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Generalized Strong Commutativity Preserving Centralizers of Semiprime Γ- Rings

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Abstract
In this paper, we introduce the concept of generalized strong commutativity (Cocommutativity) preserving right centralizers on a subset of a *Γ*-ring. And we generalize some results of a classical ring to a gamma ring.

Keywords: *Γ*-ring, semiprime *Γ*-ring, generalized strong commutativity preserving right centralizer, generalized strong Cocommutativity preserving right centralizer.

التمركزات الحافظة للابدالية القوية المعممة على حلقات كاما شبه الاولية

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الخلاصة
في هذا البحث , سوف نقدم مفهوم تعميم التبادليه القوية (المعاكسة) المحافظة المركزية لمجموعة جزئية من حلقة كاما. وايضا نعمم بعض نتائج للحلقة الى حلقة كاما.

1. Introduction

Let M and Γ be additive abelian groups. If there exists a mapping of $M \times \Gamma \times M \to M$, $(a, \alpha, b) \to a\alpha b$ which satisfies the conditions
(i) $a\alpha b \in M$ (ii) $(a + b)\alpha c = a\alpha c + b\alpha c$, $a(\alpha + \beta)c = a\alpha c + a\beta c$, $a\alpha(b+ c) = a\alpha b + a\alpha c$ (iii) $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$, then M is called a Γ-ring. Every ring M is a Γ-ring with $M = \Gamma$. However a Γ-ring need not be a ring. Gamma rings, more general than rings, were introduced by Nobusawa [1]. Bernes [2] weakened slightly the conditions in the definition of Γ-ring. Let M be a Γ-ring. Then an additive subgroup N of M is called a left (right) ideal of M if $M\Gamma N \subset N$ ($N\Gamma M \subset N$). If N is both a left and a right ideal, then we say N is an ideal of M. Suppose again that M is a Γ-ring. Then M is said to be a 2-torsion free if $2a = 0$ implies $a = 0$ for all $a \in M$. A Γ-ring M is said to be prime if $a\Gamma M\Gamma b = (0)$ with $a, b \in M$, implies $a = 0$ or $b= 0$ and semiprime if $a\Gamma M\Gamma a = (0)$ with $a \in M$ implies $a = 0$. Furthermore, M is said to be commutative Γ-ring if $a\alpha b = b\alpha a$ for all $a, b \in M$ and $\alpha \in \Gamma$. Moreover, the set $Z(M) =\{a \in M : a\alpha b = b\alpha a$ for all $\alpha \in \Gamma, b \in M\}$ is called the center of the Γ-ring M. If M is a Γ-ring, then $[a, b]_\alpha = a\alpha b - b\alpha a$ is known as the commutator of a and b with respect to α, where $a, b \in M$ and $\alpha \in \Gamma$ and $(aob)_\alpha = a\alpha b + b\alpha a$ is known as an anticommutator of a and b with respect to α . We make the basic commutator identities:
$[a\alpha b, c]_\beta = [a,c]_\beta \alpha b + a[\alpha,\beta]_c b + a\alpha[b, c]_\beta$ and $[a, b\alpha c]_\beta =[a, b]_\beta \alpha c + b[\alpha, \beta]a c + b\alpha[a, c]_\beta$, for all $a, b. c \in M$ and $\alpha, \beta \in \Gamma$. We consider the following assumption:
(A)................$a\alpha b\beta c = a\beta b\alpha c$, for all $a, b, c \in M$, and $\alpha, \beta \in \Gamma$.
According to the assumption (A), the above two identities reduce to
$[a\alpha b, c]_\beta = [a, c]_\beta \alpha b + a\alpha[b, c]_\beta$ and $[a, b\alpha c]_\beta = [a, b]_\beta \alpha c + b\alpha[a, c]_\beta$,
which we extensively used. An additive mapping $T : M \to M$ is a left(right) centralizer if $T(a\alpha b) = T(a)\alpha b$ ($T(a\alpha b) = a\alpha T(b)$) holds for all $a, b \in M$ and $\alpha \in \Gamma$. A centralizer is an additive mapping which

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is both a left and a right centralizer. We shall restrict our attention on left centralizer, since all results of right centralizers are the same as left centralizers. An additive mapping $D : M \to M$ is called a derivation if $D(a\alpha b) = D(a)\alpha b + a\alpha D(b)$ holds for all $a, b \in M$, and $\alpha \in \Gamma$ and is called a Jordan derivation if $D(a\alpha a) = D(a)\alpha a + a\alpha D(a)$ for all $a \in M$ and $\alpha \in \Gamma$. An additive mapping $T : M \to M$ is Jordan left(right) centralizer if $T(a\alpha a) = T(a)\alpha a(T(a\alpha a) = a\alpha T(a))$ for all $a \in M$, and $\alpha \in \Gamma$. Every left centralizer is a Jordan left centralizer but the converse is not true in general. An additive mappings $T : M \to M$ is called a Jordan centralizer if $T(a\alpha b + b\alpha a) = T(a)\alpha b + b\alpha T(a)$, for all $a, b \in M$ and $\alpha \in \Gamma$. Every centralizer is a Jordan centralizer but Jordan centralizer is not in general a centralizer. Bernes [2], Kyuno [3] and Luh [4] studied the structure of Γ-rings and obtained various generalizations of corresponding parts in ring theory. Borut Zalar [5] worked on centralizers of semiprime rings and prove that Jordan centralizers and centralizers of this rings coincide. Joso Vukman [6-8] developed some remarkable results using centralizers on prime and semiprime rings. In [9], Hoque and Paul proved that every Jordan centralizer of a 2.torsion free semiprime Γ–ring satisfying a certain assumption is a centralizer. Also, they proved in [10], if T is an additive mapping on a 2.torsion free semiprime Γ–ring M with a certain assumption such that $T(a\alpha b\beta a) = a\alpha T(b)\beta a$, for all $a, b \in M$ and $\alpha, \beta \in \Gamma$, then T is a centralizer. And in [11], if $2T(a\alpha b\beta a) = T(a)\alpha b\beta a + a\alpha b\beta T(a)$, for all $a, b \in M$ and $\alpha, \beta \in \Gamma$, then T is also a centralizer.

In this paper, we generalize some results of Shaker [12] and Abduljaleel [13] in gamma rings, To prove our main results, we need the following lammas.

Lemma 1.1[9] Le t M be a semiprime Γ -ring. If $a, b \in M$ and $\alpha, \beta \in \Gamma$ are such that $a\alpha x\beta b = 0$ for all $x \in M$, then $a\alpha b = b\alpha a = 0$.

Lemma 1.2 [9]Let M be a semiprime Γ-ring and $A : M \times M \to M$ biadditive mapping. If $A(a,b)\alpha w\beta B(a, b) = 0$ for all $a, b, w \in M$ and $\alpha, \beta \in \Gamma$, then $A(a, b)\alpha w\beta B(u, v) = 0$ for all $a, b, u, v \in M$ and $\alpha, \beta \in \Gamma$.

Lemma 1.3[9] Let M be a semiprime Γ -ring satisfying the assumption (A) and $a \in M$ be some fixed element. If $a\alpha[x, y]\beta = 0$ for all $a, b \in M$ *and* $\alpha, \beta \in \Gamma$, then there exists an ideal U of M such that $a \in U \subset Z(M)$ holds.

2. Generalized strong commutativity preserving centralizers on semiprime Γ-ring.

We will introduce the following definition.

Definition 2.1 . Let N be a subset of a Γ-ring M. Two right centralizers T_1 and T_2 on M are called generalized strong commutativity preserving (GSCP) on N if $[T_1(a), T_2(b)]_\alpha = [a, b]_\alpha$, for all $a, b \in N$ and $\alpha \in \Gamma$. And are called generalized commutativity preserving (GCP) on N if $[T_1(a), T_2(b)]_\alpha = 0$, for all $a, b \in N$ and $\alpha \in \Gamma$.

Example 2.2 Let $M = \left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix},\ a, b \in R, where\ R\ is\ a\ ring\ of\ intergers\right\}$and $\Gamma = \left\{\begin{pmatrix} n & 0 \\ 0 & m \end{pmatrix},\ n, m \in Z, where\ Z\ is\ a\ ring\ of\ intergers\right\}$. Then M is a Γ-ring under usual addition and multiplication of matrices. Let $N = \left\{\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix},\ b \in R\right\}$, then N is a left ideal of M. Define mappings T_1, T_2: $M \to M$ as follows:

$$T_1\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \text{ and } T_2\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -b \\ 0 & 0 \end{pmatrix}, \text{ for all } a, b \in R.$$

Then T_1 and T_2 are right centralizers on M and which are GSCP on N.

Lemma 2.3. Let M be a semiprime Γ-ring and I be a non-zero ideal of M. If T is a non-zero right centralizer on M. Then T is a non-zero on I.

Proof. Assume that $T = 0$ on I and let $a = a\alpha m$, for all $a \in I$, $\alpha \in \Gamma$, $m \in M$.

Then,

$$0 = T(a) = T(a\alpha m) = a\alpha T(m), \text{ for all } a \in I, \alpha \in \Gamma, m \in M.$$

It follows that,

$$I\Gamma T(M) = 0 \text{ and } I\Gamma M\Gamma T(M) = \{0\} \quad (1)$$

Let $P = \{ p_i, i \in \Lambda\}$ be a family of prime ideals of M such that $\cap\ p_i = \{0\}$. If p_i is a typical member of P, then by (1), it follows that

$$I\Gamma M\Gamma T(M) = \{0\} = \cap\ p_i \text{ and hence } I\Gamma M\Gamma T(M) \subseteq p_i, \text{ for all } i \in \Lambda.$$

By primeness of p_i, we have

$$\text{Either } I \subseteq p_i \text{ or } T(M) \subseteq p_i, \text{ for all } i \in \Lambda.$$

Using the fact that $\cap\ p_i = \{0\}$, we conclude that, Either $I = 0$ or $T(M) = 0$, a contradiction with assumption, then T is a non-zero on I.

Theorem 2.4. Let M be a semiprime Γ-ring of characteristic different from 2 and N be an ideal of M. If $T:M \to M$ is an additive mapping which satisfies $T(a\alpha a) = T(a)\alpha a$, for all $a \in N$, $\alpha \in \Gamma$, then M contains a central idea.

Proof. By given hypothesis, we have

$$T(a\alpha a)=T(a)\alpha a, \text{ for all } a \in N,\ \alpha \in \Gamma \quad (1)$$

Replacing a by $a + b$, where $b \in N$ in (1), we get $T((a + b)\alpha(a + b)) = T(a + b)\alpha(a + b)$,

Imply $T((a + b)\alpha a + (a + b)\alpha b) = (T(a) + T(b))\alpha(a + b)$,

Then $T(a\alpha a) + T(a\alpha b + b\alpha a) + T(b\alpha b) = T(a)\alpha a + T(a)\alpha b + T(b)\alpha a + T(b)\alpha b$,

Using (1), we have $T(a\alpha b + b\alpha a) = T(a)\alpha b + T(b)\alpha a$, for all $a, b \in N$, $\alpha \in \Gamma$ (2)

Replacing b by $a\beta b + b\beta a$, where $\beta \in \Gamma$ in (2), we get $T(a\alpha(a\beta b + b\beta a) + (a\beta b + b\beta a)\alpha a) = T(a)\alpha(a\beta b + b\beta a) + T(a\beta b + b\beta a)\alpha a$,

Thus, $T(a\alpha(a\beta b + b\beta a) + (a\beta b + b\beta a)\alpha a) =$

$$T(a)\alpha(a\beta b) + 2T(a)\alpha(b\beta a) + T(b)\beta(a\alpha a), \text{ for all } a, b \in N,\ \alpha, \beta \in \Gamma \quad (3)$$

But this can also calculated in a different way,

$T((a\alpha a)\beta b + b\beta(a\alpha a)) + 2T(a\alpha(b\beta a)) = T(a\alpha a)\beta b + T(b)\beta(a\alpha a) + 2T(a\alpha(b\beta a))$, (by(2)) Using (1), we have $T((a\alpha a)\beta b + b\beta(a\alpha a)) + 2T(a\alpha(b\beta a)) = (T(a)\alpha a)\beta b + T(b)\beta(a\alpha a) + 2T(a\alpha(b\beta a))$, for all $a, b \in N$, $\alpha, \beta \in \Gamma$ (4)

Comparing (3) and (4), we get $T(a\alpha(b\beta a)) = T(a)\alpha(b\beta a)$, for all $a, b \in N$, $\alpha, \beta \in \Gamma$ (5) Replacing a by $a + c$, where $c \in N$ in (5), we have $T((a + c)\alpha(b\beta(a + c)) = T(a + c)\alpha(b\beta(a + c))$,

Imply $T((a + c)\alpha(b\beta a + b\beta c)) = T(a + c)\alpha(b\beta a + b\beta c))$, So

$$T(a\alpha(b\beta a) + c\alpha(b\beta a) + a\alpha(b\beta c) + c\alpha(b\beta c)) = (T(a) + T(c))\alpha(b\beta a) + (T(a) + T(c))\alpha(b\beta c)$$

Thus $T(a\alpha(b\beta a)) + T(a\alpha(b\beta c) + c\alpha(b\beta a)) + T(c\alpha(b\beta c)) = T(a)\alpha(b\beta a) + T(a)\alpha(b\beta c) + T(c)\alpha(b\beta a) + T(c))\alpha(b\beta c)$, for all $a, b, c \in N$, $\alpha, \beta \in \Gamma$.

Using (5), we get $T(a\alpha(b\beta c) + c\alpha(b\beta a)) = T(a)\alpha(b\beta c) + T(c)\alpha(b\beta a)$, for all $a, b, c \in N$, $\alpha, \beta \in \Gamma$. (6)

Now, we shall compute

$J = T(a\alpha b\beta c\gamma b\alpha a + b\alpha a\beta c\gamma a\alpha b)$ in two different ways, where $\gamma \in \Gamma$. Using (5), we have

$$J = T(a\alpha((b\beta c\gamma b)\alpha a) + T(b\alpha((a\beta c\gamma a)\alpha b) = T(a)\alpha((b\beta c\gamma b)\alpha a) + T(b)\alpha((a\beta c\gamma a)\alpha b), \quad (7)$$

And using (6), we get

$$J = T((a\alpha b)\beta c\gamma(b\alpha a) + (b\alpha a)\beta c\gamma(a\alpha b)) = T(a\alpha b)\beta c\gamma(b\alpha a) + T(b\alpha a)\beta c\gamma(a\alpha b), \quad (8)$$

Comparing (7) and (8), we have $T(a\alpha b)\beta(c\gamma(b\alpha a)) - (T(a)\alpha b)\beta(c\gamma(b\alpha a)) + T(b\alpha a)\beta(c\gamma(a\alpha b)) - (T(b)\alpha a)\beta(c\gamma(a\alpha b)) = 0$,

Hence

$(T(a\alpha b) - T(a)\alpha b)\beta(c\gamma(b\alpha a)) + (T(b\alpha a) - T(b)\alpha a)\beta(c\gamma(a\alpha b)) = 0$, for all $a, b, c \in N$, $\alpha, \beta, \gamma \in \Gamma$ (9)

Equation (2) can be written as $T(a\alpha b) - T(a)\alpha b = -(T(b\alpha a) - T(b)\alpha a)$, for all $a, b \in N$, $\alpha \in \Gamma$.

Using this relation in (9), we get

$$(T(a\alpha b) - T(a)\alpha b)\beta c)\gamma[b, a]_\alpha = 0, \text{ for all } a, b, c \in N,\ \alpha, \beta, \gamma \in \Gamma. \quad (10)$$

Using Lemma 1.2, we have

$$(T(a\alpha b) - T(a)\alpha b)\beta c)\gamma[u, v]_\alpha = 0, \text{ for all } a, b, c, u, v \in N,\ \alpha, \beta, \gamma \in \Gamma. \quad (11)$$

Now fix some a, b in N and write B instead of $T(a\alpha b) - T(a)\alpha b)$, we get

$$B\Gamma c\Gamma[u, v]_\alpha = 0, \text{ for all } c, u, v \in N,\ \alpha \in \Gamma. \quad (12)$$

Applying Lemma 1.1, we have

$$B\Gamma[u, v]_\alpha = 0, \text{ for all } u, v \in N,\ \alpha \in \Gamma. \quad (13)$$

And by using Lemma 1.3, we get There exists an ideal I of M such that $B \in I \subseteq Z(M)$.

Theorem 2.5. Let M be a prime Γ-ring of characteristic different from 2 and I a non-zero ideal of M. If T_1 and T_2 be two non-zero right centralizers on M such that T_1 and T_2 are GCP on I. Then M contains a non-zero central ideal.

Proof. By the given hypothesis, we have

$$[T_1(a), T_2(b)]_\alpha = 0, \text{ for all } a, b \in I \text{ and } \alpha \in \Gamma. \quad (1)$$

Replacing a by $b\beta a$, where $\beta \in \Gamma$ in (1), we get

$$0 = [T_1(b\beta a), T_2(b)]_\alpha = [b\beta T_1(a), T_2(b)]_\alpha$$

$= [b, T_2(b)]_\alpha \beta T_1(a) + b\beta [T_1(a), T_2(b) = [b, T_2(b)]_\alpha \beta T_1(a)$, by (1).

Thus

$$[b, T_2(b)]_\alpha \beta T_1(a) = 0, \text{ for all } a, b \in I \text{ and } \alpha, \beta \in \Gamma. \quad (2)$$

Replacing a by $c\gamma a$, where $c \in I$, $\gamma \in \Gamma$ in (2), we have

$$0 = [b, T_2(b)]_\alpha \beta T_1(c\gamma a)$$
$$= [b, T_2(b)]_\alpha \beta c\gamma T_1(a), \text{ for all } a, b, c \in I \text{ and } \alpha, \beta, \gamma \in \Gamma.$$

Then

$$[b, T_2(b)]_\alpha \Gamma I \Gamma T_1(a) = 0, \text{ for all } a, b \in I \text{ and } \alpha \in \Gamma. \quad (3)$$

And hence,

$$[b, T_2(b)]_\alpha \Gamma I \Gamma M \Gamma T_1(a) = \{0\}, \text{ for all } a, b \in I \text{ and } \alpha \in \Gamma. \quad (4)$$

Since M is a semiprime, then it must contain a family $P = \{ p_i, i \in \Lambda\}$ of prime ideals of M such that $\cap p_i = \{0\}$. If p_i is a typical member of P, then by (4), it follows that

$[b, T_2(b)]_\alpha \Gamma I \Gamma M \Gamma T_1(a) = \{0\} = \cap p_i$ and hence $[b, T_2(b)]_\alpha \Gamma I \Gamma M \Gamma T_1(a) \subseteq p_i$, for all $i \in \Lambda$.

By primeness of p_i, we have

$$\text{Either } T_1(a) \in p_i \text{ or } [b, T_2(b)]_\alpha \Gamma I \subseteq p_i, \text{ for all } a, b \in I \text{ and } \alpha \in \Gamma, i \in \Lambda. \quad (5)$$

Now, using the fact that $\cap p_i = \{0\}$, we get

$$\text{Either } T_1(a) = 0 \text{ or } [b, T_2(b)]_\alpha \Gamma I = 0, \text{ for all } a, b \in I \text{ and } \alpha \in \Gamma. \quad (6)$$

Since T_1 is non-zero on M, then by **Lemma 2.3**, we have T_1 is non-zero on I.

Thus

$[b, T_2(b)]_\alpha \Gamma I = 0$, for all $b \in I$ and $\alpha \in \Gamma$, and hence $[b, T_2(b)]_\alpha \Gamma M \Gamma I = \{0\} = \cap p_i$, for all $b \in I$ and $\alpha \in \Gamma$.

Then, $[b, T_2(b)]_\alpha \Gamma M \Gamma I \subseteq p_i$, for all $b \in I$ and $\alpha \in \Gamma$, $i \in \Lambda$.

So,

$$\text{Either } I \subseteq p_i \text{ or } [b, T_2(b)]_\alpha \in p_i, \text{ for all } b \in I \text{ and } \alpha \in \Gamma, i \in \Lambda. \quad (7)$$

Using the fact that $\cap p_i = \{0\}$, we conclude that,

$$\text{Either } I = 0 \text{ or } [b, T_2(b)]_\alpha = 0, \text{ for all } b \in I \text{ and } \alpha \in \Gamma. \quad (8)$$

Since I is a non-zero ideal, then $[b, T_2(b)]_\alpha = 0$, for all $b \in I$ and $\alpha \in \Gamma$.

Therefore,

$$T_2(b\alpha b) = b\alpha T_2(b) = T_2(b)\alpha b, \text{ for all } b \in I \text{ and } \alpha \in \Gamma, \text{ by } T_2 \text{ is right centralizer.}$$

Hence

$$T_2(b\alpha b) = T_2(b)\alpha b, \text{ for all } b \in I \text{ and } \alpha \in \Gamma. \quad (9)$$

Therefore, M contains a non-zero central ideal by **Theorem 2.4.**

Corollary 2.6. Let M be a prime Γ-ring of characteristic different from 2 and I a non-zero ideal of M. If T_1 and T_2 be two non-zero right centralizers on M such that T_1 and T_2 are GCP on I. Then M is a commutative Γ-ring.

Theorem 2.7. Let M be a semiprime Γ-ring of characteristic different from 2 and I a non-zero ideal of M. If T_1 and T_2 be two non-zero right centralizers on M such that T_1 and T_2 are GSCP on I. Then M contains a non-zero central ideal.

Proof. By the given hypothesis, we have

$$[T_1(a), T_2(b)]_\alpha = [a, b]_\alpha, \text{ for all } a, b \in I \text{ and } \alpha \in \Gamma. \quad (1)$$

Replacing a by $b\beta a$, where $\beta \in \Gamma$ in (1), we get $[T_1(b\beta a), T_2(b)]_\alpha = [b\beta a, b]_\alpha$, for all $a, b \in I$ and $\alpha, \beta \in \Gamma$.

Then $[b\beta T_1(a), T_2(b)]_\alpha = [b\beta a, b]_\alpha$, for all $a, b \in I$ and $\alpha, \beta \in \Gamma$.

Thus,

$$[b, T_2(b)]_\alpha \beta T_1(a) + b\beta [T_1(a), T_2(b)]_\alpha = [b, b]_\alpha \beta a + b\beta [a, b]_\alpha$$

Using (1), we have

$$[b, T_2(b)]_\alpha \beta T_1(a) = 0, \text{ for all } a, b \in I \text{ and } \alpha, \beta \in \Gamma. \quad (2)$$

By the same method of proof in **Theorem 2.5**, we complete the proof.

Corollary 2.9. Let M be a prime Γ-ring of characteristic different from 2 and I a non-zero ideal of M. If T_1 and T_2 be two non-zero right centralizers on M such that T_1 and T_2 are GSCP on I. Then M is a commutative Γ-ring.

3. Generalized strong Cocommutativity preserving centralizers on semiprime Γ-ring.

We introduce the following definition.

Definition 3.1. Let N be a subset of a Γ-ring M. Two right centralizers T_1 and T_2 on M are said to be generalized strong Cocommutativity preserving (GSCCP) on N if $(T_1(a)oT_2(b))_\alpha = (aob)_\alpha$, for all $a, b \in N$ and $\alpha \in \Gamma$. And are called generalized Cocommutativity preserving (GCCP) on N if $(T_1(a)oT_2(b))_\alpha = 0$, for all $a, b \in N$ and $\alpha \in \Gamma$.

We shall make use of the following commutator identities by condition(A):

$(ao(b\beta c))_\alpha = (aob)_\alpha\beta c - b\beta[a, c]_\alpha = b\beta(aoc)_\alpha + [a, b]_\alpha\beta c.$

$((a\beta b)oc)_\alpha = a\beta(boc)_\alpha - [a, c]_\alpha\beta c = (aoc)_\alpha b\beta + a\beta[b, c]_\alpha.$

Theorem 3.2. Let M be a semiprime Γ-ring of characteristic different from 2, I a non-zero ideal of M and T_1, T_2 be two non-zero right centralizers on M such that T_1 and T_2 are GCCP on I. Then M contains a non-zero central ideal.

Proof. By the given hypothesis, we have

$$(T_1(a)oT_2(b))_\alpha = 0, \text{ for all } a, b \in I \text{ and } \alpha \in \Gamma. \quad (1)$$

Replacing b by $a\beta b$, where $\beta \in \Gamma$ in (1), we get

$$0 = (T_1(a)oT_2(a\beta b))_\alpha = (T_1(a)o(a\beta T_2(b)))_\alpha, \text{ for all } a, b \in I \text{ and } \alpha, \beta \in \Gamma.$$

We have $a\beta(T_1(a)oT_2(b))_\alpha + [T_1(a), a]_\alpha\beta T_2(b) = 0$, for all $a, b \in I$ and $\alpha, \beta \in \Gamma$.

by using (1), we get

$$[T_1(a), a]_\alpha\beta T_2(b) = 0, \text{ for all } a, b \in I \text{ and } \alpha, \beta \in \Gamma. \quad (2)$$

Replacing b by $c\gamma b$, where $c \in I$, $\gamma \in \Gamma$ in (2), we have

$$0 = [T_1(a), a]_\alpha\beta T_2(c\gamma b)$$
$$= [T_1(a), a]_\alpha\beta c\gamma T_2(b), \text{ for all } a, b, c \in I \text{ and } \alpha, \beta, \gamma \in \Gamma.$$

Then $[T_1(a), a]_\alpha \Gamma I \Gamma T_2(b) = 0$, for all $a, b \in I$ and $\alpha \in \Gamma$. (3)

And hence,

$$[T_1(a), a]_\alpha \Gamma I \Gamma M \Gamma T_2(b) = \{0\}, \text{ for all } a, b \in I \text{ and } \alpha \in \Gamma. \quad (4)$$

Since M is a semiprime, then it must contain a family $P = \{ p_i, i \in \Lambda\}$ of prime ideals of M such that $\cap p_i = \{0\}$. If p_i is a typical member of P, then by (4), it follows that

$[T_1(a), a]_\alpha \Gamma I \Gamma M \Gamma T_2(b) = \{0\} = \cap p_i$ and hence $[T_1(a), a]_\alpha \Gamma I \Gamma M \Gamma T_2(b) \subseteq p_i$, for all $i \in \Lambda$.

By primeness of p_i, we have

$$\text{Either } T_2(b) \in p_i \text{ or } [T_1(a), a]_\alpha \Gamma I \subseteq p_i, \text{ for all } a, b \in I \text{ and } \alpha \in \Gamma, i \in \Lambda. \quad (5)$$

Using the fact that $\cap p_i = \{0\}$, we conclude that,

$$\text{Either } T_2(b) = 0 \text{ or } [T_1(a), a]_\alpha \Gamma I = 0, \text{ for all } a, b \in I \text{ and } \alpha \in \Gamma. \quad (6)$$

Since T_2 is non-zero on M, then by **Lemma 2.3**, we have T_2 is non-zero on I.

Thus

$$[T_1(a), a]_\alpha \Gamma I = 0, \text{ for all } a \in I \text{ and } \alpha \in \Gamma, \text{ and hence}$$
$$[T_1(a), a]_\alpha \Gamma M \Gamma I = \{0\} = \cap p_i, \text{ for all } a \in I \text{ and } \alpha \in \Gamma.$$

Then,

$$[T_1(a), a]_\alpha \Gamma M \Gamma I \subseteq p_i, \text{ for all } a \in I \text{ and } \alpha \in \Gamma, i \in \Lambda.$$

So,

$$\text{Either } I \subseteq p_i \text{ or } [T_1(a), a]_\alpha \in p_i, \text{ for all } a \in I \text{ and } \alpha \in \Gamma, i \in \Lambda. \quad (7)$$

Using the fact that $\cap p_i = \{0\}$, we have,

$$\text{Either } I = 0 \text{ or } [T_1(a), a]_\alpha = 0, \text{ for all } a \in I \text{ and } \alpha \in \Gamma. \quad (8)$$

Since I is a non-zero ideal, then $[T_1(a), a]_\alpha = 0$, for all $a \in I$ and $\alpha \in \Gamma$.

Therefore,

$$T_1(a)\alpha a = a\alpha T_1(a) = T_1(a\alpha a), \text{ for all } a \in I \text{ and } \alpha \in \Gamma. \quad (9)$$

Hence

$$T_1(a\alpha a) = T_1(a)\alpha a, \text{ for all } a \in I \text{ and } \alpha \in \Gamma. \quad (10)$$

Therefore, M contains a non-zero central ideal by **Theorem 2.4.**

Corollary 3.3. Let M be a prime Γ-ring of characteristic different from 2, I a non-zero ideal of M and T_1, T_2 be two non-zero right centralizers on M such that T_1 and T_2 are GCCP on I. Then M is a commutative Γ-ring.

Theorem 3.4. Let M be a semiprime Γ-ring of characteristic different from 2, I a non-zero ideal of M and T_1, T_2 be two non-zero right centralizers on M such that T_1 and T_2 are GSCCP on I. Then M contains a non-zero central ideal.

Proof. By the given hypothesis, we have

$$(T_1(a)oT_2(b))_\alpha = (aob)_\alpha, \text{ for all } a, b \in I \text{ and } \alpha \in \Gamma. \quad (1)$$

Replacing b by $a\beta b$, where $\beta \in \Gamma$ in (1), we get

$$(T_1(a)oT_2(a\beta b))_\alpha = (ao(a\beta b))_\alpha, \text{ for all } a, b \in I \text{ and } \alpha, \beta \in \Gamma.$$

Imply,

$$(T_1(a)o(a\beta T_2(b)))_\alpha = (ao(a\beta b))_\alpha, \text{ for all } a, b \in I \text{ and } \alpha, \beta \in \Gamma.$$

Then,

$$a\beta(T_1(a)oT_2(b))_\alpha + [T_1(a), a]_\alpha \beta T_2(b) = a\beta(aob)_\alpha + [a, a]_\alpha \beta b$$

Using (1), we get

$$[T_1(a), a]_\alpha \beta T_2(b) = 0, \text{ for all } a, b \in I \text{ and } \alpha, \beta \in \Gamma. \quad (2)$$

By the same method of proof in **Theorem 3.2**, we complete the proof.

Corollary 3.5. Let M be a prime Γ-ring of characteristic different from 2, I a non-zero ideal of M and T_1, T_2 be two non-zero right centralizers on M such that T_1 and T_2 are GSCCP on I. Then M is a commutative Γ-ring.

Now, by using similar techniques as in the **Theorems 2.7** and **3.4**, we get the following result.

Theorem 3.6. Let M be a semiprime Γ-ring of characteristic different from 2, I a non-zero ideal of M and T_1, T_2 be two non-zero right centralizers on M. Then M contains a non-zero central ideal, if one of the following conditions holds:

(i) $[T_1(a), T_2(b)]_\alpha + [a, b]_\alpha = 0$, for all $a, b \in I$ and $\alpha \in \Gamma$.

(ii) $(T_1(a)oT_2(b))_\alpha + (aob)_\alpha = 0$, for all $a, b \in I$ and $\alpha \in \Gamma$.

(iii) $[T_1(a), T_2(b)]_\alpha = (aob)_\alpha$, for all $a, b \in I$ and $\alpha \in \Gamma$.

(iv) $[T_1(a), T_2(b)]_\alpha + (aob)_\alpha = 0$, for all $a, b \in I$ and $\alpha \in \Gamma$.

References

1. Nobusawa, N. **1964**. On a generalization of the ring theory, *Osaka J. Math.*, (1): 81-89.

2. Barnes, W. E. **1966**. On the Γ-rings of Nobusawa, *Pacific J. Math.* **18**(3): 411-422.

3. Kyuno, S. **1978**. On prime gamma ring, *Pacific J. Math.***75**: 185-190.

4. Luh, J. **1969**. On the theory of simple Γ-rings, *Michigan Math. J.*, **16**(1): 65-75.

5. Zalar, B. **1991**. On centralizers of semiprime rings, *Comment. Math. Univ. Carolinae*, **32**: 609-614.

6. Vukman, J. **1997**. Centralizers in prime and semiprime rings, *Comment. Math. Univ. Carolinae*, **38**: 231-240.

7. Vukman, J. **1999.** An identity related to centralizers in semiprime rings, *Comment. Math. Univ. Carolinae*, **40**(3): 447-456.

8. Vukman, J. **2001**. Centralizers on semiprime rings, *Comment. Math. Univ. Carolinae*, **42**(2): 237-245.

9. Hoque, M. F. and Paul, A. C. **2011**. On centralizers of semiprime gamma rings, *International Mathematical Forum*, **6**(13): 627- 638.

10. Hoque, M. F. and Paul, A. C. **2013**. Centralizers of semiprime gamma rings, *Italian J. Pure and Applied Mathematics*, **30**: 289- 302.

11. Hoque, M. F. and Paul, A. C**. 2012**. An equation related to centralizers in semiprime gamma rings, *Annals of Pure and Applied Mathematics*, **1**(1): 84- 90.

12. Haker, H. A. **2005**. Centralizers on prime and semiprime rings,Department of Mathematics, University of Baghdad, Baghdad, Iraq.

13. Abduljaleel, A. A. **2017**. Commutativity results on prime and semiprime rings, M.Sc. Thesis. Department of Mathematics, University of Baghdad, Baghdad, Iraq.