On The Generalized Type and Generalized Lower Type of Entire Function in Several Complex Variables With Index Pair (*p*, *q*)

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Abstract

In the present paper, we will study the generalized (p,q)-type and generalized lower (p,q)-type of an entire function in several complex variables with respect to the proximate order with index pair (p,q) are defined and their coefficient characterizations are obtained.

Keywords: Entire function, generalized type, generalized lower type,index pair.

حول اعمام النوع واعمام النوع الادنى لدالة كلية ذات متغيرات معقدة متعددة مع دليل الزوج (*p*, *q*)

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الخلاصة:

في بحثنا هذا سوف ندرس اعمام النوع (*p*, *q*) واعمام النوع الادنى (*p*, *q*) لدالة كلية ذات متغيرات معقدة متعددة بالنسبة الى تقريب الرتبة لدليل الزوج (*p*, *q*) من خلال تعريفها على المعاملات المميزة.

1 Introduction

Kumar and Gupta [1] let $f(z_1,z_2,\ldots,z_n)$ be an entire function $z=(z_1,z_2,\ldots,z_n)\in C^n$. Let G be a region in R_+^n (positive hyper octant) and let $G_R\subset C^n$ denote the region obtained from G by a similarity transformation about the origin, with ratio of similitude R. Let $d_t(G)=\sup_{z\in G}|z|^t$, where $|z|^t=|z_1|^{t_1}|z_2|^{t_2}\ldots|z_n|^{t_n}$, and let ∂G denote the boundary of the region G. Let

$$f(z)=f(z_1,z_2,\ldots,z_n)=\sum_{t_1,t_2,\ldots,t_n=0}^{\infty}a_{t_1\ldots t_n}z_1^{t_1}\ldots z_n^{t_n}=\sum_{\|t\|=0}^{\infty}a_t z^t\ ,\ \|t\|=t_1+t_2+\cdots+t_n\text{, be the}$$

power series expansion of the function $f(z)$. Let $M_{f,G}(R)=\max_{z\in G_R}|f(z)|$.

To characterize the growth of *f*, order ρ_G and type T_G of f are defined as .

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$$\rho_G=\lim_{R\to\infty}\sup\frac{\log\log M_{f,G}(R)}{\log R}, \text{ and } T_G=\lim_{R\to\infty}\sup\frac{\log M_{f,G}(R)}{R^{\rho_G}}.$$

For $R>0$, the maximum term $\mu_{f,G}(R)$ of entire function $f(z)$ is defined as (see [2] and [3])

$$\mu_{f,G}(R)=\max_{\|t\|\geq 0}\{|a_t|d_t(G)\ R^{\|t\|}\}.$$

For entire function $f(z)=\sum_{\|t\|=0}^{\infty}a_t z^t$, A.A. Gol'dberg [4,Th.1] obtained the order and type in terms of the coefficients of its Taylor expansion as

$$\rho_G=\lim_{\|t\|\to\infty}\sup\frac{\|t\|\log\|t\|}{-\log|a_t|}$$

and

$$(e\rho_G T_G)^{1/\rho_G}=\lim_{\|t\|\to\infty}\sup\{\|t\|^{1/\rho_G}[|a_t|d_t(G)]^{1/\|t\|}\},(0<\rho_G<\infty)$$

where $d_t(G)=\max_{r\in G}(r^t);r^t=r_1^{t_1}r_2^{t_2}\ldots r_n^{t_n}$.

The concept of (p,q)-order, lower order (p,q)-order, (p,q)-type and lower (p,q)-type of an entire function $f(z_1,z_2,\ldots,z_n)$ having an index pair (p,q), was introduced by Juneja *et* al. ([5] , [6]). Thus $f(z)$ is said to be of (p,q)-order ρ_G and lower (p,q)-order λ_G if

$$\lim_{R\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log^{[p]}M_{f,G}(R)}{\log^{[q]}R}=\begin{matrix}\rho_G(p,q)\\ \lambda_G(p,q)\end{matrix} \qquad (1.1)$$

where *p* and *q* are integers such that $p\geq q\geq 1$. If $b\leq\rho_G(p,q)\leq\infty$, where $b=1$, if $p=q$ and $b=0$ if $p>q$, then the (p,q)-type T_G and lower (p,q)-type t_G is given by

$$\lim_{R\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log^{[p-1]}M_{f,G}(R)}{(\log^{[q-1]}R)^{\rho_G(p,q)}}=\begin{matrix}T_G(p,q)\\ t_G(p,q)\end{matrix}$$

and $\log^{[m]}x=\exp^{[-m]}x=\log(\log^{[m-1]}x)=\exp(\exp^{[-m-1]}x),m=0,\pm1,\pm2,\cdots$ provided that $0<\log^{[m-1]}x<\infty$ with $\log^{[0]}x=\exp^{[0]}x=x$.

The growth of a function $f(z)$ can be studied in terms of its order ρ_G and type T_G, but these concepts are inadequate to compare the growth of those functions which are of the same order and of infinite type. Hence, for a refinement of the above growth scale, one may utilize proximate order the concept of which is [7] as follows:

A function $\rho_G(R)$ defined on $(0,\infty)$ is said to be a proximate order of an entire function with index pair (p,q) if it satisfies the properties: $\lim_{R\to\infty}\rho_G(R)=\rho_G$ and $\lim_{R\to\infty}\Lambda_{[q]}(R)\ \rho'(R)=0$, where $\Lambda_{[q]}(R)=\log^{[q]}R\ldots\log R.R$.

Now, we define the generalized (p,q)-type T_G^* and generalized lower (p,q)-type t_G^* of $f(z)$ with respect to a given proximate order $\rho_G(R)$ as

$$\lim_{R\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log^{[p-1]} M_{f,G}(R)}{(\log^{[q-1]} R)^{\rho_G(R)}} = \begin{matrix} T_G^* \\ t_G^* \end{matrix}, (0 \le t_G^* \le T_G^* \le \infty). \qquad (1.2)$$

A proximate order $\rho_G(R)$ is called a proximate order of an entire function $f(z)$ with index (p,q) if T_G^* is non-zero and finite and the function $f(z)$ is said to be of perfectly regular (p,q) growth with respect to its proximate order $\rho_G(R)$ if $T_G^* = t_G^*$.

In the present paper we obtain coefficient characterizations of generalized (p,q)-type T_G^* and generalized lower (p,q)-type t_G^* of the entire function $f(z)$.

By [7] $(\log^{[q-1]} R)^{\rho_G(R)}$ is a monotonically increasing function of R for $0 < R_0 < R < \infty$, so we define a single valued real function $\chi(k)$ of k for $k > k_0$ such that

$$k = (\log^{[q-1]} R)^{\rho_G(R)-A} \Leftrightarrow \log^{[q-1]} R = \chi(k). \qquad (1.3)$$

Then we have the following :

Lemma 1.1. Let $\rho_G(R)$ be a proximate order with index pair (p,q) and let $\chi(k)$ be defined as in (1.3). Then

$$\lim_{k\to\infty} \frac{d \log \chi(k)}{d \log k} = \frac{1}{\rho_G - A} \qquad (1.4)$$

and for every η with $o < \eta < \infty$

$$\lim_{k\to\infty} \frac{\chi(\eta k)}{\chi(k)} = \eta^{1/(\rho_G - A)} \qquad (1.5)$$

where $A = 1$ when $(p,q) = (2,2)$

= 0 otherwise.

Proof.

$$\frac{d \log \chi(k)}{d \log k} = \frac{d(\log^{[q]} R)}{d\{(\rho_G(R) - A)\log^{[q]} R\}}$$

$$= 1/[\rho_G(R) - A + \Lambda_{[q]}(R)\,\rho_G'(R)].$$

passing to the limits $k \to \infty$ we obtain (1.4).

Again,

$$\frac{\chi(\eta k)}{\chi(k)} = \eta^{1/(\rho_G - A)},$$

taking limits we get (1.5).

Lemma 1.2. Let $f(z) = \sum_{\|t\|=1}^{\infty} a_t z^t$ be an entire function having proximate order $\rho_G(R)$ with index pair (p,q). Let T_G^* and t_G^* be the generalized (p,q)-type and generalized lower (p,q)-type of $f(z)$ with respect to a proximate order $\rho_G(R)$. Then

$$\lim_{R\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log^{[p-1]} \mu_{f,G}(R)}{(\log^{[q-1]} R)^{\rho_G(R)}} = \begin{matrix} T_G^* \\ t_G^* \end{matrix}. \qquad (1.6)$$

Proof: By the maximum term in [8] and by using the type and lower type [6], we have

For $R > 0$, the maximum term $\mu_{f,G}(R)$ of entire function $f(z)$ is defined as

$$\mu_{f,G}(R) = \mu_{f,G}(R, f) = \max_{\|t\|\geq 0}\{\|a_t\| R^{\|t\|}\}$$

and

$$\lim_{R\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \mu_{f,G}(R)}{R^{\rho_G(R)}} = \begin{matrix}T_G^*\\ t_G^*\end{matrix}$$

Then from [6], we get (1.6).

2 Main Result

Theorem 2.1. If $f(z) = \sum_{\|t\|=1}^{\infty} a_t z^t$ is an entire function with proximate order $\rho_G(R)$ and (p,q)-order ρ_G with index pair (p,q), then the generalized (p,q)-type T_G^* of $f(z)$ with respect to the proximate order $\rho_G(R)$ is given by

$$T_G^*/M = \lim_{\|t\|\to\infty} \sup\left[\frac{\chi(\log^{[p-2]}\{\|t\|\alpha_{\|t\|}\})}{\log^{[q-1]}\{-(1/\|t\|)\log(|a_t|d_t(G))\}}\right]^{\rho_G - A}, \qquad (2.1)$$

where

$$M = \begin{cases}(\rho_G - 1)^{\rho_G - 1}/\rho_G^{\rho_G} & if \quad (p,q) = (2,2)\\ 1/e\rho_G & if \quad (p,q) = (2,1)\\ 1 & if \quad for\ all\ other\ index\ pair\ (p,q).\end{cases}$$

and

$$\alpha_{\|t\|} = \begin{cases}(t_1^{t_1} t_2^{t_2} \cdots t_n^{t_n})^{1/\|t\|}/\|t\| & ;\ t_1, t_2, \dots, t_n \geq 1,\ for\ (p,q) = (2,1)\\ 1 & ;\ t_1, t_2, \dots, t_n \geq 1,\ for\ 2 \leq q \leq p < \infty\\ 0 & ;\ at\ least\ one\ t_1, t_2, \dots, t_n = 0.\end{cases}$$

Proof. From (1.6) for every $\varepsilon > 0$ and for all

$R > R_0 (0 < R_0 = R_0(\varepsilon) < R < \infty)$

$$\log M_{f,G}(R) < \exp^{[p-2]}\{(T_G^* + \varepsilon)(\log^{[q-1]} R)^{\rho_G(R)}\},$$

for all R such that $0 < R_0 < R < \infty$,

$$\log|a_t|d_t(G) \leq \exp^{[p-2]}\{(T_G^* + \varepsilon)(\log^{[q-1]} R)^{\rho_G(R)}\} - \|t\| \log R. \qquad (2.2)$$

Now choose *R* such that

$$(\log^{[q-1]} R)^{\rho_G(R) - A} = \frac{1}{T_g^* + \varepsilon} \log^{[p-2]}(\|t\|/\rho_G). \qquad (2.3)$$

For $(p,q) \neq (2,2)$, (2.3) is reduced to

$$(\log^{[q-1]} R)^{\rho_G(R)} = \frac{1}{T_G^* + \varepsilon} \log^{[p-2]}(\|t\|/\rho_G),$$

which gives that

$$k = \frac{1}{T_G^* + \varepsilon} \log^{[p-2]}(\|t\|/\rho_G) \text{ and } \log^{[q-1]} R = \chi\left(\frac{1}{T_G^* + \varepsilon} \log^{[p-2]}(\|t\|/\rho_G)\right).$$

Using the results (2.2) yields

$$\frac{\chi(\log^{[p-2]}\|t\|)}{\log^{[q-2]}\left\{-\frac{1}{\|t\|}\log|a_t|d_t(G)\right\}} < \frac{\chi(\log^{[p-2]}\|t\|)}{\chi\left(\frac{1}{T_G^*+\varepsilon}\log^{[p-2]}(\|t\|/\rho_G)\right)+o(1)}.$$

Passing to limits, we have (using (1.5))

$$\limsup_{\|t\|\to\infty}\left[\frac{\chi(\log^{[p-2]}\|t\|)}{\log^{[q-2]}\{-(1/\|t\|)\log|a_t|d_t(G)\}}\right]^{\rho_G} \le T_G^*(p\ge 3). \qquad (2.4)$$

For $(p,q)=(2,2)$, the equation (2.3) becomes

$$(\log R)^{\rho_G(R)-1} = \|t\|/\rho_G(T_G^*+\varepsilon)],$$

which implies that

$k=\|t\|/\rho_G(T_G^*+\varepsilon)$ and $\log R=\chi(\|t\|/\rho_G(T_G^*+\varepsilon))$.

Hence, (2.2) is written as

$$\frac{\chi(\|t\|)}{-(1/\|t\|)\log|a_t|d_t(G)} < \frac{\chi(\|t\|)}{\chi\left(\frac{\|t\|}{\rho_G(T_G^*+\varepsilon)}\right)\left(1-\frac{\{\|t\|/(T_G^*+\varepsilon)\}^{p(\|t\|)}}{\rho_G^{1+p(\|t\|)}\chi(\|t\|/\rho_G(T_G^*+\varepsilon))}\right)},$$

where

$p(\|t\|)=1/(\rho_G(R)-1)$ and $1+p(\|t\|)=\rho_G(R)/(\rho_G(R)-1)$.

Since

$$\lim_{\|t\|\to\infty}\frac{\chi(\|t\|)}{\chi(\|t\|/\rho_G(T_G^*+\varepsilon))} = (\rho_G T_G^*)^{1/(\rho_G-1)} \qquad \text{(since ε is very small)}$$

and

$$\lim_{\|t\|\to\infty}\frac{(\|t\|/(T_G^*-\varepsilon))^{p(\|t\|)}}{\rho_G^{1+p(\|t\|)}\chi(\|t\|/\rho_G(T_G^*+\varepsilon))} = \frac{1}{\rho_G}$$

so

$$\limsup_{\|t\|\to\infty}\left[\frac{\|t\|\chi(\|t\|)}{-\log|a_t|d_t(G)}\right]^{\rho_G-1} \le \frac{\rho_G^{\rho_G}}{(\rho_G-1)^{\rho_G-1}}T_G^*. \qquad (2.5)$$

Again, for $(p,q)=(2,1)$, (2.3) is reduced to

$$\|t\|/\rho_G(T_G^*+\varepsilon)=R^{\rho_G(R)}$$

which gives

$$k=R^{\rho_G(R)} \Leftrightarrow R=\chi(k).$$

Equation (2.2) is converted into

$$\frac{\chi(\|t\|)}{(|a_t|d_t(G))^{-1/\|t\|}} < \frac{\chi(\|t\|)}{e^{-1/\rho_G}\chi(\|t\|/\rho_G(T_G^*+\varepsilon))}.$$

Passing to limits we have

$$\lim_{\|t\|\to\infty}\sup\left(\frac{\chi(\|t\|)}{(|a_t|d_t(G))^{-1/\|t\|}}\right)^{\rho_G} \le T_G^* e\rho_G. \qquad (2.6)$$

Equations (2.4), (2.5) and (2.6) combine into

$$\lim_{\|t\|\to\infty}\sup\left[\frac{\chi(\log^{[p-2]}\|t\|\alpha_{\|t\|})}{\log^{[q-2]}\{-(1/\|t\|)\log|a_t|d_t(G)\}}\right]^{\rho_G-A} \le T_G^*/M. \qquad (2.7)$$

To prove the reverse inequality, let

$$\lim_{\|t\|\to\infty}\sup\left[\frac{\chi(\log^{[p-2]}\|t\|\alpha_{\|t\|})}{\log^{[q-2]}\{-(1/\|t\|)\log|a_t|d_t(G)\}}\right]^{\rho_G-A} = \beta/M.$$

For any $\varepsilon > 0$, we have for all $\|t\| > m_0 = m_0(\varepsilon)$

$$|a_t|d_t(G)R^{\|t\|} < \exp\left[-\|t\|\exp^{[q-2]}\left(\chi\left(\frac{M}{\alpha}\log^{[p-2]}\|t\|\alpha_{\|t\|}\right)\right)+\|t\|\log R\right],$$

where $\alpha = \beta + \varepsilon$

So,

$$\log\mu_{f,G}(R) < \max_{\|t\|\ge 0}\left[-\|t\|\exp^{[q-2]}\left(\chi\left(\frac{M}{\alpha}\log^{[p-2]}\|t\|\alpha_{\|t\|}\right)\right)+\|t\|\log R\right]. \qquad (2.8)$$

For $(p,q)\ne(2,1)$ and (2, 2), using (1.4) it can be easily seen that the maximum value on the right-hand side is attained for

$$\|t\| = \left[\exp^{[p-2]}\left(\alpha\left\{\log^{[q-2]}\left(\frac{\rho_G}{1+\rho_G}\log R\right)\right\}^{\rho_G(R)}\right)\right].$$

Thus, for R sufficiently large we get from (2.8)

$$\frac{\log^{[q-1]}\mu_{f,G}(R)}{(\log^{[q-1]}R)^{\rho_G(R)}} < \alpha\frac{[\log^{[q-2]}(\rho_G(1+\rho_G)^{-1}\log R)]^{\rho_G(R)}}{(\log^{[q-1]}R)^{\rho_G(R)}} + o(1).$$

Proceeding to limits

$$T_G^* \le \alpha. \qquad (2.9)$$

Consider when $(p,q)=(2,1)$. Let $\|t\| = \alpha(Re^{-1/\rho_G})/M$, equation (2.8) is then reduced to

$$\frac{\log \mu_{f,G}(R)}{R^{\rho_G(R)}} < \frac{\alpha}{\rho_G M} e - \rho_G(R)/\rho_G$$

and passing to limits we get

$$T_G^* \le \alpha . \qquad (2.10)$$

If $(p,q)=(2,2)$, in order to get the maximum value of the right-hand side of the inequality (2.8) $\|t\|$ is given by

$$\|t\| = \frac{\alpha}{M}\left(\frac{\rho_G - 1}{\rho_G}\right)^{\rho_G(R)-1} (\log R)^{\rho_G(R)-1},$$

which reduces (2.8) to

$$\frac{\log \mu_{f,G}(R)}{(\log R)^{\rho_G(R)}} < \frac{\alpha}{M} \frac{(\rho_G - 1)^{\rho_G(R)-1}}{\rho_G{}^{\rho_G(R)}} .$$

On taking limits we get

$$T_G^* \le \alpha. \qquad (2.11)$$

(2.9), (2.10) and (2.11) give

$T_G^* \le \alpha = (\beta + \varepsilon).$

Since this inequality holds for every $\varepsilon > 0$, so $T_G^* \le \beta$. This and (2.7) together prove the theorem.

Taking $\rho_G(R) = \rho_G$ and $\chi(k) = k^{1/(\rho_G - A)}$, we have the following corollary which gives a formula for the (p,q)-type T_G of the entire function $f(z)$.

Theorem 2.2. Let $f(z) = \sum_{\|t\|=0}^{\infty} a_t z^t$ be an entire function having the proximate order $\rho_G(R)$ and (p,q)-order ρ_G such that

$$\phi(\|t\|) = |a_t / a_{t+1}|,$$

forms a non-decreasing function of $\|t\|$ for $\|t\| > m_0$. Then the generalized lower (p,q)-type t_G^* of $f(z)$ is given by

where M, A and $\alpha_{\|t\|}$ are the same as given in Theorem 2.1.

Proof. Since by hypothesis, $\phi(\|t\|)$ is a non-decreasing function of $\|t\|$ for $\|t\| > m_0$. We have $\phi(\|t\|) > \phi(\|t\| - 1)$ for infinitely many values of $\|t\|$; otherwise $f(z)$ ceases to be an entire function. So $\phi(\|t\|) \to \infty$ as $\|t\| \to \infty$.

When $\phi(\|t\|) > \phi(\|t\| - 1)$, the term $a_t z^t$ becomes maximum and then

$\mu_{f,G}(R) = |a_t| R^{\|t\|}$, $v(R) = \|t\|$ for $\phi(\|t\| - 1) \le R < \phi(\|t\|)$.

First, let $0 < t_G^* < \infty$, in view of Lemma 1.2. , for any ε satisfying $0 < \varepsilon < t_G^*$ and for all $R > R_0 = R_0(\varepsilon)$ we get

$$\log \mu_{f,G}(R) > \exp^{[p-2]}[(t_G^* - \varepsilon)(\log^{[q-1]} R)^{\rho_G(R)}]. \qquad (2.12)$$

Let $a_{m_1} z^{m_1}$ and $a_{m_2} z^{m_2}$ $(m_1 > m_0, \phi(m_1 - 1) > R_0)$.

be two consecutive maximum terms of $f(z)$. Then since $\phi(\|t\|)$ is a non-decreasing function of $\|t\|$ for $\|t\| > m_0$, we have for $m_1 \le \|t\| \le m_2 - 1$,

$$\phi(m_0) = \phi(m_1 + 1) = \ldots = \phi(\|t\|) = \ldots = \phi(m_2 - 1) \qquad (2.13)$$

And

$$|a_t| R^{\|t\|} = |a_{m_2}| R^{m_2} \text{ for } R = \phi(\|t\|). \qquad (2.14)$$

Hence, (2.12),(2.13) and (2.14) give

$$\log|a_t| d_t(G) + \|t\| \log \phi(\|t\|) > \exp^{[p-2]}[(t_g^* - \varepsilon)(\log^{[q-1]} \phi(\|t\|))^{\rho_G(\phi(\|t\|))}]$$

or,

$$X \equiv \frac{\{\chi(\log^{[p-2]}\|t\|\alpha_{\|t\|})\}^{\rho_G - A}}{\exp[(\rho_G - A)\log^{[q-1]}\{-(1/\|t\|)\log|a_t|d_t\}]} .$$

$$> \frac{\{\chi(\log^{[p-2]}\|t\|\alpha_{\|t\|})\}^{\rho_G - A}}{\exp[(\rho_G - A)\log^{[q-1]}\{\log\phi(\|t\|) - (1/\|t\|)\exp^{[p-2]}\{(t_G^* - \varepsilon)(\log^{[q-1]}\phi(\|t\|))^{\rho_G(\phi(\|t\|))}\}\}]}$$

.(2.15)

We note that the minimum value of the function

$$S(R) = \frac{\{\chi(\log^{[p-2]}\|t\|\alpha_{\|t\|})\}^{\rho_G - A}}{\exp[(\rho_G - A)\log^{[q-1]}\{\log R - (1/\|t\|)\exp^{[p-2]}\{(t_G^* - \varepsilon)(\log^{[q-1]} R)^{\rho_G(R)}\}\}]}$$

is attained at a point $R = R_0$ satisfying

$$\frac{E_{[p-2]}\{(t_G^* - \varepsilon)(\log^{[q-1]} R)^{\rho_G(R)}\}}{\Lambda_{[q-1]}(R)} = \|t\|/R\rho_G . \qquad (2.16)$$

For $(p,q) = (2,1)$, (2.16) gives $R^{\rho_G(R)} = \|t\|/(t_G^* - \varepsilon)\rho_G \Leftrightarrow R = \chi(\|t\|/(t_G^* - \varepsilon)\rho_G)$.

Hence

$$X > \min_{0<R<\infty} S(R) = \min \frac{(\chi(\|t\|\alpha_{\|t\|}))^{\rho_G}}{\exp[\rho_G\{\log R - (t_G^* - \varepsilon) R^{\rho_G(R)}/\|t\|\}]}$$

$$= e[\chi(\|t\|)/\chi(\|t\|/(t_G^* - \varepsilon)\rho_G)]^{\rho_G}$$

$$\approx e\rho_G(t_G^* - \varepsilon). \qquad (2.17)$$

For $(p,q) = (2,2)$, (2.16) becomes

$$(\log R)^{\rho_G(R)-1} = \frac{\|t\|}{\rho_G(t_G^* - \varepsilon)} \Leftrightarrow \log R = \chi(\|t\|/(t_G^* - \varepsilon)\rho_G).$$

Hence,

$$\min_{0<R<\infty} S(R) = (\chi(\|t\|)/\chi(\|t\|/(t_G^* - \varepsilon)\rho_G))^{\rho_G - 1}\{\rho_G/(\rho_G - 1)\}^{\rho_G - 1}.$$

$$\approx \frac{\rho_G^{\rho_G}}{(\rho_G - 1)^{\rho_G - 1}}(t_G^* - \varepsilon) \qquad (2.18)$$

For $(p,q)\neq(2,2)$ and (2,1), (2.16) is reduced to

$$(\log^{[q-1]} R)^{\rho_G(R)} = \frac{1}{t_G^* - \varepsilon}\log^{[p-2]}(\|t\|/\rho_G) \Leftrightarrow \log^{[q-1]} R = \chi\left(\frac{1}{t_G^* - \varepsilon}\log^{[p-2]}(\|t\|/\rho_G)\right).$$

So

$$\begin{aligned}\min_{0<R<\infty} S(R) &= (\chi(\log^{[p-2]}\|t\|\alpha_{\|t\|}))^{\rho_G} \Big/ \exp\{\rho_G \log^{[q]}(R\,e^{-1/\rho_G})\} \\ &\cong \{\chi(\log^{[p-2]}\|t\|\alpha_{\|t\|}) \big/ (\log^{[q-1]} R)\}^{\rho_G} \\ &= \left\{\chi(\log^{[p-2]}\|t\|\alpha_{\|t\|}) \Big/ \chi\left(\frac{1}{t_G^* - \varepsilon}\log^{[p-2]}(\|t\|/\rho_G)\right)\right\} \\ &\approx t_G^* - \varepsilon. \end{aligned} \quad (2.19)$$

(2.15),(2.17),(2,18) and (2.19) combine into

$$\lim_{\|t\|\to\infty}\inf X \geq t_G^*/M. \quad (2.20)$$

The inequality (2.20) is obvious if $t_G^* = 0$. When $t_G^* = \infty$, above arguments with an arbitrarily large number in place of $(t_G^* - \varepsilon)$ leads to

$$\lim_{\|t\|\to\infty}\inf X = \infty.$$

We now prove that strict inequality cannot hold in (2.20). for if it holds, then there exists a number $\delta(\delta > t_G^*)$ such that

$$\frac{\delta}{M} = \lim_{\|t\|\to\infty}\inf\left[\frac{\chi(\log^{[p-2]}\|t\|\alpha_{\|t\|})}{\log^{[q-2]}\{-(1/\|t\|)\log|a_t|d_t(G)\}}\right]^{\rho_G - A}.$$

Let δ_1 be such that $\delta > \delta_1 > t_G^*$, then for all $\|t\| > m_0$

$$\log|a_t|d_t(G) > -\|t\|\exp^{[q-2]}\left[\frac{\chi(\log^{[p-2]}\|t\|\alpha_{\|t\|})}{(\delta_1/M)^{1/(\rho_G - A)}}\right].$$

Therefore, for sufficiently large R and $\|t\|$ we have

$$\log M_{f,G}(R) > -\|t\|\exp^{[q-2]}\left[\frac{\chi(\log^{[p-2]}\|t\|\alpha_{\|t\|})}{(\delta_1/M)^{1/(\rho_G - A)}}\right] + \|t\|\log R. \quad (2.21)$$

For $(p,q)=(2,1)$, choose $\|t\| = [\rho_G\delta_1 R^{\rho_G(R)}]$, then in view of Lemma 1.1. ,

$$\log M_{f,G}(R) > -\|t\|\log\left[\frac{\chi(\|t\|\alpha_{\|t\|})}{(e\rho_G\delta_1)^{1/\rho_G}}\right] + \|t\|\log\chi(\|t\|/\rho_G\delta_1)$$

or,

$$\frac{\log M_{f,G}(R)}{R^{\rho_G(R)}} > \delta_1.$$

Passing to limits

$$t_G^* \geq \delta_1. \tag{2.22}$$

In case $(p,q)=(2,2)$, choose

$$(\log R)^{\rho_G(R)-1} = \frac{M\|t\|}{\delta_1\{(\rho_G-1)/\rho_G\}^{\rho_G-1}} = k,$$

Then (2.21) is reduced to

$$\log M_{f,G}(R) > \|t\|[\log R - \chi(\|t\|)(M/\delta_1)^{1/(\rho_G-1)}]$$
$$\approx \frac{\|t\|}{\rho_G}\log R$$

or,

$$\frac{\log M_{f,G}(R)}{(\log R)^{\rho_G(R)}} > \delta_1$$

which gives on passing to limits

$$t_G^* \geq \delta_1. \tag{2.23}$$

Further, consider $(p,q) \neq (2,1)$ and (2, 2) if $\|t\|$ is given by

$$\log^{[p-2]}(\|t\|/\rho_G) = \delta_1(\log^{[q-1]} R/e^{\varepsilon})^{\rho_G(R/e^{\varepsilon})} \Leftrightarrow \log^{[q-1]} R/e^{\varepsilon} = \chi\left(\frac{\log^{[p-2]}(\|t\|/\rho_G)}{\delta_1}\right);$$

then

$$\log M_{f,G}(G) > \|t\|\left\{\log R - \exp^{[q-2]}\left[\frac{\chi(\log^{[p-2]}\|t\|)}{\delta_1^{1/\rho_G}}\right]\right\}$$
$$= \|t\|\left[\varepsilon + \exp^{[q-2]}\left\{\frac{\chi(\log^{[p-2]}(\|t\|/\rho_G))}{\delta_1^{1/\rho_G}}\right\} - \exp^{[q-2]}\left\{\frac{\chi(\log^{[p-2]}\|t\|)}{\delta_1^{1/\rho_G}}\right\}\right]$$

or,

$$\frac{\log^{[p-1]} M_{f,G}(R)}{(\log^{[q-1]} R)^{\rho_G(R)}} > \delta_1 + o(1).$$

Proceeding to limits we have

$$t_G^* \geq \delta_1. \tag{2.24}$$

So (2.22),(2.23) and (2.24) are formed in to

$$t_G^* \geq \delta_1.$$

which is a contradiction. Hence the proof of the theorem is complete.

Corollary 2.3. Let $f(z)=\sum_{\|t\|=0}^{\infty} a_t z^t$ be an entire function having the (p,q)-order ρ_G and lower (p,q)-type $t_G (0 \le t_G < \infty)$ such that $\phi(\|t\|)$ is non-decreasing function of $\|t\|$ for $\|t\| > m_0$, then

$$t_G / M = \liminf_{\|t\|\to\infty} \frac{\log^{[p-2]}\|t\|\alpha_{\|t\|}}{\{\log^{[q-2]}(-(1/\|t\|)\log|a_t|)\}^{\rho_G - A}} .$$

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