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Some Fixed Point Theorems in Intuitionistic Fuzzy Metric Space

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Abstract
Mathematics and the applied sciences both heavily rely on fixed point (FP) theory. Additionally, this theory has several applications in integral equations and differential equations to guarantee the solutions' existence and uniqueness. FP theory relies mainly on the Banach contraction principle. Since this idea first appeared, it has gained a lot of attention and there has been a lot of development in this field. In this paper, the concept of generalized Kannan-type($\mathcal{GKT}$) mapping is presented in intuitionistic fuzzy metric space(IFM space), and the FP theory is proven. The results contain extensions of FP theory in IFM-space which include the Caccioppoli FP theorem. Additionally, an instance is provided to illustrate the practical significance of the research's results

Keywords: Intuitionistic fuzzy metric, Fixed point theorem, Cauchy sequence, Generalized Kannan-type mappings.

بعض نظريات النقطة الصامدة في الفضاء المتري الضبابي الحدسي

رغد ابراهيم صبري
فرع الرياضيات وتطبيقات الحاسوب, قسم العلوم التطبيقية, الجامعة التكنولوجية, بغداد, العراق

الخلاصة
تعتمد الرياضيات والعلوم التطبيقية بشكل كبير على نظرية النقطة الصامدة. بالإضافة إلى ذلك، فإن لهذه النظرية عدة تطبيقات في المعادلات التكاملية والمعادلات التفاضلية لضمان وجود الحلول وتفردها. تعتمد نظرية النقطة الصامدة بشكل أساسي على مبدأ الانكماش باناخ. منذ ظهور هذه الفكرة لأول مرة، حظيت باهتمام كبير وحدث تطور كبير في هذا المجال. في هذا البحث، تم عرض مفهوم الدوال المعمم من نوع كانان في الفضاء المتري الضبابي الحدسي ، وتم إثبات نظرية النقطة الصامدة. تحتوي النتائج على امتدادات لنظرية النقطة الصامدة في الفضاء المتري الضبابي الحدسي والتي تشمل نظرية النقطة الصامدة كاشيوبولي. بالإضافة إلى ذلك، يتم توفير مثال لتوضيح الأهمية العملية لنتائج البحث.

1. Introduction

Functional analysis is a theoretical field of mathematics that emerged from classical analysis. Currently, functional analytic techniques and conclusions have significant importance in many areas of mathematics and their practical implementations see [1-8]

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A new age of researching FP theory in fuzzy metric spaces (FMS) has begun with the introduction of the idea of these spaces. Numerous authors have developed numerous approaches for FMS, including [9-11]. For instance, Kramosil and Michalek [12] generalized Menger's notion of probabilistic metric spaces to the fuzzy situation in 1975 to propose the notion of FMS. On the other hand, the idea of an intuitionistic fuzzy set was first presented and investigated by Atanassov in 1983. Jin Han Park [13] proposes the definition of IFMspace utilizing the concepts of intuitionistic fuzzy sets. In this space, he also developed a Hausdorff topology and demonstrates how any metric produces an intuitionistic fuzzy metric. Numerous mathematicians, including [14-17], etc., developed several FP theories in the IFM space.

In the present work, FP theory for a G$\mathcal{KT}$ mapping has proven, and Caccioppoli's FP is extended in IFM space.

The paper has the following structure. Following the preliminary content, in Section 3 FP theory for a G$\mathcal{KT}$ mapping has proven on IFM space. After that Caccioppoli's FP is extended in IFM space. The fuzzy Caccioppoli FP theory is supported by the presented example.

2. Preliminaries

This section includes the terms and results that will be used throughout this paper.

Definition 2.1:[18] If a binary operation $\circledast: [0,1] \times [0,1] \to [0,1]$ fulfills the following conditions for all $s, c, d, e \in [0,1]$, then it is called a t-norm:

(i) $1 \circledast c = c$,

(ii) $c \circledast e = e \circledast c$,

(iii) $c \circledast (d \circledast e) = (c \circledast d) \circledast e$,

(iv) If $c \leq e$ and $d \leq s$ then $c \circledast d \leq e \circledast s$.

Definition 2.2:[18] If a binary operation $\odot: [0,1] \times [0,1] \to [0,1]$ fulfills the following conditions for all $s, c, d, e \in [0,1]$, then it is called a t-conorm:

(i) $0 \odot c = c$,

(ii) $c \odot e = e \odot c$,

(iii) $c \odot (d \odot e) = (c \odot d) \odot e$,

(iv) If $c \leq e$ and $d \leq s$ then $c \odot d \leq e \odot s$.

Definition 2.3:[19] An operation $\circledast: \prod_{\mathrm{i}=1}^{\mathrm{n}}[0,1] \to [0,1]$ is continuous t-norm of the n^{th} order if ([0,1],$\circledast$) is commutative topological monoid with

$$c_1 \circledast c_1 \circledast \ldots \circledast c_n \leq d_1 \circledast d_2 \circledast \ldots \circledast d_n$$

whenever $c_i \leq d_i$ for each $c_i, d_i \in [0,1]$; $i = 1, 2, \ldots, n$.

Definition 2.4:[13] A 5-tuple $(L, \tilde{n}, \tilde{m}, \circledast, \odot)$, is termed as IFM space if $\tilde{n}, \tilde{m}$ are fuzzy sets on $L^2 \times (0, \infty)$ fulfill the requirements:

(1) $\tilde{m}(x, y, r) + \tilde{n}(x, y, r) \leq 1; \forall r > 0$ and $x, y \in L$;

(2) $\tilde{m}(x, y, 0) = 0$;

(3) $\tilde{m}(x, y, r) = 1$ if and only if $x = y$;

(4) $\tilde{m}(x, y, r) = \tilde{m}(y, x, r)$;

(5) $\tilde{m}(x, y, r) \circledast \tilde{m}(y, z, s) \leq \tilde{m}(x, z, r + s) \quad \forall r, s > 0$ and $z \in L$;

(6) $\tilde{m}(x, y, .): (0, \infty) \to [0,1]$;

(7) $\lim_{r \to \infty} \tilde{m}(x, y, r)$=1;

(8) $\tilde{n}(x, y, 0) = 1$;

($\tilde{n}_1$) $\tilde{n}(x, y, r) = 0$ if and only if $x = y$;

($\tilde{n}_2$) $\tilde{n}(x, y, r) = \tilde{n}(y, x, r)$;

($\tilde{n}_3$) $\tilde{n}(x, y, r) \odot \tilde{n}(y, z, s) \geq \tilde{n}(x, z, r + s)$;

($\tilde{n}_4$) $\tilde{n}(x, y, .): (0, \infty) \to [0,1]$ is right continuous;
($\tilde{n}_5$) $\lim_{r \to \infty} \tilde{n}(x, y, r) = 0$.
where L is an arbitrary set , $\circledast$ is a continuous t-norm and $\odot$ is a continuous t-conorm.

Definition 2.5:[13] Let $(L, \tilde{n}, \tilde{m}, \circledast, \odot)$ be an IFM space. Then
(1) $\{x_n\}$ is called convergent to $x \in L$ if $\lim_{,n \to \infty} \tilde{m}(x_n,\ x, r) = 1$ and $\lim_{,n \to \infty} \tilde{n}(x_n,\ x, r) = 0$ for all $r > 0$.
(2) $\{x_n\}$ is called Cauchy if $\lim_{m,n \to \infty} \tilde{m}(x_n,\ x_m, r) = 1$ and $\lim_{m,n \to \infty} \tilde{n}(x_n,\ x_m, r) = 0$ for all $r > 0$.

3. Main result

In this part, FP theorem for a $G\mathcal{KT}$ mapping is proven in IFM space. After that Caccioppoli's FP is extended in IFM space.
At first, some notations are introduced which are essential for this current work.

Notation1: Let $\mathfrak{R}_1$ stand for the collection of each function $\hat{\theta}: [0,1] \times [0,1] \to [0,1]$, having the properties:
1) $\hat{\theta}$ is increasing and continuous,
2) $\hat{\theta}(d, d) > d$ for all $0 < d < 1$,
3) $\hat{\theta}(1,1) = 1;\ \hat{\theta}(0,0) = 0$.

Notation2: Let $\mathfrak{R}_2$ stand for the collection of each function $\hat{\theta}: [0,1] \times [0,1] \to [0,1]$, that possess the following properties:
1) $\hat{\theta}$ is decreasing and continuous,
2) $\hat{\theta}(d, d) < d$ where $0 < d < 1$,
3) $\hat{\theta}(1,1) = 1;\ \hat{\theta}(0,0) = 0$.

Definition 3.1: Let $(L, \tilde{n}, \tilde{m}, \circledast, \odot)$ be an IFM space and let $\hat{\theta}_1 \in \mathfrak{R}_1$ and $\hat{\theta}_2 \in \mathfrak{R}_2$. A mapping $\mathbb{f}: L \to L$ is termed as generalized Kannan-type mapping(briefly $G\mathcal{KT}$ mapping) if for every $x, y \in L$

$$\tilde{m}(\mathbb{f}(x), \mathbb{f}(y), t) \geqslant \hat{\theta}_1\left(\tilde{m}\left(x, \mathbb{f}(x), \frac{r}{a}\right), \tilde{m}\left(y, \mathbb{f}(y), \frac{s}{b}\right) \right) \tag{1}$$

and

$$\tilde{n}(\mathbb{f}(x), \mathbb{f}(y), t) \leq \hat{\theta}_2\left(\tilde{n}\left(x, \mathbb{f}(x), \frac{r}{a}\right), \tilde{n}\left(y, \mathbb{f}(y), \frac{s}{b}\right) \right) \tag{2}$$

where $r, s > 0$ and $a, b > 0$ with $t = r + s$ and $0 < a + b < 1$.
Before establishing the main theorem, it is necessary to show the following lemma

Lemma 3.2: Let $(L, v, \tilde{m}, \circledast, \odot)$ be IFM space and $\mathbb{f}$ be a $G\mathcal{KT}$ self-map on L. Let $\lim_{t \to \infty} \tilde{m}(x, y, t) = 1$ and $\lim_{t \to \infty} \tilde{n}(x, y, t) = 0$ for all $x, y \in L$ and $x_n = \mathbb{f}(x_{n-1})$ be an iterative sequence generated by $x_\circ \in L$ for all $n \in Z^+$, then

$$\lim_{n \to \infty} \tilde{m}(x_{n+1}, x_n, t) = 1$$

and

$$\lim_{n \to \infty} \tilde{n}(x_{n+1}, x_n, t) = 0$$

for all $t > 0$.

Proof: Let $x_\circ \in L$, $x_n = \mathbb{f}(x_{n-1})$ and $r, s, a,$ and b be positive real numbers with $0 < a + b < 1$. From the inequality (1), for $t = r + s$:

$$\begin{aligned}\tilde{m}(\mathrm{x}_{n+1},\mathrm{x}_n,t) &= \tilde{m}(\mathbb{f}(\mathrm{x}_n),\mathbb{f}(\mathrm{x}_{n-1}),t)\\ &\geq \hat{\theta}_1\left(\tilde{m}\left(\mathrm{x}_n,\mathbb{f}(\mathrm{x}_n),\frac{r}{a}\right),\tilde{m}\left(\mathrm{x}_{n-1},\mathbb{f}(\mathrm{x}_{n-1}),\frac{s}{b}\right)\right)\\ &= \hat{\theta}_1\left(\tilde{m}\left(\mathrm{x}_n,\mathrm{x}_{n+1},\frac{r}{a}\right),\tilde{m}\left(\mathrm{x}_{n-1},\mathrm{x}_n,\frac{s}{b}\right)\right)\\ &= \hat{\theta}_1\left(\tilde{m}\left(\mathrm{x}_{n+1},\mathrm{x}_n,\frac{r}{a}\right),\tilde{m}\left(\mathrm{x}_n,\mathrm{x}_{n-1},\frac{s}{b}\right)\right)\end{aligned} \tag{3}$$

and

$$\begin{aligned}\tilde{n}(\mathrm{x}_{n+1},\mathrm{x}_n,t) &= \tilde{n}(\mathbb{f}(\mathrm{x}_n),\mathbb{f}(\mathrm{x}_{n-1}),t)\\ &\leq \hat{\theta}_2\left(\tilde{n}\left(\mathrm{x}_n,\mathbb{f}(\mathrm{x}_n),\frac{r}{a}\right),\tilde{n}\left(\mathrm{x}_{n-1},\mathbb{f}(\mathrm{x}_{n-1}),\frac{s}{b}\right)\right)\\ &= \hat{\theta}_2\left(\tilde{n}\left(\mathrm{x}_n,\mathrm{x}_{n+1},\frac{r}{a}\right),\tilde{n}\left(\mathrm{x}_{n-1},\mathrm{x}_n,\frac{s}{b}\right)\right)\\ &= \hat{\theta}_2\left(\tilde{n}\left(\mathrm{x}_{n+1},\mathrm{x}_n,\frac{r}{a}\right),\tilde{n}\left(\mathrm{x}_n,\mathrm{x}_{n-1},\frac{s}{b}\right)\right)\end{aligned} \tag{4}$$

for all $t > 0$, putting $r = \frac{at}{a+b}$, $s = \frac{bt}{a+b}$ and $c = a + b$ in (3), obtain:

$$\tilde{m}(\mathrm{x}_{n+1},\mathrm{x}_n,t) \geq \hat{\theta}_1\left(\tilde{m}\left(\mathrm{x}_{n+1},\mathrm{x}_n,\frac{t}{c}\right),\tilde{m}\left(\mathrm{x}_n,\mathrm{x}_{n-1},\frac{t}{c}\right)\right)$$

and (5)

$$\tilde{n}(\mathrm{x}_{n+1},\mathrm{x}_n,t) \leq \hat{\theta}_2\left(\tilde{n}\left(\mathrm{x}_{n+1},\mathrm{x}_n,\frac{t}{c}\right),\tilde{n}\left(\mathrm{x}_n,\mathrm{x}_{n-1},\frac{t}{c}\right)\right).$$

Now to demonstrate that the following inequality is valid:

$$\tilde{m}\left(\mathrm{x}_{n+1},\mathrm{x}_n,\frac{t}{c}\right) \geq \tilde{m}\left(\mathrm{x}_n,\mathrm{x}_{n-1},\frac{t}{c}\right) \quad\text{and}\quad \tilde{n}\left(\mathrm{x}_{n+1},\mathrm{x}_n,\frac{t}{c}\right) \leq \tilde{n}\left(\mathrm{x}_n,\mathrm{x}_{n-1},\frac{t}{c}\right) \tag{6}$$

for all $t > 0;\ n \in \mathrm{Z}^+$.

The proof will be done by contradiction, assuming that there is $t > 0$ with
$\tilde{m}\left(\mathrm{x}_{n+1},\mathrm{x}_n,\frac{t}{c}\right) < \tilde{m}\left(\mathrm{x}_n,\mathrm{x}_{n-1},\frac{t}{c}\right)$ and $\tilde{n}\left(\mathrm{x}_{n+1},\mathrm{x}_n,\frac{t}{c}\right) > \tilde{n}\left(\mathrm{x}_n,\mathrm{x}_{n-1},\frac{t}{c}\right)$.

By properties of $\hat{\theta}_1$ and the inequality (5), obtain:

$$\begin{aligned}\tilde{m}(\mathrm{x}_{n+1},\mathrm{x}_n,t) &\geq \hat{\theta}_1\left(\tilde{m}\left(\mathrm{x}_{n+1},\mathrm{x}_n,\frac{t}{c}\right),\tilde{m}\left(\mathrm{x}_n,\mathrm{x}_{n-1},\frac{t}{c}\right)\right)\\ &\geq \hat{\theta}_1\left(\tilde{m}\left(\mathrm{x}_{n+1},\mathrm{x}_\mathrm{n},\frac{t}{\mathrm{c}}\right),\tilde{m}\left(\mathrm{x}_{n+1},\mathrm{x}_n,\frac{t}{\mathrm{c}}\right)\right)\\ &> \tilde{m}\left(\mathrm{x}_{n+1},\mathrm{x}_n,\frac{t}{\mathrm{c}}\right)\\ &> \tilde{m}(\mathrm{x}_{n+1},\mathrm{x}_n,t)\end{aligned}$$

and by properties of $\hat{\theta}_2$ and the inequality (5), get:

$$\begin{aligned}\tilde{n}(\mathrm{x}_{n+1},\mathrm{x}_n,t) &\leq \hat{\theta}_2\left(\tilde{n}\left(\mathrm{x}_{n+1},\mathrm{x}_n,\frac{t}{c}\right),\tilde{n}\left(\mathrm{x}_n,\mathrm{x}_{n-1},\frac{t}{c}\right)\right)\\ &\leq \hat{\theta}_2\left(\tilde{n}\left(\mathrm{x}_{n+1},\mathrm{x}_n,\frac{t}{\mathrm{c}}\right),\tilde{n}\left(\mathrm{x}_{n+1},\mathrm{x}_n,\frac{t}{\mathrm{c}}\right)\right)\\ &< \tilde{n}\left(\mathrm{x}_{n+1},\mathrm{x}_n,\frac{t}{\mathrm{c}}\right)\\ &< \tilde{n}(\mathrm{x}_{n+1},\mathrm{x}_n,t).\end{aligned}$$

Thus, a contradiction exists. Therefore, inequalities (5) and (6) imply that the required inequality is as follows:

$$\tilde{m}(\mathrm{x}_{n+1},\mathrm{x}_n,t) \geq \tilde{m}\left(\mathrm{x}_n,\mathrm{x}_{n-1},\frac{t}{\mathrm{c}}\right) \quad\text{and}\quad \tilde{n}(\mathrm{x}_{n+1},\mathrm{x}_n,t) \leq \tilde{n}\left(\mathrm{x}_n,\mathrm{x}_{n-1},\frac{t}{\mathrm{c}}\right)$$

for $t > 0; n \in \mathrm{Z}^+$.

When applying the process of induction to the inequality stated above, observe that

$$\tilde{m}(\mathrm{x}_{n+1},\mathrm{x}_n,t) \geq \tilde{m}\left(\mathrm{x}_1,\mathrm{x}_0,\frac{t}{\mathrm{c}^n}\right) \quad\text{and}\quad \tilde{n}(\mathrm{x}_{n+1},\mathrm{x}_n,t) \geq \tilde{n}\left(\mathrm{x}_1,\mathrm{x}_0,\frac{t}{\mathrm{c}^n}\right) \quad n \in \mathrm{Z}^+$$

Additional assumption on intuitionistic fuzzy metric implies that $\lim_{n\to\infty} \tilde{m}\left(x_1, x_0, \frac{t}{c^n}\right) = 1$ and $\lim_{n\to\infty} \tilde{n}\left(x_1, x_0, \frac{t}{c^n}\right) = 0$. Therefore, by evaluating the limit as n tends to infinity, it is possible to derive that

$\lim_{n\to\infty} \tilde{m}(x_{n+1}, x_n, t) = 1$ and $\lim_{n\to\infty} \tilde{n}(x_{n+1}, x_n, t) = 0$.
The results of this study demonstrate in the following.

Theorem 3.3: Let $(L, \tilde{n}, \tilde{m}, \circledast, \odot)$ be a complete IFM space, such that
(i) $\circledast$ is the 3-rd order t –norm(minimum) and $\odot$ is the 3-rd order t-conorm (maximum)
(ii) $\lim_{t\to\infty} \tilde{m}(x, y, t) = 1$ and $\lim_{t\to\infty} \tilde{n}(x, y, t) = 0$ for all $x, y \in L$,
(iii) $\mathbb{f}: L \to L$ be a GKT mapping.
Then $\mathbb{f}$ possesses a unique FP.

Proof: Consider $x_\circ \in L$, $x_n = \mathbb{f}(x_{n-1})$ that was generated in the previous lemma. In order to establish that $\{x_n\}$ is a Cauchy. Assuming it is not, hence by definition, $\exists\ \epsilon$ where $0 < \epsilon < 1$ for which find $t > 0$ and subsequences $\{x_{m_{(k)}}\}$ and $\{x_{n_{(k)}}\}$ of $\{x_n\}$ with $n_{(k)} > m_{(k)} > k$ for all positive integers k such that $\tilde{m}\left(x_{m_{(k)}}, x_{n_{(k)}}, t\right) \le 1 - \varepsilon$ and $\tilde{n}\left(x_{m_{(k)}}, x_{n_{(k)}}, t\right) \ge \varepsilon$.
So, for all $r, s > 0$ with $t = r + s$ and $a, b > 0$ with $0 < a + b < 1$, obtain :

$$\begin{aligned}1 - \varepsilon &\ge \tilde{m}\left(x_{m_{(k)}}, x_{n_{(k)}}, t\right)\\ &= \tilde{m}\left(\mathbb{f}(x_{m_{(k)-1}}), \mathbb{f}(x_{n_{(k)-1}}), t\right)\\ &\ge \hat{\theta}_1(\tilde{m}\left(x_{m_{(k)-1}}, \mathbb{f}(x_{m_{(k)-1}}), \frac{r}{a}\right), \tilde{m}\left(x_{n_{(k)-1}}, \mathbb{f}(x_{n_{(k)-1}}), \frac{s}{b}\right))\\ &\ge \hat{\theta}_1\left(\tilde{m}\left(x_{m_{(k)-1}}, x_{m_{(k)}}, \frac{r}{a}\right), \tilde{m}\left(x_{n_{(k)-1}}, x_{n_{(k)}}, \frac{s}{b}\right)\right).\end{aligned}$$

Therefore,

$$1 - \varepsilon \ge \hat{\theta}_1(\tilde{m}\left(x_{m_{(k)-1}}, x_{m_{(k)}}, \frac{r}{a}\right), \tilde{m}\left(x_{n_{(k)-1}}, x_{n_{(k)}}, \frac{s}{b}\right)) \quad \text{where } \hat{\theta}_1 \in \Re_1, \tag{7}$$

and

$$\begin{aligned}\varepsilon &\le \tilde{n}\left(x_{m_{(k)}}, x_{n_{(k)}}, t\right)\\ &= \tilde{n}\left(\mathbb{f}(x_{m_{(k)-1}}), \mathbb{f}(x_{n_{(k)-1}}), t\right)\\ &\le \hat{\theta}_2(\tilde{n}\left(x_{m_{(k)-1}}, \mathbb{f}(x_{m_{(k)-1}}), \frac{r}{a}\right), \tilde{n}\left(x_{n_{(k)-1}}, \mathbb{f}(x_{n_{(k)-1}}), \frac{s}{b}\right))\\ &\le \hat{\theta}_2\left(\tilde{n}\left(x_{m_{(k)-1}}, x_{m_{(k)}}, \frac{r}{a}\right), \tilde{n}\left(x_{n_{(k)-1}}, x_{n_{(k)}}, \frac{s}{b}\right)\right).\end{aligned}$$

Therefore,

$$\varepsilon \le \hat{\theta}_2(\tilde{n}\left(x_{m_{(k)-1}}, x_{m_{(k)}}, \frac{r}{a}\right), \tilde{n}\left(x_{n_{(k)-1}}, x_{n_{(k)}}, \frac{s}{b}\right)) \quad \text{where } \hat{\theta}_2 \in \Re_2. \tag{8}$$

By Lemma 3.2, for all $t > 0$,
$\lim_{n\to\infty} \tilde{m}(x_{n+1}, x_n, t) = 1$ and $\lim_{n\to\infty} \tilde{n}(x_{n+1}, x_n, t) = 0$.
So, it can choose k large enough such that

$$\tilde{m}\left(x_{m_{(k)-1}}, x_{m_{(k)}}, \frac{r}{a}\right) > 1 - \varepsilon \text{ and } \tilde{m}\left(x_{n_{(k)-1}}, x_{n_{(k)}}, \frac{s}{b}\right) > 1 - \varepsilon \tag{9}$$

and

$$\tilde{n}\left(x_{m_{(k)-1}}, x_{m_{(k)}}, \frac{r}{a}\right) < \varepsilon \text{ and } \tilde{n}\left(x_{n_{(k)-1}}, x_{n_{(k)}}, \frac{s}{b}\right) < \varepsilon. \tag{10}$$

Therefore, from (7), (8), (9), (10) and the definition of $\hat{\theta}_1$ and $\hat{\theta}_2$ it is inferred that,

$1-\varepsilon \geq \hat{\theta}_1(1-\varepsilon, 1-\varepsilon) > 1-\varepsilon$ and $\varepsilon \leq \hat{\theta}_2(\varepsilon, \varepsilon) < \varepsilon$

which is a contradiction. Hence, $\{x_n\}$ is a Cauchy and the completeness of IFM space indicates that $\lim_{n\to\infty} x_n = x$ for some $x \in X$.

Now, to assert that x is FP. Given the assumption that it is not, therefore there is $t > 0$ with $0 < \tilde{m}(x, \mathbb{f}(x), t) < 1$.

Since $0 < b < 1$, it can be choose $\rho_1, \rho_2, r, s > 0$ such that

$$t = \rho_1 + \rho_2 + r + s \text{ and } \frac{s}{b} > t. \tag{11}$$

Then

$$\tilde{m}(x, \mathbb{f}(x), t) \geq \tilde{m}(x, x_n, \rho_1) \circledast \tilde{m}(x_n, x_{n+1}, \rho_2) \circledast \tilde{m}(x_{n+1}, \mathbb{f}(x), r+s)$$

$$\geq \tilde{m}(x, x_n, \rho_1) \circledast \tilde{m}(x_n, x_{n+1}, \rho_2) \circledast \hat{\theta}_1(\tilde{m}\left(x_n, \mathbb{f}(x_n), \frac{r}{a}\right), \tilde{m}\left(x, \mathbb{f}(x), \frac{s}{b}\right))$$

$$\geq \tilde{m}(x, x_n, \rho_1) \circledast \tilde{m}(x_n, x_{n+1}, \rho_2) \circledast \hat{\theta}_1(\tilde{m}\left(x_n, x_{n+1}, \frac{r}{a}\right), \tilde{m}\left(x, \mathbb{f}(x), \frac{s}{b}\right)) \tag{12}$$

and

$$\tilde{n}(x, \mathbb{f}(x), t) \leq \tilde{n}(x, x_n, \rho_1) \odot \tilde{n}(x_n, x_{n+1}, \rho_2) \odot \tilde{n}(x_{n+1}, \mathbb{f}(x), r+s)$$

$$\leq \tilde{n}(x, x_n, \rho_1) \odot \tilde{n}(x_n, x_{n+1}, \rho_2) \odot \hat{\theta}_2(\tilde{n}\left(x_n, \mathbb{f}(x_n), \frac{r}{a}\right), \tilde{n}\left(x, \mathbb{f}(x), \frac{s}{b}\right))$$

$$\leq \tilde{n}(x, x_n, \rho_1) \odot \tilde{n}(x_n, x_{n+1}, \rho_2) \odot \hat{\theta}_2(\tilde{n}\left(x_n, x_{n+1}, \frac{r}{a}\right), \tilde{n}\left(x, \mathbb{f}(x), \frac{s}{b}\right))$$

where $\hat{\theta}_1 \in \mathfrak{R}_1$ and $\hat{\theta}_2 \in \mathfrak{R}_2$.

According to Lemma 3.2 and the convergence of $\{x_n\}$, there is $N1$ (positive integer) such that for each $n > N1$,

$$\tilde{m}(x, x_n, \rho_1) \circledast \tilde{m}(x_n, x_{n+1}, \rho_2) \circledast \tilde{m}\left(x_n, x_{n+1}, \frac{r}{a}\right) > \tilde{m}(x, \mathbb{f}(x), t),$$

$$\tilde{n}(x, x_n, \rho_1) \odot \tilde{n}(x_n, x_{n+1}, \rho_2) \odot \tilde{n}\left(x_n, x_{n+1}, \frac{r}{a}\right) < \tilde{n}(x, \mathbb{f}(x), t)$$

Then from (11) and (12), it follows that,

$$\tilde{m}(x, \mathbb{f}(x), t) > \tilde{m}(x, \mathbb{f}(x), t) \circledast \hat{\theta}_1(\tilde{m}(x, \mathbb{f}(x), t), \tilde{m}\left(x, \mathbb{f}(x), \frac{s}{b}\right))$$

$$\geq \tilde{m}(x, \mathbb{f}(x), t) \circledast \hat{\theta}_1(\tilde{m}(x, \mathbb{f}(x), t), \tilde{m}(x, \mathbb{f}(x), t))$$

$$\geq \tilde{m}(x, \mathbb{f}(x), t)$$

and

$$\tilde{n}(x, \mathbb{f}(x), t) > \tilde{n}(x, \mathbb{f}(x), t) \odot \hat{\theta}_2(\tilde{n}(x, \mathbb{f}(x), t), \tilde{n}\left(x, \mathbb{f}(x), \frac{s}{b}\right))$$

$$\geq \tilde{n}(x, \mathbb{f}(x), t) \odot \hat{\theta}_2(\tilde{n}(x, \mathbb{f}(x), t), \tilde{n}(x, \mathbb{f}(x), t))$$

$$\geq \tilde{n}(x, \mathbb{f}(x), t)$$

which is a contradiction.

Hence, $\tilde{m}(x, \mathbb{f}(x), t) = 1$ and $\tilde{n}(x, \mathbb{f}(x), t) = 0$ for all $t > 0$, therefore, x is a FP for $\mathbb{f}$. Assume that $\mathbb{f}$ admits two FPs x and u. In light of the aforementioned assertions about on a, b, and r, s, for each $t > 0$, obtain:

$$\tilde{m}(x, u, t) = \tilde{m}(\mathbb{f}(x), \mathbb{f}(u), t)$$

$$\geq \hat{\theta}_1(\tilde{m}\left(x, \mathbb{f}(x), \frac{r}{a}\right), \tilde{m}\left(u, \mathbb{f}(u), \frac{s}{b}\right))$$

$$= \hat{\theta}_1(\tilde{m}\left(x, x, \frac{r}{a}\right), \tilde{m}\left(u, u, \frac{s}{b}\right))$$

$$= \hat{\theta}_1(1,1) = 1,$$

and

$$\tilde{n}(x, u, t) = \tilde{n}(\mathbb{f}(x), \mathbb{f}(u), t)$$

$$\geq \hat{\theta}_2(\tilde{n}\left(x, \mathbb{f}(x), \frac{r}{a}\right), \tilde{n}\left(u, \mathbb{f}(u), \frac{s}{b}\right))$$

$$= \hat{\theta}_2(\tilde{n}\left(x,x,\frac{r}{a}\right),\tilde{n}\left(u,u,\frac{s}{b}\right))= \hat{\theta}_1(0,0) = 0.$$

Thus x $=$ u.

The Caccioppoli FP theorem in IFM space is now stated and shown.

Theorem 3.4: Assume that $(L,\tilde{n},\widetilde{m},\circledast,\odot)$ is complete IFM space and $\mathbb{T}: L \to L$ is a mapping satisfies:

For any positive integer n and $t > 0$,

$\widetilde{m}(\mathbb{T}^n x,\mathbb{T}^n y,k_n t) \geq \widetilde{m}(x,y,t)$

and $\tilde{n}(\mathbb{T}^n x,\mathbb{T}^n y,k_n t) \leq \tilde{n}(x,y,t)$, (13)

for all $x,y \in L$, $k_n > 0$ being independent of x, y. If $k_n \to 0$, then $\mathbb{T}$ possesses a unique FP in X.

proof: Assume $x \in L$; $x_n = \mathbb{T}^n x$; $n \in N$. Now, $\{x_n\}$ is a sequence of points of L such that $x_1 = \mathbb{T}x$, $x_2 = \mathbb{T}x_1, \dots, x_{n+1} = \mathbb{T}x_n$; $n \in N$.

$$1 \geq \widetilde{m}\left(x_n,x_{n+p},t\right)$$

$$1 \geq \widetilde{m}\left(x_n,x_{n+1},\frac{t}{p}\right) \circledast \widetilde{m}\left(x_{n+1},x_{n+2},\frac{t}{p}\right) \circledast \dots \circledast \widetilde{m}\left(x_{n+p-1},x_{n+p},\frac{t}{p}\right)$$

$$\geq \widetilde{m}\left(x,x_1,\frac{t}{pk_n}\right) \circledast \widetilde{m}\left(x,x_1,\frac{t}{pk_{n+1}}\right) \circledast \dots \circledast \widetilde{m}\left(x,x_1,\frac{t}{pk_{n+p-1}}\right) \quad \text{by (13)}$$

and

$$0 \leq \tilde{n}\left(x_n,x_{n+p},t\right)$$

$$0 \leq \tilde{n}\left(x_n,x_{n+1},\frac{t}{p}\right) \odot \tilde{n}\left(x_{n+1},x_{n+2},\frac{t}{p}\right) \odot \dots \odot \tilde{n}\left(x_{n+p-1},x_{n+p},\frac{t}{p}\right)$$

$$\leq \tilde{n}\left(x,x_1,\frac{t}{pk_n}\right) \odot \tilde{n}\left(x,x_1,\frac{t}{pk_{n+1}}\right) \odot \dots \odot \tilde{n}\left(x,x_1,\frac{t}{pk_{n+p-1}}\right) \quad \text{by (13)}$$

$\lim_{n\to\infty} \widetilde{m}\left(x_n,x_{n+p},t\right)=1$ and $\lim_{n\to\infty} \tilde{n}\left(x_n,x_{n+p},t\right) = 0$ as $n \to \infty$

for all $t > 0, p > 0$ so $\{x_n\}$ is a Cauchy. Because L is complete there is $y \in L$ with $x_n \to y$ as $n \to \infty$. Thus,

$$1 \geq \widetilde{m}\left(y,\mathbb{T}y,t\right) \geq \widetilde{m}\left(y,x_{n+1},\frac{t}{2}\right) \circledast \widetilde{m}\left(x_{n+1},\mathbb{T}y,\frac{t}{2}\right)$$

$$\geq \widetilde{m}\left(x_{n+1},y,\frac{t}{2}\right) \circledast \widetilde{m}\left(x_n,y,\frac{t}{2k_1}\right) \quad \text{by (13),}$$

and

$$0 \leq \tilde{n}\left(y,\mathbb{T}y,t\right) \leq \tilde{n}\left(y,x_{n+1},\frac{t}{2}\right) \odot \tilde{n}\left(x_{n+1},\mathbb{T}y,\frac{t}{2}\right)$$

$$\leq \tilde{n}\left(x_{n+1},y,\frac{t}{2}\right) \odot \tilde{n}\left(x_n,y,\frac{t}{2k_1}\right) \quad \text{by (13).}$$

As $n \to \infty$ for all $t > 0$, $\widetilde{m}\left(y,\mathbb{T}y,t\right) = 1$ and $\tilde{n}\left(y,\mathbb{T}y,t\right) = 0$. Thus $\mathbb{T}y = y$ a FP of $\mathbb{T}$.

To demonstrate uniqueness, consider $w \in L$ such that $\mathbb{T}w = w$. Get that $\mathbb{T}^n y = y$, $\mathbb{T}^n w = w$ for all $n \in N$.

Now,

$$1 \geq \widetilde{m}\left(y,w,t\right) \geq \widetilde{m}\left(\mathbb{T}^n y,\mathbb{T}^n w,t\right) \geq \widetilde{m}\left(y,w,\frac{t}{k_n}\right)$$

and,

$$0 \leq \tilde{n}\left(y,w,t\right) \leq \tilde{n}\left(\mathbb{T}^n y,\mathbb{T}^n w,t\right) \leq \tilde{n}\left(y,w,\frac{t}{k_n}\right)$$

As $n \to \infty$ for all $t > 0$, obtain that $\widetilde{m}(y,\mathbb{T}y,t) = 1$ and $\tilde{n}\left(y,\mathbb{T}y,t\right) = 0$. Thus $y = w$.

Example 3.5: Let $L = [0,1]$ and $\vartheta(\mathrm{x}, y) = |\mathrm{x} - y|$ for every $\mathrm{x}, y \in L$. Then (L, ϑ) is a complete metric space. Consider m *and* ñ to be a fuzzy set in $L^2 \times (0, \infty)$ specified by:
$\widetilde{m}(\mathrm{x}, y, t) = \frac{t}{t+\vartheta(\mathrm{x},y)}$ and $\tilde{n}(\mathrm{x}, y, t) = \frac{\vartheta(\mathrm{x},y)}{t+\vartheta(\mathrm{x},y)}$ if $t > 0$ and $\widetilde{m}(\mathrm{x}, y, 0) = 0$ with $a \circledast b = \min\{a, b\}$ and $a \odot b = \max\{a, b\}$ for every $a, b \in [0,1]$.
$(L, \tilde{n}, \widetilde{m}, \circledast, \odot)$ is a complete IFM space[20]. Let $\mathbb{T}: \mathrm{x} \to \mathrm{x}$ be given by $\mathbb{T}\mathrm{x} = \frac{\mathrm{x}}{5}$ for each $\mathrm{x} \in L$.
Now,

$$\widetilde{m}(\mathbb{T}^n\mathrm{x};\ \mathbb{T}^n y;\ k^n t) = \frac{k^n t}{k^n t + \vartheta(\mathbb{T}^n\mathrm{x}, \mathbb{T}^n y)}$$

$$= \frac{\frac{t}{2^n}}{\frac{t}{2^n} + |\mathbb{T}^n\mathrm{x} - \mathbb{T}^n y|} \text{ with } k = \frac{1}{2}$$

$$= \frac{t}{t + (\frac{2}{5})^n|\mathrm{x} - y|} \geq \frac{t}{t + \vartheta(\mathrm{x}, y)} = \widetilde{m}(\mathrm{x}, y, t)$$

for every $\mathrm{x}, y \in L, t > 0, n > 0$.
and

$$\tilde{n}(\mathbb{T}^n\mathrm{x};\ \mathbb{T}^n y;\ k^n t) = \frac{\vartheta(\mathbb{T}^n\mathrm{x}, \mathbb{T}^n y)}{k^n t + \vartheta(\mathbb{T}^n\mathrm{x}, \mathbb{T}^n y)}$$

$$= \frac{|\mathbb{T}^n\mathrm{x} - \mathbb{T}^n y|}{\frac{t}{2^n} + |\mathbb{T}^n\mathrm{x} - \mathbb{T}^n y|} \text{ with } k = \frac{1}{2}$$

$$= \frac{|\mathrm{x} - y|}{(\frac{5}{2})^n\ t + |\mathrm{x} - y|} \leq \frac{\vartheta(\mathrm{x}, y)}{t + \vartheta(\mathrm{x}, y)} = \tilde{n}(\mathrm{x}, y, t)$$

Also, $k^n = \frac{1}{2^n} \to 0$. As a result, the requirements of Theorem 3.4 are fulfilled. The unique FP of $\mathbb{T}$ is zero.

4. Conclusions

This study presents the idea of G$\mathcal{KT}$ mappings in the IFM space. The existence of FP theorem in IFM space is then proven. After that, in the same space, Caccioppoli's FP theorem is proved and a specific example is given to highlight the advantages of the outcomes.

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