)}{\frac{i!}{(i-q-1)!}\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{p-q}\right)\ \Gamma_{p,i}^{\kappa,\mu,\nu,\eta}} \theta_i w^i$$

$$= w^{-p} - \sum_{i=p}^{\infty}[w^{-p} - f_i(w)]\theta_i = \left(1 - \sum_{i=p}^{\infty}\theta i\right) w^{-p} + \sum_{i=p}^{\infty}\theta \quad _i f_i(w)$$

$$= \theta_{p-1}w^{-p} + \sum_{i=p}^{\infty}\theta i f_i(w)$$

This completes the proof .

8. Weighted Mean And Arithmetic Mean

Definition (8.1): Let f and $g$belong to Σ_p. The weighted mean $E_q$of *f(w)* and $g(w)$ is offered by

$$E_q(w) = \frac{1}{2}[(1-q)f(w) + (1+q)g(w)], \quad (0 < q < 1).$$

The following theorem shows the weighed mean for this class.

Theorem (8.1): Let f and$g$be the class$\Sigma_p(\gamma,\alpha,\beta,\mu,\nu,\eta)$, then the weighted mean of f and $g$is also in the class $\Sigma_p(\gamma,\alpha,\beta,\mu,\nu,\eta)$.

Proof: By definition (8.1), we have

$$E_q(w) = \frac{1}{2}[(1-q)f(w) + (1+q)g(w)]$$

$$=\frac{1}{2}\left[(1-q)\left(w^{-p}-\sum_{i=p}^{\infty}a_i w^i\right)+(1-q)\left(w^{-p}-\sum_{i=p}^{\infty}b_i w^n\right)\right]$$
$$=w^{-p}-\sum_{i=p}^{\infty}\quad\frac{1}{2}[(1-q)a_i\ +(1+q)b_i]w^i,$$
since f and g be the class $\Sigma_p(\gamma,\alpha,\beta,\mu,\nu,\eta)$. So by theorem (2.1), we have
$\sum_{i=p}^{\infty}\Gamma_{p,i}^{\kappa,\mu,\nu,\eta}\frac{i!}{(i-q-1)!}\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{i-q}\right)a_i\leq\frac{(p+q-1)!}{(p-1)!}(\beta(\gamma(p+q)+\alpha)-(1-q))$,
and
$\sum_{i=p}^{\infty}\Gamma_{p,i}^{\lambda,\mu,\nu,\eta}\frac{i!}{(i-q-1)!}\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{i-q}\right)b_i\leq\frac{(p+q-1)!}{(p-1)!}(\beta(\gamma(p+q)+\alpha)-(1-q))$.
Hence ,
$$\sum_{i=p}^{\infty}\Gamma_{p,i}^{\lambda,\mu,\nu,\eta}\frac{i!}{(i-q-1)!}\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{i-q}\right)\frac{1}{2}\left((1-q)a_i\ +(1+q)b_i\right)$$
$$=\frac{1}{2}(1-q)\sum_{i=p}^{\infty}\Gamma_{p,i}^{\kappa,\mu,\nu,\eta}\frac{i!}{(i-q-1)!}\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{i-q}\right)a_i$$
$$+\frac{1}{2}(1+q)\sum_{i=p}^{\infty}\Gamma_{p,i}^{\kappa,\mu,\nu,\eta}\frac{i!}{(i-q-1)!}\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{i-q}\right)b_i$$
$$\leq\frac{1}{2}(1-q)\frac{(p+q-1)!}{(p-1)!}(\beta(\gamma(p+q)+\alpha)-(1-q))$$
$$+\frac{1}{2}(1+q)\frac{(p+q-1)!}{(p-1)!}(\beta(\gamma(p+q)+\alpha)-(1-q)).$$
This shows that $E_q\in\Sigma_p(\gamma,\alpha,\beta,\mu,\nu,\eta)$.
In the following theorem, we will expound that the class $\Sigma_p(\gamma,\alpha,\beta,\mu,\nu,\eta)$.is closed under arithmetic mean.
Theorem (8.2): Let $f_1(w),f_2(w),\ldots\ldots,f_s(w)$ that defined by
$$f_k(w)=w^{-p}-\sum_{i=p}^{\infty}a_{i,k}w^i,\quad\left(a_{i,k}\geq 0,k=1,2,\ldots,s,i\geq p\right),\tag{8.1}$$
are in the class $\Sigma_p(\gamma,\alpha,\beta,\mu,\nu,\eta)$,then the arithmetical mean of $f_k(z),(k=1,3,\ldots,s)$, that is defined by
$$h(w)=\frac{1}{s}\sum_{i=p}^{s}f_k(w),\tag{8.2}$$
is also in the class $\Sigma_p(\gamma,\alpha,\beta,\mu,\nu,\eta)$.
Proof: By (8.1) and (8.2), we can write
$$h(w)=\frac{1}{s}\sum_{k=1}^{s}\left(w^{-p}-\sum_{i=p}^{\infty}a_{i,k}w^i\right)=w^{-p}-\sum_{p}^{\infty}(\frac{1}{s}\sum_{k=1}^{s}a_{i,k})w^i\ .$$
Since $f_k\in\ \Sigma_p(\gamma,\alpha,\beta,\mu,\nu,\eta)$. For every $(k=1,3,\ldots,s)$ so by theorem (2.1), we have
$$\sum_{i=p}^{\infty}\Gamma_{p,i}^{\kappa,\mu,\nu,\eta}\frac{i!}{(i-q-1)!}\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{i-q}\right)(\frac{1}{s}\sum_{k=1}^{s}\ a_{i,k})$$

$$= \frac{1}{s}\sum_{k=1}^{s}\left(\sum_{i=p}^{\infty}\Gamma_{p,i}^{\kappa,\mu,\nu,\eta}\frac{i!}{(i-q-1)!}\left(1+\beta\gamma+\frac{p+1+\beta\gamma}{i-q}\right)a_{i,k}\right)$$

$$\leq \frac{1}{s}\sum_{k=1}^{s}\frac{(p+q-1)!}{(p-1)!}(\beta(\gamma(p+q)+\alpha)-(1-q))$$

$$= \frac{(p+q-1)!}{(p-1)!}(\beta(\gamma(p+q)+\alpha)-(1-q)).$$

This ends the proof.

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