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$\mu -$ Centralizers with Commutativity Results of Prime and Semiprime $\Gamma -$ Rings

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Abstract

Let A be a 2- torsion free prime $\Gamma -$ ring, and L be a square closed Lie ideal of A i.e. $x\beta x \in L$ for all $x \in L$ and $\beta \in \Gamma$, suppose that μ is an endomorphism of A, and f is a right $\mu -$ centralizer which is not the identity map on L, satisfying $f([x,b]_\gamma) = \mp[x,b]_\gamma$ (or $f(x\delta r) = \mp x\delta r$) for all $x, b \in L$, $\gamma \in \Gamma$, then $L \subseteq Z(A)$.

Keywords: Prime rings, Semiprime rings, Centralizers, $\mu -$ Centralizers, Lie ideals.

تمركزات−μ مع نتائج الابداليه على حلقات كاما الاولية و شبه الاوليه

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الخلاصه

لتكن Aحلقة كاما الاولية طليقة الالتواء من النوع 2 و L مثالي لي مربع مغلق بحيث $x\beta x \in L$ لكل $x \in L$ و $\beta \in \Gamma$. لنفرض ان μ هي تشاكل شامل ل A و f هي تمركز-μ يميني بحيث لايكون تطبيق محايد على L. اذا . for all $x, b \in L$, $\gamma \in \Gamma$, $f([x,b]_\gamma) = \mp[x,b]_\gamma$ (or $f(x\delta r) = \mp x\delta r$) فانL⊆Z(A)

1.Introduction

Let A and Γ be two additive abelian groups. Suppose that there is a mapping from A $\Gamma A \rightarrow$ A, such that $(a, \alpha, c) \rightarrow a\alpha c$ which satisfies the following conditions

(i) $d\alpha m \in A$.

(ii) $(d+m)\alpha q = d\alpha q + m\alpha q$

$d(\alpha+\gamma)m = d\alpha m + d\gamma m.$

$d\alpha(m+q) = d\alpha m + d\alpha q$

(iii) $(d\alpha m)\gamma q = d\alpha(m\gamma q)$.

For all $d, m, q \in A$ and $\alpha, \gamma \in \Gamma$, then A is called a $\Gamma -$ rin [1].

Every ring A is a $\Gamma -$ ring with A = Γ. However, a $\Gamma -$ ring need not be a ring. Gamma rings, more general than rings, were introduce by Bernes [1].

Let A be $a\,\Gamma -$ ring. A right (left) ideal of A is an additive subgroup J of A such that $J\Gamma A \subset J(A\Gamma J \subset J)$. If J is both left and right ideal, then we say J is an ideal or two-sided of A [2]. $A\,\Gamma -$ ring A is a 2- torsion free if 2x= 0 implies x= 0 for all x∈ A. An ideal P_1 of a $\Gamma -$ ring A is said to be prime [2] if for any ideal H and M of A. H ΓM $\subseteq P_1$ implies H⊆P_1 or

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M⊆P_1. An ideal P_2 of a $\Gamma -$ ring A is said to be semiprime if for any ideal H of A, H ΓH⊆P_2, implies H⊆P_2. A $\Gamma -$ ring A is called a prime if $h\Gamma A\Gamma m = (0)$ implies h=0 or m=0 where $h, m \in A$. As well as a $\Gamma -$ ring A is called a semiprime, if $h\Gamma A\Gamma h = (0)$ implies h=0 or $h \in A$ [2].

Furthermore, A is said to be commutative $\Gamma -$ ring if h$\beta m = m\ \beta h$ for all h, m $\in A$ and $\beta \in \Gamma$. The set $Z(A) = \{h \in A : h\gamma m = m\gamma h, m \in A\}$ is called center of the $\Gamma -$ ring A. If A is a $\Gamma -$ ring, then $[x, m]_\delta = x\delta m - m\delta x$ for all $x, m \in A, \delta \in \Gamma$ is called the commutator [3] of *x* and *m* with respect to δ.

An additive mapping F: $A \to A$ is a left $(right)$ centralizer [4] if $F(x\delta r) = F(x)\delta r\big(F(x\delta r)\big) = x\delta F(r)$ holds for all x, $r \in A$ and $\delta \in \Gamma$. Let A be $a\,\Gamma -$ ring, if there exists $n \in Z^+$ such that $nx = 0$ for all $x \in A$, then the smallest positive integer with this property is called the characteristic of A, which is dented by Char (A)=n, if no exist then char(A)=0 [2]. An additive subgroup J of A is said to be a Lie ideal of A if whenever j∈ J, $a \in A$ and $\beta \in \Gamma$, then $[j, a]_\beta \in$ J, and if $u\alpha u$, for all $u \in U$, a Lie ideal of this type is called sequer closed Lie ideal [5]. Many researchers have studied centralizers and derivations in prime and semiprime Γ- rings [6-16].

The purpose of this paper is to prove that when A be a 2- torsion free prime $\Gamma -$ ring and L be a square closed Lie ideal of A such that $x\beta x \in L$ for all $x \in L$ and $\beta \in L$, if we suppose that μ is an endomorphism of A, and f is a right $\mu -$ centralizer which is not the identity map on L. If $f\big([x, b]_\gamma\big) = \mp[x, b]_\gamma($ if $f(x\delta r) = \mp x\delta r)$ for all $x, b \in L$, $\gamma \in \Gamma$, then $L \subseteq Z(A)$.
Throughout this paper, we use the condition *m* α *n* β *t* = *m* β *n* α *t*, for all *m*, *n*, *t* ∈ A and α, β ∈ Γ and this is represented by property (*).

2. Basic Concepts and Fundamentals Results

In this section we will give the following definition and consequence results.

Definition 2.1: [4] Let A be a 2- torsion free semiprime $\Gamma -$ ring and let μ be an endomorphism of A, an additive mapping $T: A \to A$ is called left (right) $\mu -$ centralizer if $T(x\alpha y) = T(x)\mu(y)\big(T(x\alpha y) = \mu(x)\alpha T(y)\big)$ holds for all $x, y \in$ A, $\alpha \in \Gamma$,
if T is left and right $\mu -$ centralizer then it is natural to call T a μ- centralizer.

Remark 2.2:
Let A be $a\,\Gamma -$ ring and $r, s, t \in$ A then $\beta, \delta \in \Gamma$

(i) $[r\beta s, t]_\delta = r\beta[s, t]_\delta + [r, t]_\delta\beta\ s.$

(ii) $[r, s\beta\ t]_\delta = S\beta[r, t]_\delta + [r, s]_\delta\beta\ t.$

(iii) $(r\beta s) \circ_\delta t = r\beta(s\ \circ \delta t) - [r, t]_\delta\beta\ s$
$= (x \circ \beta t)\delta s + r\beta[s, t]_\delta.$

(iv) $r \circ \beta(s\delta t) = (x\beta s)\delta t - S\beta[r, t]_\delta$
$= S\beta(r\delta t) + [r, s]_\delta\beta t.$

(v) $(r\beta s)\delta x\ = r\beta(s\delta x).$

Lemma 2.3: [5]
If a prime $\Gamma -$ ring A contains a commutative non-zero right ideal. Then A is commutative.

Lemma 2.4: [5]
Let A be a semiprime $\Gamma -$ ring and *I* be any non-zero one sided ideal of A then $Z(I) \subseteq Z(A)$.

Lemma 2.5: [7]
If $L \nsubseteq Z(A)$ is a Lie ideal of A, and if $x\beta L\delta a = 0$, then x =0 or a =0 for all $x, a \in A$, $\beta, \delta \in \Gamma$.

Lemma 2.6: [8]
Let A be a 2- torsion free prime $\Gamma -$ ring and L be a non-zero Lie ideal of A. If L is a commutative Lie ideal of A then $L \subseteq Z(A)$.

3. Main Results

Lemma 3.1:
Let A be a semiprime $\Gamma -$ ring and *I* be anon-zero ideal of A if f is a non-zero right centralizer on *I*, then f is non-zero on A.

Proof: Let $f(a) = 0$ for $a \in I$, we let $f(a\beta\, k) = 0$ where $a \in I, k \in A, \beta \in \Gamma$.
It follows that

$$L\beta\, f(k) = \{0\} \text{ and } a\beta\, f(k) = 0 \text{ hence } I\Gamma f(k) = \{0\}. \quad (1)$$

Since A semiprime it must contain a family $P = \{Pr: r \in \Lambda\}$ of a prime ideal such that $\cap P_r = \{0\}$, (see [2], p.121 for more detailes).
If P atypical member of P, then from (1) it follows that
$L\Gamma\, f(A) = \{0\} = \cap \mathrm{P}_r$ and hence $L\Gamma\, f(A) \subseteq \mathrm{P}_r$. By primness of P

$$\text{either } I \subseteq P \text{ or } f(A) \subseteq P. \quad (2)$$

Now using the tact that $\cap \mathrm{P_r} = \{0\}$, we conclude that

$$\text{either } I = 0 \text{ or } f(A) = 0. \quad (3)$$

Since *I* is non-zero ideal then we get $f(A) = 0$.

Theorem 3.2:
Let A be a semiprime $\Gamma -$ ring *I* non-zero ideal of A and *f* be a non-zero right centralizer on A, such that $f(I) \subseteq Z(A)$. Then $I \subseteq Z(A)$.

Proof:
Since $f(I) \subseteq Z(A)$, then

$$[f(a), x]_\beta = 0 \text{ , for all } a, x \in I,\ \beta \in \Gamma. \quad (1)$$

Replacing a by $z\ \alpha\ a$ in (1), we get

$$[f(z\alpha a), x]_\beta = 0, \text{ for all } z, a, x, \in I\ , \alpha, \beta \in \Gamma. \quad (2)$$

So, $[z\alpha f(a), x]_\beta = 0$. By Remark 2.2 (i), we have

$$z\alpha[f(a), x]_\beta + [z, x]_\beta \alpha f(a) = 0.$$

Using (1), we get

$$[z, x]_\beta \alpha f(a) = 0 \quad \text{for all } z, x, a \in I, \alpha, \beta \in \Gamma. \quad (3)$$

Replacing x by $y\delta x$ in (3), we get

$$[z, y\delta x]_\beta \alpha f(a) = 0, \text{ for all } x, y, z, a \in I, \alpha, \beta, \gamma \in \Gamma \quad (4)$$

From Remark 2.2 (ii), we have
$y\delta[z, x]_\beta\ \alpha f(a) + [z, y]_\beta\ \delta x \alpha f(a) = 0.$
Using (3), we get

$$[z, y]_\beta \Gamma\ L\ \Gamma\ f(a) = 0 \text{ for all } a, z, y\ \in I,\ \beta \in \Gamma. \quad (5)$$

Since A semiprime $\Gamma -$ ring, it must contain a family $P = \{P_r: r \in \wedge\}$ of prime ideal such that $\cap Pr = \{0\}$, (see [2], p. 121) if P is atypical member of P, then form (5), it follows that $[z, y]_\beta \Gamma L \Gamma\, f(a) = \{0\} = \cap P_r$.
And hence $[Z, y]_\beta \Gamma I \Gamma\, f(a) \subseteq P$.
By primness of P.
Either

$$f(a) \in P \text{ or } [z, y]_\beta\ \Gamma \mathrm{L}\ \subseteq P \text{ for all } z, y\ \in I, \beta \in\ \Gamma. \quad (6)$$

Now using the fact that $\cap P_r = \{0\}$ we conclude that either

f(a)=0 or $[z,y]_\beta \Gamma I = 0$ for all $z,y \in I$, $\beta \in \Gamma$. (7)
Since f is a non-zero on A, then by Lemma 3.1, f is a non-zero on I
$[z,y]_\beta I = 0$ and hence $[z,y]_\beta \, \Gamma I = \{0\} = \cap P_r$.
Then either

$$I \subseteq P \text{ or } [z,y]_\beta \in P \text{ for all } z,y \in I, \quad \beta \in \Gamma. \quad (8)$$

Using the fact that $\cap P_r = \{0\}$, we conclude that either

$$I = 0 \text{ or } [z,y]_\beta = 0 \,, \text{ for all } z,y \in I, \ \beta \in \Gamma. \quad (9)$$

Since L non-zero ideal. Then $[z,y]_\beta = 0$ for all $z,y \in I$, $\beta \in \Gamma$.
Therefore, I is commutative and hence $I \subseteq Z(A)$ by Lemma 2.4.

Corollary 3.3:
Let A be a prime $\Gamma -$ ring, I be a non-zero ideal of A and f be a non-zero centralizer on A such that $f(I) \subseteq Z(A)$, then A is commutative.

Lemma 3.4:
Let L be a square closed Lie ideal of A, then $2a\gamma\, x \in L$ for all $a,x \in L, \gamma \in \Gamma$.
Proof:
Since L is square closed Lie ideal of A, we get
$u\gamma v + v\gamma u = (u+v)\gamma(u+v) - u\gamma u - v\gamma v$ for all $u,v \in L$ and $\gamma \in \Gamma$
That is

$$u\gamma v + v\gamma u \in L. \quad (1)$$

Also, we have

$$u\gamma v - v\gamma u \in L. \quad (2)$$

From (1) and (2), we get $2u\gamma v \in L$.

Theorem 3.5:
Let A be a 2- torsion free prime $\Gamma -$ ring and L be a square closed Lie ideal of A such that $x\beta x \in L$ for all $x \in L$ and $\beta \in \Gamma$, suppose that μ is an endomorphism of A and f is a right $\mu -$ centralizer such that μ is not the identity map on L. If $f\big([x,b]_\gamma\big) = [x,b]_\gamma$ for all $x,b \in L$, $\gamma \in \Gamma$ then $L \subseteq Z(A)$.
Proof:
By the given hypothesis, we have.

$$f\big([x,b]_\gamma\big) = [x,b]_\gamma \text{ for all } x,b \in L, \gamma \in \Gamma \quad (1)$$

By using Lemma 2.4 replacing b by $2x\alpha b$ in (1), we get

$$f\big([x, 2x\,\alpha\, b]_\gamma\big) = [x, 2x\,\alpha\, b]_\gamma \quad \text{for all } x,b \in L\ \alpha,\gamma \in \Gamma \quad (2)$$

Using the fact that Char A≠2, and from Remark 2.2, we get

$$f\big(x\alpha[x,b]_\gamma\big) = x\alpha[x,b]_\gamma,$$

which implies that

$$\mu(x)\alpha f[x,b]_\gamma = x\alpha[x,b]_\gamma.$$

By using (1), we get

$$\mu(x)\alpha[x,b]_\gamma = x\alpha[x,b]_\gamma.$$

And hence

$$(\mu(x) - x)\alpha[x,b]_\gamma = 0 \text{ for all } x,b \in L, \alpha,\gamma \in \Gamma \quad (3)$$

By Lemma 2.4 replacing b by $2b\beta z$ in (3), we get

$$(\mu(x) - x)\alpha[x, 2b\beta z]_\gamma = 0\,, \text{ for all } x,b,z \in L\ \alpha,\gamma,\beta \in \Gamma. \quad (4)$$

Using the fact that Char A $\neq 2$ and from Remark 2.2 (ii), we get
$(\mu(x) - x)\alpha\, b\beta[x,z]_\gamma + (\mu(x) - x)\alpha\,[x,z]_\gamma\, \beta z = 0.$
Using (3), we get

$$(\mu(x)-x)\alpha\, b\, \beta[x,z]_\gamma = 0, \text{ for all } x,b,z \in L\ \alpha,\gamma,\beta \in \Gamma, \quad (5)$$

so,

$$(\mu(x)-x)\Gamma L\, \Gamma[x,z]_\gamma = 0.$$

From Lemma 2.5 we have

$$(\mu(x)-x) = 0 \text{ or } [x,z]_\gamma = 0.$$

Since μ is not the identity map on L, then $(\mu(x)-x) \neq 0$.

And hence

$$[x,z]_\gamma = 0, \quad \text{for all } x,z \in L, \ \gamma \in \Gamma. \quad (6)$$

So, L is commutative and hence $L \subseteq Z(A)$ by Lemma 2.6.

Corollary 3.6:

Let A be a 2- torsion free prime $\Gamma-$ ring I be an ideal of A suppose that μ is an endomorphism of A and f is aright μ -centralizer such that μ is not the identity map on I if $f([d,e]_\beta) = [d,e]_\beta$, for all $d,e \in I, \beta \in \Gamma$, then A is a commutative.

The proof of the following theorm is similar to Theorem 3.5.

Theorem 3.7:

Let A be a 2- torsion free prime $\Gamma-$ ring and L be a square closed Lie ideal of A such that $b\alpha b \in L$, for all $b \in L, \alpha \in \Gamma$. Suppose that μ is an endomorphism of A and f is a right $\mu-$ centralizer such that μ is not the identity map on L. If $f([b,y]_\beta) = -[b,y]_\beta$, for all $b,y \in L, \beta \in \Gamma$, then $L \subseteq Z(R)$.

Corollary 3.8:

Let A be a 2- torsion free Prime $\Gamma-$ring and I be an ideal of A Suppose that μ is an endomorphism of A and f is aright μ- centralizer such that a is not the identity map on I if $f([c,d]_\gamma)] = -[c,d]_\gamma$ for all $c,d \in I, \gamma \in \Gamma$ then A is commutative.

Theorem 3.9:

Let A be a 2- torsion free prime $\Gamma-$ ring and L be a square closed Lie ideal of A such that $x\beta x \in L$, for all $x \in L$ and $\beta \in \Gamma$. Suppose that μ is an endomorphism of A and f is a right μ-centralizer such that μ is not the identity map on L. If $f(x\alpha b) = x\alpha b$ for all $x,b \in L, \alpha \in \Gamma$ then $L \subseteq Z(A)$.

Proof: By the given hypothesis, we have

$$f(x\alpha b) = x\alpha b, \text{ for all } x,b \in L, \ \alpha \in \Gamma. \quad (1)$$

By using Lemma 2.4 replacing x by $2b\gamma x$ in (1), we get

$$f((2b\gamma x)\alpha b) = (2b\gamma x)\alpha b \text{ for all } x,b \in L, \ \alpha,\gamma \in \Gamma. \quad (2)$$

Using the fact that A is 2- torsion free

$$f((b\gamma x)\alpha b) = (b\gamma x)\alpha b.$$

By Remark 2.2 this can be rewritten as

$$f(b\gamma(x\alpha b) = b\gamma(x\alpha b),$$

which implicit that

$$\mu(b)\gamma f(x\alpha b) = b\gamma(x\alpha b).$$

By using (1), we get

$$\mu(b)\gamma(x\alpha b) = b\gamma(x\alpha b),$$

and hence

$$(\mu(b)-b)\gamma(x\alpha b) = 0, \quad \text{for all } x,b \in L\ \gamma,\alpha,\beta \in \Gamma. \quad (3)$$

Using Lemma 2.4 replacing x by $2x\beta y$ in (3), we get

$$(\mu(b)-b)\gamma(2x\beta y)\alpha b) = 0, \quad \text{for all } x,b,y \in L\ \gamma,\alpha,\beta \in \Gamma. \quad (4)$$

Using the fact that char $A \neq 2$, and from Remark 2.2, we get.

$$(\mu(b) - b)\gamma(x\alpha b)\beta y + (\mu(b) - b)\gamma x\beta[y, b]_\delta = 0\,. \quad (5)$$

Using (3), we get

$(\mu(b) - b)\gamma x\beta[y, b]_\delta = 0,$ for all $x, b, y \in L,\ \gamma, \beta, \delta \in \Gamma,$

so,

$$(\mu(b) - b)\Gamma L\Gamma[y, b]_\delta = 0.$$

By Lemma 2.5, we have either $(\mu(b) - b) = 0$ or $[y, b]_\delta = 0$.

Since μ is not the identity map on L, then $\mu(b) - b \neq 0$.

And hence

$$[y, b]_\delta = 0, \text{ for all } y, b \in L,\ \delta \in \Gamma. \quad (6)$$

So, L is commutative and hence $L \subseteq Z(A)$ by Lemma 2.6.

Corollary 3.10:

Let A be a 2- torsion free prime Γ −ring and I is an ideal of A. Suppose that μ is a right μ-centralizer such that μ is not the identity map on I. If $f(k\delta y) = k\delta y$ for all $k, y \in I,\ \delta \in \Gamma$, then A is a commutative.

The proof of the following theorm is similar to Theorem 3.9.

Theorem 3.11:

Let A be a 2- torsion free prime Γ −ring and L be a square closed Lie ideal of A, such that $x\alpha x \in L$ for all $x \in L$ and $\alpha \in \Gamma$ suppose that μ is an endomorphism of A and f is aright μ-centralizer such that μ is not the identity map on L. If $f(x\delta r) = -x\delta r$ for all $\delta \in \Gamma,\ x, r \in L$, then $L \subseteq Z(A)$.

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