*DOI: 10.24996/ijs.2025.66.12.31*

ISSN: 0067-2904

# Generalization of the Carsti's Condition on Partial b- Metric Spaces

**Liqaa Jameel Khaleel*, Suad Naji Kadhim**
*Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq*

Received: 22/7/2024 Accepted: 11/12/2024 Published: 30/12/2025

**Abstract**:
Caristi's Theorem plays as an important rule to guaranties the existence of the fixed point for a single valued mapping and a Set-valued mapping that defined on complete partial b-metric spaces. In our paper some new generalizations of Caristi's condition have been introduced and we use them in them in our work to give the guaranties in which the single and set-valued mappings have a fixed point on partial b- metric spaces. As well as, some applications have been studied to illustrate the mechanism of using these generalizations of Caristi's condition.

**Keywords:** Partial b-metric space, Caristi's Condition, Set-valued mapping, Houssdorff partial b- metric space.

## تعميم شرط كارستي في الفضاءات المترية الجزئية من النوع b

**لقاء جميل خليل * , سعاد ناجي كاظم**
قسم الرياضيات , كلية العلوم , جامعة بغداد , بغداد , العراق

**الخلاصة**
نظرية كارستي تلعب دور مهم في اثبات وجود النقطة الثابتة للدوال المفردة والمتعددة المعرفة في الفضاءات المترية الجزئية .
أعطينا في بحثنا هذا بعض التعميمات لشرط كارستي و استخدمنا هذه التعميمات في اثبات امتلاك التطبيقات المفردة . b والمتعددة للنقاط الثابتة المعرفة على الفضاءات المترية الجزئية من النوع –
كما درسنا بعض التطبيقات لتوضيح الية استخدام الصيغ الجديدة لشرط كارستي في برهان وجود النقاط الثابتة في هذا الفضاء

## 1. Introduction

Caristi in 1976 [1] proved the existence for a fixed point of a function f which defined on a metric space $X$ if it satisfies the condition: $d(x, f(x)) \leq \varphi(x) - \varphi(f(x))$ for all $x \in X$.
Czerwik in 1993 [2] introduced the concept of the b-metric space by modifying the triangle inequality for the metric space.

In 1994 [3] Matthews introduced the notion of the partial metric space by adding the non-self-distance property and proved the Banach contraction principle in this space. Aydi H. et

*Email: Liqaa.khaleel@sc.uobaghdad.edu.iq

al. in 2013 [4] gave some notes on the modified proof of Caristi's fixed point theorem on partial metric spaces.

Shukla in 2014 [5] defined the partial b-metric by connecting with the Czerwiks space and Mathews' space. Franz-Viktor Kuhlmann et al. in 2018 [6] showed that a metric space is complete if and only if all of its Caristi–Kirk ball spaces are spherically complete.
Hakan Karayılan, Mustafa Telci in 2019 [7] extended the generalized Caristi's fixed point theorem point theorem on complete fuzzy metric spaces.

In 2019 [8] Bota M.et al. established some fixed point $(\alpha-\psi)$- ciric -type contractive multivalued operators in b-metric spaces. In 2020 [9] the existence of a fixed point for set valued mappings with some modification of the Banach construction principle in Mathews' space was proved.

Panel Takashi Ishizuka in 2021 [10] considered the complexity of finding a fixed point whose existence is guaranteed by an order-theoretic such as Caristi and Brondsted's fixed point theorems. Nattawut Pholasa et al. in 2021 [11] studied Caristi's fixed point theorem in complete Mv-metric space.
EL Kouch Youness et al. in 2021 [12] gave a generalization version of some new fixed point theorems of mappings satisfying Caristi type conditions. Md. Abdul Mannan et al. in 2021 [13] discussed Caristi's fixed point theorem for mappings on normed spaces. Fatemeh Lael et al. in 2022 [14] provided a brief proof for Caristi-Kirk fixed point result for single and set-valued mappings in cone metric spaces. Nikola Savanovic et al. in the same year [15] discussed some theorems in b-metric spaces
Piotr Nowakowski and Filip Turobo´s in 2023 [16] proved several properties, with convergence in semi-metric spaces and put several open questions connected with this notion. In this work, the existence of fixed points for single valued and set valued mappings defined on complete partial b-metric spaces that satisfying Caristi's condition are established with new generalization and some applications.

## 2. Preliminaries

In this section we recall all the basic concepts that we need in this work such as the definitions of b - metric, partial metric and partial b-metric spaces, as follows: (See [2, 3, 5]).

**Definition 2.1: [2]**
The space b-metric space $(Y,d,s)$ is a set $Y \neq \emptyset$ and $d: Y\, x\, Y \rightarrow (0\,,\infty)$ such that:
$(Mb1) d(a,b)=0$ if and only if $a=b$ for all $a,b \in Y$;
$(Mb2)$ $d(a,b)=d(b,a)$ for all $a,b \in Y$;
$(Mb3)$ $d(a,b) \leq s[d\,(a,c)+d\,(c,b)]$, where $s \geq 1$ and $a,b,c \in Y$.

**Definition 2.2: [3]**
A partial metric space $(Y\,,P)$ is a set $Y \neq \emptyset$ and a function $p: \; Y \times Y \rightarrow [0,+\infty)$ satisfies:
$p1)$ Non-negativity and a self-distance: $0 \leq p\,(a,a) \leq p(a,b)$,
$p2)$ Indistancy implies equality, if $p\,(a,a)=p(b,b)=p(a,b)$ then $a=b$ ,
$p3)$ Symmetry, $p\,(a,b)=p\,(b,a)$,
$p4)$ Triangular property: $p\,(a,c) \leq p\,(a,b)+p(b,c)-p(b,b)$.

**Definition 2.3: [5]**
A partial b-metric space$(Y, Pb)$ is a set $Y$ and a function $Pb : Y \times Y \rightarrow [0, +\infty)$ such that for all $u, v, w \in Y$:
$(Pb1)$ $a = b$ if and only if $Pb(a, a) = pb(a, b) = pb(b, b)$;
$(Pb2)$ $Pb(a, a) \leq pb(a, b)$;
$(Pb3)$ $pb(a, b) = pb(b, a)$;
$(Pb4) Pb(a, b) \leq s[pb(a, c) + pb(c, b)] - pb(c, c)$, $s \geq 1$, $a, b, c \in Y$.

**Definition 2.4: [16]**
The sequence $\{y_n\}$ in $(Y, Pb)$ is:
1- Converges if $\lim_{n \to \infty} pb(y_n, y) = pb(y, y)$, $y \in Y$.
2- Cauchy if $\lim_{n,m \to \infty} pb(y_n, y_m)$ exists and it is finite.
3- $(Y, Pb)$ is complete if every Cauchy sequence satisfies:
$\lim_{n,m \to \infty} pb(y_n, y_m) = \lim_{n \to \infty} pb(y_n, y) = pb(y, y)$, $y \in Y$.

**Definition 2.5: [17]**
Let $CBpb(Y)$ be all the closed bounded subsets of $(Y, pb)$, then for all $M, N \in CBpb(Y)$, Housedorff partial b-metric space which denoted by $Hpb$ on $(Y, pb)$ is defined by :
$Hpb(M, N) = \max\{\partial pb(M, N), \partial pb(N, M)\}$,
where:
$\partial pb(M, N) = \sup\{\beta pb(u, N) : u \in M\}$;
$\partial pb(N, M) = \sup\{\beta pb(v, M) : v \in N\}$;
$\beta pb(u, N) = \inf\{pb(u, \grave{u}), \grave{u} \in N\}$.

**Definition 2.6: [17]**
Let $(Y, pb)$ be a partial b- metric space. A single valued mapping $T : Y \rightarrow Y$ is said to be a Caristi mapping if there exists a lower semi continuous bounded below function $h: Y \rightarrow (-\infty, +\infty)$ such that $pb(y, Ty) \leq h(y) - h(Ty)$ for $y \in Y$.

**Definition 2.7: [17]**
Let $(Y, pb)$ be a partial b- metric space. The mapping $G : Y \rightarrow CB(Y)$ is said to be a Caristi mapping if there exists a lower semi continuous bounded below function $h: Y \rightarrow (-\infty, +\infty)$ such that
$pb(a, b) \leq h(a) - h(b)$ for $a \in Y$ and for each $b \in Gy$.
Fixed point theorem with Caristi's condition in partial b- metric spaces will be established.

The two important lemmas which have an essential rule in the proof of the mean results in this section will be necessary to recall.

**Lemma 2.8: [18]**
Let $(Y, pb)$ be a partial b-metric space, and $M, N$ be closed bounded subsets of $Y$. For all $u \in M$, then there exists $y = y(u) \in N$ and $k > 1$ with $pb(u, y) \leq k \partial pb(M, N)$.

**Lemma 2.9: [18]**
Let $(Y, pb)$ be a partial b-metric space and $A, B, C$ be three closed bounded subsets in $Y$:
a- $Hpb(A, A) \leq Hpb(A, B)$,
b- $Hpb(A, B) = Hpb(B, A)$,
c- $Hpb(A, B) \leq s[Hpb(A, C) + Hpb(C, B)] - \inf_{c \in C} pb(c, c)$.

To show the prove, see [19].

## 3. Main results

New generalizations of the Caristi's condition have been established and used for proving the existence of a fixed point for single and set valued mappings on partial b- metric spaces.

The generalization of Theorem 5, [20] will be considered in the following theorem, such that we prove that the co-domain of the function h is extended for all the real numbers, while in Theorem 5, [20] proved the co-domain of the corresponding function was restricted to the only real positive numbers.

**Theorem 3.1:**

Let $(Y, pb)$ be a partial b- metric space, $h: Y \to (-\infty, +\infty)$ be bounded below and lower semi continuous function. Let $T: Y \to Y$ satisfied $pb(y, Ty) \leq h(y) - h(Ty)$ for all $y$ in $Y$, then there exists a point $u$ in $Y$ such that $u = Tu$.

**Proof:**

Let $u \in Y$. Define a set $W$ as:

$W = \{ y \in Y \text{ such that } pb(u, y) \leq \quad h(u) - h(y)\}$.

It is clear that $W$ is a non-empty subset of $Y$.

Now, for each $y \in W$,

$$pb(u, y) \leq h(u) - h(y) \quad \ldots \quad (3.1)$$

But T satisfy the Caristi's condition, we have

$$h(Ty) \leq h(y) - pb(y, Ty). \quad \ldots \quad (3.2)$$

From Relations (3.1) and (3.2), one can get the following relation

$$\begin{aligned} h(Ty) &\leq h(y) - pb(y, Ty) + h(u) - h(y) - pb(u, y) \\ &\leq h(u) - [pb(y, Ty) + pb(u, y)] \\ &\leq h(u) - [s[pb(y, u) + pb(u, Ty)] - pb(u, u) + pb(u, y)] \\ &\leq h(u) - [s[pb(y, u) + pb(u, Ty)] + pb(u, y)] \\ &\leq h(u) - spb(y, u) - spb(u, Ty) - pb(u, y) \\ &\leq h(u) - [spb(y, u) + spb(u, Ty) - pb(u, u) + pb(u, u)] \\ &\leq h(u) - [pb(u, Ty) + pb(u, u)] \\ &\leq h(u) - pb(u, Ty) - pb(u, u). \end{aligned}$$

Thus, $h(Ty) \leq h(u) - pb(u, Ty)$, which means $Ty \in W$.

Claim, $u = T u$.

Suppose $u \neq T u \quad \forall u \in W$.

Then, for each $u \in W$, there exists $a \in$ W such that $u \neq a$ and

$$pb(a, u) \leq h(a) - h(u).$$

Hence, there exists a $\in$ W, with

$$h(\text{a}) = \inf h(u) \leq h(T\text{a}).$$

Then:

$$\begin{aligned} 0 < pb(\text{a}, \text{Ta}) &\leq h(\text{a}) - h(T\text{a}) \\ &\leq h(T\text{a}) - h(T\text{a}) = 0. \end{aligned}$$

Which is a contradiction, since when $Tu = u$, means $u$ is a fixed point for $T$.

Fixed point theorem with Caristi's condition for set-valued mappings in partial b- metric spaces will be established in the following lemma:

**Theorem 3.2:**
Let $(Y, pb)$ be a partial metric space, $G : Y \to CB(Y)$, such that for each $y \in Y$, there exists $w \in G(y)$ such that $pb(y,w) \leq h(y) - h(w)$. Where $h: Y \to (-\infty, +\infty)$ is bounded below and lower semi continuous function. Then $G$ has a fixed point.

**Proof:**
Let $G : Y \to CB(Y)$, $w \in G(y)$. Then, for every $y \in Y$, we have
$$pb(y,w) \leq h(y) - h(w).$$
Let $M = \{ y \in Y : h(y) \leq h(y_0) - pb(y_0, y) - pb(y_0, y_0) \}$ for some $y_0 \in Y$.
Clearly, from our hypothesis that $M \neq \emptyset$.
One can get:
$$h(w) + pb(y,w) \leq h(y) \leq h(y_0) - pb(y_0, y) - pb(y_0, y_0).$$
So, $h(w) \leq h(y_0) - [pb(y_0, y) + pb(y_0, y_0) + pb(y,w)]$
$$h(w) \leq h(y_0) - pb(y_0, w) - pb(y,y) - pb(y_0, y_0)$$
$$h(w) \leq h(y_0) - pb(y_0, w).$$
This implies $w \in M$. By using Theorem 3.4 one can obtain $w \in Y$ such that $w \in Gw$.

## 4. Generalization of Caristi's condition

The generalization Caristi's fixed point theorem for a single valued mapping will be stated and proved in this section.

**Theorem 4.1:**
If $T$ is a self-single valued mapping defined on a complete partial b- metric space $Y$ such that $pb(y, Ty) \leq \varphi\big(f(y)\big)\big(f(y) - f(Ty)\big)$ for each $y \in Y$, where $f : Y \to (-\infty, +\infty)$ is a lower semi continuous bounded below function and $\varphi : (-\infty, +\infty) \to (0, +\infty)$ is a non-decreasing function. Then there exists $y_0 \in Y$ such that $T y_0 = y_0$.
**Proof:**
Let $u \in Y$ and $M = \{y \in Y : f(y) \leq f(u)\}$.
That is $\inf f(y) \leq f(u)$.
Clearly, $M$ is not an empty set, and this is by the condition of the theorem.
For any $y \in Y$, and by our assumption, we have $f(Ty) \leq f(u)$ and hence $Ty \in M$.
By contradiction we obtain $Ty \neq y$ for all $y \in M$. So, there exists $v \in M, Tv \in M$ and $Tv \neq v$. By assumption, we have $p(v, Tv) \leq \varphi(f(v))(f(v) - f(Tv))$.
Cauchy sequence $\{u_n\}$ in $Y$ will be construct as follows:
Let $u_1 = u$.
Define $Su_n = \{y \in Y : p(u_n, y) \leq \varphi\big(f(u_n)\big)\big(f(u_n) - f(y)\big)\}$.
Then choose $u_{n+1} \in Su_n$ such that
$$f(u_{n+1}) \leq \inf\{f(y), y \in Su_n\} + \frac{1}{n}.$$
That is
$$f(u_{n+1}) \leq f(u_n) \quad \text{for each } n \in \mathrm{N}.$$
This means $\{f(u_n)\}$ is non-increasing. Since $f$ is bounded below and lower semi continuous. Then, $\lim_{n\to\infty} f(u_n)$ exists and finite.
Now, claim that $\{u_n\}$ is a Cauchy sequence. Indeed, for $n, m \in N$ such that $m > n$, one can get $p(u_n, u_m) \leq \varphi\big(f(u_n)\big)\big(f(u_n) - f(u_m)\big)$. But $\varphi$ is a non-decreasing function, so
$$\varphi\big(f(u_n)\big) \leq \varphi\big(f(u_{n-1})\big)$$
$$\leq \varphi\big(f(u_{n-2})\big)$$

$\vdots$
$\leq \varphi(f(u_1))$.
Then, $\lim_{n,m\to\infty} p(u_n, u_m)$ exists and finite, which means $\{u_n\}$ is a Cauchy sequence.
By completeness of $(Y, pb)$ the Cauchy sequence $\{u_n\}$ converges to some $u_0$ in $Y$.
By using the relation:
$\inf\{f(y), y \in Y\} \leq f(u_0)$, one can find $v_1 \in Y$ *with* $v_1 \neq u_0$ such that
$p(u_0, v_1) \leq \varphi(f(u_0))(f(u_0) - f(v_1))$. But, $f$ is lower semi continuous function and $\{u_n\}$ converges to $u_0$.
So,
$$f(u_0) \leq \lim_{n\to\infty} \inf f(u_n) \leq f(u_i) \text{ for all } i \in N.$$
Then, for all $n, m \in N$ such that $m > n$:
$pb(u_n, u_m) \leq \varphi(f(u_n))(f(u_n) - f(u_0))$.
And
$pb(u_0, v_1) \leq \varphi(f(u_0))(f(u_0) - f(v_1))$
$\leq \varphi(f(u_i))(f(u_0) - f(v_1))$ for all $i \in N$.
Thus, $f(u_0) \geq f(v_1)$ …(4.1)

On the other hand, by last property for partial b- metric spaces, one can get,
for each $n \in N$,
$pb(u_n, v_1) \leq pb(u_n, u_0) + pb(u_0, v_1) - pb(u_0, u_0)$
$\leq \varphi(f(u_n))(f(u_n) - f(v_1))$.
Hence, $v_1 \in S(u_n)$ for each $n \in N$,
so, $f(u_0) \leq f(u_{n+1}) \leq f(v_1) + \frac{1}{n}$ for all $n \in N$
$f(u_0) \leq f(v_1)$. …(4.2)
By Relations (1) and (2), we get $f(u_0) = f(v_1)$. Also, we can obtain, $p(u_0, v_1) = 0$.
Again, there exists $v_2 \in Y$ with $v_1 \neq v_2$ such that
$pb(v_1, v_2) \leq \varphi(f(v_1))(f(v_1) - f(v_2))$.
Then,
$f(v_1) \geq f(v_2)$ and for each $n \in N$, we have
$p(u_n, v_2) \leq pb(u_n, v_1) + pb(v_1, v_2) - pb(v_1, v_1)$
$\leq pb(u_n, v_1) + pb(v_1, v_2)$
$\leq \varphi(f(u_n))(f(u_n) - f(v_2))$.
That means, $v_2 \in Su_n$ for each $n \in N$, and so
$f(u_0) \leq f(u_{n+1}) \leq f(v_1) + \frac{1}{n}$ for all $n \in N$
Then $f(u_0) \leq f(v_1)$,
and hence, $f(u_0) = f(v_1)$.
Also, we can obtain, $p(v_1, v_2) = 0$. Again, by using the fourth property of a partial b-metric space, we get
$pb(u_0, v_2) \leq pb(u_0, v_1) + pb(v_1, v_2) - pb(v_1, v_1)$
$\leq pb(u_0, v_1) + pb(v_1, v_2) = 0$.
So, $pb(u_0, v_2) = 0$, and we have $v_1 = v_2$ and that is a contradiction
Therefore, there exists $u_0 \in Y$ such that
$$f(u_0) = \inf\{f(y), y \in Y\} \leq f(Tu_0).$$
Then
$pb(u_0, T u_0) \leq \varphi(f(u_0))(f(u_0) - f(T u_0))$, means $pb(u_0, T u_0) = 0$, thus $u_0$ is a fixed point for $T$, and this completes the proof.

Now, the generalization Caristi's type mapping of set-valued mappings in complete partial b-metric spaces will be stated and prove as follows.

**Theorem 4.2:**
Let $G$ be a mapping defined from a complete partial b- metric space $Y$ into the family of all closed bounded subsets of $Y$. Suppose that for each $a \in Y$, there exists $b \in Gy$ such that $pb(a,b) \leq \varphi(f(a))(f(a) - f(b))$,
where $f : Y \rightarrow (-\infty,+\infty)$ is a lower semi continuous bounded below function and $\varphi : (-\infty,+\infty) \rightarrow (0,+\infty)$ is a non-decreasing function.
Then there exists $a_0 \in Y$ such that $a_0 \in G(a_0)$.
**Proof:**
Let $b \in GY$, such that

$$pb(a,b) \leq \varphi(f(a))(f(a) - f(b)).$$

Then, by applying Theorem 3.4, there exists $a_0 \in Y$ such that $a_0 \in Ga_0$.

## 5. Some applications

In this section, two prepositions and an example will be studied as applications for generalization of Caristi's condition in complete partial b- metric spaces.

**Proposition 5.1:**
Let $M: Y \rightarrow CB(Y)$ be a set – valued mapping defined on a partial b- metric and $M_\delta(y) : Y \rightarrow CB(y)$ such that $M_\delta(y) = \left\{q \in My : pb(q,My) \geq \frac{pb(q,y)}{(1+\delta)}\right\}$ for each $y \in Y$ .Where $\delta \in \left(0, \frac{1}{r} - 1\right)$ and $r \in (0,1)$ .
If $M_\delta(y) \cap B \neq \emptyset$ for all $y \in B$ , $B$ is a non-empty closed subset of $Y$, then $M$ has a fixed point in $B$.

**Proof:**
Let $J_y = M_\delta(y) \cap B$ for each $y \in B$. Since $M_\delta(y) \cap B \neq \emptyset$, then one can choose $w \in J_y$ such that for each $y \in B$

$$pb(y,w) \leq (1+\delta)pb(y,My) .$$

Since $pb(w,Mw) \leq H(My,Mw), \ldots (5.1)$
and $H(My,Mw) \leq r\, pb(y,w) \quad \ldots (5.2)$
Then, from (5.1) and (5.2) one can say

$$pb(w,Mw) \leq r\, pb(y,w).$$

Now, the inequality:

$$\begin{aligned} pb(y,My) - pb(w,Mw) &\geq pb(y,My) - r\, pb(y,w) \\ &\geq \frac{1}{1+\delta} pb(y,w) - r\, pb(y,w) \\ &\geq \left(\frac{1}{1+\delta} - r\right) pb(y,w). \end{aligned}$$

Then,

$$pb(y,w) \leq \left[\frac{1}{1+\delta} - r\right]^{-1} [pb(y,My) - pb(w,Mw)]$$

$$\leq \left[\frac{1}{1+\delta} - r\right]^{-1} pb(y,My) - \left[\frac{1}{1+\delta} - r\right]^{-1} pb(w,Mw).$$

Let $\omega(y) = \left[\frac{1}{1+\delta} - r\right]^{-1} pb(y,My)$ , $\omega(w) = \left[\frac{1}{1+\delta} - r\right]^{-1} pb(w,Mw)$
Then, $pb(y,w) \leq \omega(y) - \omega(w)$. So, there exists $z \in B$ such that $z \in J_z$.

This implies $z \in Mz \cap B$ , so $z \in Mz$, and hence $M$ has a fixed point in $B$

**Proposition 5.2:**
Let $(Y, Pb)$ be a partial metric space and let $G : Y \to CB(Y)$, if

1- there exists a lower semi continuous function
$h : [0,\infty) \to [0,1)$ , $h(y) = \frac{1}{b-a} pb(y, Gy)$ , $0 < a < b < 1$ , for any $y \in Y$.
2- $\lim_{r \to t^+} sup\, h(r) < 1$ , $t \in [0,\infty)$.
3- There exists $w \in G(y)$ satisfying
$bp(y,w) \le pb(y,Gy)$ and $pb(w,Gw) \le apb(y,w)$.
Then $G$ has a fixed point, so there exists $z \in Y$ such that $z \in Gz$.

**Proof:** As

$$pb(y,Gy) = \frac{1}{b-a}[(b-a)pb(y,Gy)] = \frac{1}{b-a}[bp(y,Gy) - apb(y,Gy)]$$

then by condition 3, there exists $w \in G(y)$ such that

$$pb(w,Gw) \le apb(y,w).$$

So, $pb(y,Gy) \le \frac{1}{b-a}[bp(y,Gy) - pb(w,Gw)]$.
But, $bp(y,Gy) \le pb(y,Gy)$.
Hence, $pb(y,Gy) \le \frac{1}{b-a}[pb(y,Gy) - pb(w,Gw)]$
$= \frac{1}{b-a}\big(pb(y,Gy)\big) - \frac{1}{b-a}\big(pb(w,Gw)\big)$.
That is,

$$pb(y,Gy) \le h(y) - h(Gy).$$

Hence, $G$ satisfied Caristi's condition, so according to Theorem 3.1, it has a fixed point.

**Example 5.3:**
Let $Y = [0, 2]$ and $pb(a,b) = \max\{a,b\}$ for any $a, b$ in $Y$. Then
$(Y, pb)$ is a complete partial $b-$metric space

Now, if we define $G : Y \to CB(Y)$ as :

$$G(a) = \begin{cases} b \in Y: \frac{a}{3} \le b \le \frac{a}{2}, a \in Y \cap Q; \\ b \in Y: \frac{a}{5} \le b \le \frac{a}{4}, a \in Y - Q. \end{cases}$$

Where $CB(Y)$ the set of all closed bounded subsets of $Y$, and $Q$ the set of rational numbers.
Let $\mu(a) = 2a$, then for any $a \in Y$ and $b \in G(a)$,

$$p\,b(a,Ga) = \inf\{pb(a,b), b \in G(a)\} = \inf\{\max\{a,b\}, \text{for each } a \in G(a)\} = a.$$

So,

$$pb\big(a,G(a)\big) \le \mu(a) - \mu\big(G(a)\big).$$

Hence, by using Theorem 3.6, $G(a)$ has a fixed point. It's clear that $0 \in G(0)$.

## 6. Conclusions

We show in our work, that the single valued mapping can have a fixed point in a complete partial b-metric space if it is satisfied new generalization for Caristi's condition. Also, we have been discussed the existence of fixed points for a set valued mapping defined on complete partial b-metric spaces if it is satisfied same specific Caristi's condition. Some applications are illustrated the previous results.

**References**

**[1]** J. Caristi," Fixed Point Theorems for Mappings Satisfying Inwardness Conditions", *Transactions of the American Mathematical Society*, vol. 284, pp. 241-251, 1976.

**[2]** S. Czerwik,"Contraction mappings in b-metric spaces", *Acta Mathematica Informatica Universitatis Ostraviensis*, vol. 1, pp. 5–11, 1993.

**[3]** S. Matthews,"Partial Metric topology" The American Proc. 8th Summer Conference on General Topology and Application. *Annals of the New York Academy of Sciences*, vol. 728, pp. 183–197, 1994.

**[4]** C. Klin - eam ," Modified proof of Caristi's fixed point theorem on partial metric spaces", *Journal of Inequalities and Applications*, vol. 210, no.1, 2013.

**[5]** S. Shukla " Partial b-Metric Spaces and Fixed Point Theorems " *Mediterranean Journal of Mathematics*, vol. 11, pp.703-711, 2014.

**[6]** F. Viktor Kuhlmann, K. Kuhlmann, M. Paulsen," Caristi–Kirk Fixed Point Theorem from the point of view of ball spaces ", *Journal of Fixed Point Theory and Applications*, vol.20, no.107, 2018.

**[7]** K. Hakan and T. Mustafa, "Caristi type fixed point theorem in fuzzy metric spaces", *Hacettepe Journal of Mathematics & statistics.* vol.48, no.1, pp.73-86, 2019.

**[8]** H. Qawaqneh, M. Salmi , W. Shatanawi and H. Aydi , "Fixed Point Results for Multi-valued Contractions in b-Metric Spaces and an Application", *Advances in PureMathematics*, vol.7, no. 2, 2019.

**[9]** L. Jameel and B. Ahmed, "Fixed Point Theorem for Set Valued Mapping with Rational Condition" *Iraqi Journal of Science*, vol.61, no.4, pp.805-810, 2020.

**[10]** T. Ishizuka , "On the complexity of finding a Caristi's fixed point", *Information Processing Letters*, vol. 170, pp.400-408, 2021.

**[11]** N. Pholasa , S. Antal, I. Uddin and U. Gairola, "Caristi Fixed Point Theorem in Mv-Metric Spaces ", *Dynamic Systems and Applications*, vol.30, no.3, pp. 439-448, 2021.

**[12]** El. Youness , M. Elmiloudi, M. Aamri and J. Mouline, "Carist Type Fixed Point Theorems in Generlized Metric Spaces ", *Advanced Fixed Point Theory*, vol.11, no.1, 2021.

**[13]** M. Abdul Mannan, M. Hossain, H. Akter and S. Mondal, "A Study of Caristi's Fixed Point Theorem on Normed Space and Its Applications", *Advances in Pure Mathematics,* vol.11, no.3, pp.169-179, 2021.

**[14]** F. Lael, N. Saleem and R. George, "Caristi's Fixed Point Theorem in Cone Metric Space", *Journal of Function Spaces,* vol.9, no.1, pp. 1-6, 2022.

**[15]** N. Savanovic , I. Arandeloviˊc and Z. Mitroviˊc,"The Results on Coincidence and Common Fixed Points for a New Type Multivalued Mappings in b-Metric Spaces", *Mathematics*, vol. 10, no. 6, 2022.

**[16]** P. Nowakowski and F. Turobo, "Applications of ball space theory : fixed point theorems in semimetric spaces and ball convergence", *Journal of Fixed Point Theory and Applications*, vol.25, no.13, pp.1-27, 2023.

**[17]** V. Parvaneha and Z. Kadelburgb, "Extended Partial b-Metric Spaces and some Fixed Point Results", *Filomat*, vol.32, no.8, pp. 2837–2850, 2018.

**[18]** A. Saeed, M. Nazam, H. Hamed, A. Hussain, K. Javed and M. Arshad, "Existence fixed-point theorems in the partial b-metric spaces and an application to the boundary value problems", *AIMS Mathematics*, vol.7, no. 5, pp. 8188-8205, 2022.

**[19]** F. Ahmed," Some fixed point results for multivalued contractive mappings in partial b-metric Spaces", *Journal of Advanced Mathematics studied*, vol. 9, pp. 208–225, 2016.

**[20]** E. Karapinar, "Generalizations of Caristi Kirk's Theorem on Partial Metric Spaces", *Fixed Point Theory and Applications*, vol. 1, no. 4, pp.1-7, 2011.