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# The Impact of Fear and Anti-Predator Behavior on the Dynamics of Stage-Structure Prey–Predator Model With a Harvesting

**Azhar Abbas Majeed**
*Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq*

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**Abstract**
In this paper, an ecological model with stage-structure in prey population, fear, anti-predator and harvesting are suggested. Lotka-Volterra and Holling type II functional responses have been assumed to describe the feeding processes . The local and global stability of steady points of this model are established. Finally, the global dynamics are studied numerically to investigate the influence of the parameters on the solutions of the system, especially the effect of fear and anti-predation.

**Keywords**: Ecological model, Prey-predator model, Fear, Anti-predator.

# تأثير الخوف وسلوك مضاد الافتراس على ديناميكيات نموذج فريسة بمراحل عمرية – ومفترس بوجود الحصاد

**أزهار عباس مجيد**
قسم الرياضيات , كلية العلوم, جامعة بغداد, بغداد, العراق

**الخلاصة**
في هذا البحث تم اقتراح ودراسة نظام بيئي متضمن وجود مراحل عمرية في مجتمع الفريسة ، الخوف , مضاد الافتراس والحصاد ، تم افتراض ووصف عمليات التغذية وفق دوال استجابة من نوع لوتيكا فولتيرا وهولنك النوع الثاني. نقاط الاتزان و الاستقرارية (المحلية والشاملة ) لهذا النظام حققت , اخيرا الديناميكيات الشاملة درست عددياً لمعرفة تأثير المعلمات خاصة الخوف ومضاد الافتراس على حلول النظام .

## 1. Introduction:

In a predation world, there is a natural biological interaction between many organisms, in which a living being is killed, another being becomes its prey, and when the prey is discovered, the predator evaluates whether he must snatch it, as this may include a trick that is waiting for him. Sometimes predation occurs several times and then prey is pursued until the prey is killed by the predator completely, then any inedible parts are removed, and predators are very specialized through a number of sharp senses for instance hearing, vision or pungent odor, and in the world of predation, animals, Vertebrate (strong) and invertebrate (weak) alike, they have jaws or sharp claws to holding and slicing their prey, and predation has a strong selective influence on prey, like prey and predator in an arms race, predation has been a major driver of evolution, since the beginning of the geological ages, since at least the Cambrian period.

Obviously, if we were asked about the meaning of predation, we would answer with the encyclopedic definition of the word, which states that predation is the process of attack by a predator on alive prey and damage it, and its time does not last long. Moreover, the confrontation between the predator and the prey does not last long. In general, a predator is not always be an animal, but it can be a plant that captures small insects to feed on. Most of the predators are carnivores. Specifically, they include vegetable and carnivorous organisms on their list. In some situations, predation may not be the killing of alive prey, but rather, a predator may swallow the prey, while it is alive, and as long as it swallows it completely, it can swallow part of it and cast other parts.

There are many means that animals use to protect themselves from predators examples include the release of toxic substances. For instance, we find some animals, such as the millipede, circumvent themselves when exposed to attack, and secrete a substance that causes sensitivity to the eye and skin of the attacker, stealth and identification with the surrounding environment: Examples of such animals, cancer spider, which has the shape and color of the flower petals, which makes it difficult to see when standing still on the flower, warning using bright body colors: One of the most prominent examples of this kind of animal is the poisonous strawberry frog, whose colors are bright; to warn predators, and to alert them to its toxicity. Simulation of poisonous animals occurs when some animals may be non-poisonous, but they color in a similar way to poisonous animals, so that they protect themselves in case they have the same predators, and examples of that include some types of insects that imitate or emulate the color of the wasp, or the bees [1-7].

Growth is the stages of fertilizing the egg and the stages of its development to produce a small child that goes through different age stages until it is able to reproduce again. The embryo's journey begins with the fertilized egg in the womb to turn into a fully developed fetus, but the stages of the development of the fetus differ in animals from humans. It gives birth to a baby, called baby animals, and some of them lay eggs, such as poultry and reptiles. Also, the process of pollinating an egg differs. Among animals, what fertilizes the egg internally, including what is pollinated after the female lays eggs, as in fish female begin the rapid growth process into mature organisms able to reproduce again. Therefore, the age factor has an important influence on the level of growth and reproduction, see [8-12].

Another important factor that plays an important role in the life of living beings is fear which is the feeling caused by the threat to which living organisms are exposed and in turn it causes a change in the body's organic and metabolic functions and ultimately leads to a change in performance such as flight, freezing, or hiding toward painful procedures that the individual envisions, Fear is closely linked to tension, but we must differentiate between fear and tension, which happens as a result of dangers that are seen as not controllable or inescapable [4-7]. The fear reply helps the instinct to survive by generating appropriate behavioral responses in organisms. Zhang et.al [7], proposed an ecological model to examine the fear effect on the prey-predator model incorporating refuge for prey.

In addition to all of the above, there are many models that have taken into account the harvest [13-16].

In this paper, an ecological model with a stage-structure of prey with fear and anti-predator involving harvesting of all species has been examined. Two types of functional response are assumed in this model Lotka-Volterra and Holling type II for feeding processes . The local and global stability of this model are studied analytically and numerically.

## 2. Mathematical Model

Consider the following ecological model

$$\frac{dM_1}{dt} = \frac{u_1 M_2}{1 + LM_3}(1 - M_2) - u_2 M_1 - \frac{u_3 M_1 M_3}{u_4 + M_1} - u_5 M_1,$$
$$\frac{dM_2}{dt} = u_2 M_1 - u_6 M_2 M_3 - u_7 M_2, \qquad 1$$
$$\frac{dM_3}{dt} = \frac{u_8 M_1 M_3}{u_4 + M_1} + u_9 M_2 M_3 - u_{10} M_2 M_3 - u_{11} M_3 - u_{12} M_3.$$

Interpretation of model parameters are explained in the following table :

**Table 1:** Interpretation of the parameters and variables

| Parameter | Representation |
|---|---|
| $M_1(t)$ | The number of juvenile prey individuals at time $t$ |
| $M_2(t)$ | The number of adult prey individuals at time $t$ |
| $M_3(t)$ | The number of predator individuals at time $t$ |
| $u_1$ | The growth rate of juvenile prey |
| $L$ | Fear rate |
| $\boldsymbol{u_2}$ | Grown up rate |
| $u_3$ | The rate of attacks on the juvenile prey |
| $u_4$ | Half saturation rate |
| $u_5$ | The juvenile prey harvest rate |
| $u_6$ | *The rate of attacks on the adult prey* |
| $u_7$ | The harvesting rate of adult prey |
| $u_i\ i = 8,9,10$ | The uptake rates of food from the juvenile , adult prey, and the anti-predator rate of adult prey respectively |
| $u_i, i = 11,12$ | Predator disease rate in the absence of food and the predator harvest rate respectively |

**Theorem (2.1):** The solutions of model $(1)$ are uniformly bounded.

**Proof**.

Let $(M_1(t), M_2(t), M_3(t))$ be a solution of model (1) with initial condition $(M_1(0), M_2(0), M_3(0))$. Let $Z(t) = M_1(t) + M_2(t) + M_3(t)$.

Therefore,

$$\frac{dZ}{dt} < \mathrm{u}_1 M_2(1 - M_2) - (u_3 - u_8)\frac{M_1 M_3}{u_4 + M_1} - (u_6 - u_9)M_2 M_3 - u_5 M_1 - u_7 M_2$$
$$-(u_{11} + u_{12})M_3.$$

Now, hence from the natural facts $u_8 < u_3$, and $u_9 < u_6$ , thus

$$\frac{dZ}{dt} \le \frac{\mathrm{u}_1}{4} - mZ, \qquad \text{where } \mathrm{m} = \min\{u_5, u_7, u_{11} + u_{12}\}.$$

Now, by the comparison theorem [10], we get:

$$Z(t) \le \frac{\mathrm{u}_1}{4m} + \left(Z(0) - \frac{\mathrm{u}_1}{4m}\right) e^{-mt} \quad .$$

Thus $0 \le Z(t) \le \frac{\mathrm{u}_1}{4m}$ as $t \to \infty$, and the proof is complete ■

## 3. The equilibrium points (EPs):

System (1) has three steady points as explained below.

**1)** The trivial EP $E_0 = (0,0,0)$ always exist.

**2)** The EP $E_2 = (\bar{M}_1, \bar{M}_2, 0)$, where:

$\bar{M}_1 = \frac{u_7}{u_1}\left[\frac{u_1u_2-(u_2+u_5)u_7}{u_1u_2+(u_2+u_5)u_7}\right]$, and $\bar{M}_2 = \frac{u_1u_2-(u_2+u_5)u_7}{u_1u_2+(u_2+u_5)u_7}$ exist provided that:

$$u_1u_2 < (u_2+u_5)u_7 . \quad (3.2a)$$

**3)** The EP $E_3 = (M_1^*, M_2^*, M_3^*)$ exists if and only if :

$$\frac{u_1M_2}{1+LM_3}(1-M_2) - u_2M_1 - \frac{u_3M_1M_3}{u_4+M_1} - u_5M_1 = 0 \quad (3.3a)$$

$$u_2M_1 - u_6M_2M_3 - u_7M_2 = 0 \quad (3.3b)$$

$$\frac{u_8M_1M_3}{u_4+M_1} + u_9M_2M_3 - u_{10}M_2M_3 - u_{11}M_3 - u_{12}M_3 = 0 \quad (3.3c)$$

From Eq. $(3.3c)$, we get,

$$M_2 = \frac{g_2M_1+u_4g_3}{u_4g_1+g_1M_1}. \quad (3.3d)$$

By replacing Eq. (3.3d) in Eq. (3.3b) it follows:

$$M_3 = \frac{u_2g_1M_1^2 + \{u_2u_4g_1 - u_7g_2\}M_1 - u_4u_7g_3}{u_6g_2M_1 + u_4u_6g_3}. \quad (3.3e)$$

Also, from Eqs. (3.3d) , (3.3$e$) and (3.3$a$), we get :

$$G_1M_1^7 + G_2M_1^6 + G_3M_1^5 + G_4M_1^4 + G_5M_1^3 + G_6M_1^2 + G_7M_1 + G_8 = 0 \quad (3.3g)$$

where:

$G_1 = -u_2^2u_3Lg_1^4,$

$G_2 = g_1^2\begin{bmatrix} -u_6g_4\{u_6g_2^2 + u_2u_4Lg_1\, g_2\} - \\ 2u_2u_3Lg_1\{2u_2u_4g_1 - u_7g_2\} \end{bmatrix}$

$G_3 = [u_1u_6^2g_2^3\{g_1 - g_2\} - u_4u_6^2g_4{g_1}^2g_2\{g_2 + 2\{2g_3 - u_8\} - u_4u_6L{g_1}^2(u_2 + u_5)\{2u_2u_4g_1\{g_3 + 2g_2\} - g_2\{u_2u_4g_1 - u_7g_2\} - u_3L{g_1}^2\{u_2^2u_4^2 + 4u_2g_1\{u_2u_4g_1 - u_7g_2\} + [u_2u_4g_1 - u_7g_2]^2 - 2u_2u_4u_7g_1g_3\}],$

$$\begin{aligned} G_4 = [&u_1u_6^2g_2[g_1\{g_2(1+u_4)\} + 3g_3\} - g_2\{2u_2g_3g_2\}] \\ &- {u_4}^2{u_6}^2{g_1}^2g_4[2u_4g_2(2g_3 - u_8) + u_2u_4g_1(3g_3 - u_8) + (2g_3 - u_8)] \\ &- u_4L{g_1}^2[u_6g_1(3g_3 - u_8) + 2u_2u_3u_4g_1\{u_2u_4g_1 - u_7g_2\} \\ &+ 2u_3\{[u_2u_4g_1 - u_7g_2]^2 - 2u_2u_4u_7g_1g_3\} - 2u_3u_7g_3\{u_2u_4g_1 - u_7g_2\} \\ &- u_3u_6^2g_1{g_2}^2\}], \end{aligned}$$

$$\begin{aligned} G_5 = \big[&u_1u_4u_6^2g_2[g_1\{2u_2g_2g_3 + {g_3}^2 + (2g_3 - u_8)[u_4g_2 + 2u_2g_3\}] \\ &- g_3g_2[u_4g_3 + 2u_2(g_3 + u_8) + u_4(3g_3 - 2u_8)] - u_4^2u_6^2(u_2 \\ &+ u_5)g_1[g_1\{g_3\{u_4g_3 + 4u_2\} + u_4{g_2}^2\} + 2g_3\{u_4g_3 + u_2g_3 - u_8\}] \\ &- u_4^2u_6Lg_4{g_1}^2[u_2u_4^2g_1g_3 + 3u_4g_2\{u_2u_4g_1 - u_7g_2\} \\ &+ u_4u_7\big(g_3(3g_3 - u_8) + (3g_3 - u_8)\big) - u_3g_1[u_6^2g_2\{u_4g_2 + 2u_2g_3\} \\ &+ u_4^2Lg_1\{[u_2u_4g_1 - u_7g_2]^2 - 2u_2u_4u_7g_1g_3\} - 4u_7g_3\{u_2u_4g_1 - u_7g_2\}\big], \end{aligned}$$

$$\begin{aligned} G_6 = [&u_1u_4^2u_6^2g_3[g_1\{(2g_3 - u_8)\{2u_2g_2 + g_3\} + g_2\{u_4g_2 + g_3\} - g_3g_2\{2u_2(3g_3 - 2u_8) \\ &+ u_4 - u_4(g_3 + u_{12})\}] - u_4^2u_6^2g_4g_1g_3[2u_4\{u_4g_3 + u_2g_2\} + g_1g_3] \\ &- u_4^3u_6L{g_1}^2g_3g_4[u_4\{u_2u_4g_1 - u_7g_2\} + u_7\{2g_3 + g_2\} + 1 \\ &- u_3u_4g_1g_3[u_6^2\{2u_2g_2 + g_3\} - 2u_4^2u_7Lg_1\{u_2u_4g_1 - u_7g_2\}]] \end{aligned}$$

$G_7 = [u_1 u_4^3 u_6^2 {g_3}^2 [g_1\{2u_2 g_2 + g_3\} - g_3\{u_4(3g_3 - 2u_8) + 2u_2 g_2\}] + u_4^2 u_6^2 g_1 g_3 [2\{u_4 g_3 + u_2 g_2\} - u_4^3 g_1 g_3 g_4] - u_4^5 u_6 u_7 L {g_1}^2 {g_3}^2 g_4],$

$G_8 = u_1 u_4^5 u_6^2 {g_3}^4.$

$where\ g_1 = (u_9 - u_{10})\ ,\ g_2 = (u_{11} + u_{12} - u_8)\ , g_3 = (u_{11} + u_{12})\ and\ g_4 = (u_2 + u_5).$

So, eq. $(3.3g)$ has a unique positive root, namely $M_1^*$ if:

$$g_1 > 0\ , \tag{3.3h}$$

$$g_2 > 0 \tag{3.3i}$$

$$\frac{u_7 g_2}{u_2 u_4} < g_1 < g_2 \tag{3.3j}$$

$$2u_2(3g_3 - 2u_8) + u_4 > u_4(g_3 + u_{12}) \tag{3.3k}$$

$$H_1^2 > 2u_3 u_7 g_3 H_1 + H_3, \tag{3.3l}$$

$$H_1 < \min\left\{\frac{u_6^2 H_2}{2u_4^2 u_7 L g_1}\ ,\frac{u_4^3 g_1 g_3 g_4}{2}\right\}, \tag{3.3m}$$

$$g_1 < \min\{D_1, D_2, D_3, D_4\}, \tag{3.3n}$$

where:

$H_1 = u_2 u_4 g_1 - u_7 g_2,$

$H_2 = 2u_2 g_2 + g_3,\ H_2 = 2u_2 u_4 u_7 g_1 g_3,\ \ D_1 = \frac{2u_2 g_3 {g_2}^2}{g_2(1+u_4)+3g_3}\ ,$

$D_2 = \frac{g_2 g_3[u_4 g_3 + 2u_2(g_3+u_8) + u_4(3g_3 - 2u_8)]}{2u_2 g_2 g_3 + {g_3}^2 + (2g_3 - u_8)(u_4 g_2 + 2u_2 g_3)}, D_3 = \frac{g_2 g_3\{2u_2(3g_3 - 2u_8) + u_4 - u_4(u_8 + g_3)\}}{\{(2g_3 - u_8)H_2 g_2\{u_4 g_2 + g_3\}}\ , and$

$D_4 = \frac{g_3\ \{u_4(3g_3 - 2u_8) + 2u_2 g_2\}}{H_2}$

So, $E_3 = (M_1^*, M_2^*, M_3^*)$ where $M_2^* = M_2(M_1^*), M_3^* = M_3(M_1^*)$ which are positive, if in addition to conditions (3.3h)-(3.3j)the following condition holds:

$$u_2 g_1 {M_1^*}^2 + H_1 M_1^* > u_4 u_7 g_3 \tag{3.3p}$$

## 4. Local Stability Analysis.

The stability of model (1) are discussed as follows:

The Jacobean matrix $J(M_1, M_2, M_3)$ of model (1) can be written:

$$J = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix}, \tag{4.1}$$

where

$J_{11} = -(u_2 + u_5) - \frac{u_3 u_4 M_3}{(u_3 + M_1)^2} < 0\ \ , J_{12} = \frac{u_1(1 - 2M_2)}{1 + LM_3}\ \ ,$

$J_{13} = \frac{-u_1 L M_2(1 - M_2)}{(1 + LM_3)^2} - \frac{u_3 M_1}{u_4 + M_1} < 0, J_{21} = u_2\ , J_{22} = -u_6 M_3 - u_7 < 0\ , J_{23} = -u_6 M_2,$

$J_{31} = \frac{u_4 u_8 M_3}{(u_4 + M_1)^2}\ , J_{32} = (u_9 - u_{10}) M_3\ , J_{33} = \frac{u_8 M_1}{u_4 + M_1} + (u_9 - u_{10}) M_2 - (u_{11} + u_{12})\ .$

### 4. 1 Stability of $E_0$

At $E_0$ the Jacobian matrix ( JM) is:

$J_0 = J(E_0)$

$$= \begin{bmatrix} -(u_2 + u_5) & u_1 & 0 \\ u_2 & -u_7 & 0 \\ 0 & 0 & -(u_{11} + u_{12}) \end{bmatrix}. \tag{4.1a}$$

Then the characteristic equation (CEs) of $J_0$ is agreed with:

$[\ \lambda^2 - \mathrm{tr(A)}\ \lambda + \mathrm{Det(A)}\ ]\ [-(u_{11} + u_{12}) - \lambda\ ] = 0,$

where:

$tr(A) = \lambda_{0M_1} + \lambda_{0M_2} = -(u_2 + u_5 + u_7) < 0$

$Det(A) = \lambda_{0M_1}.\lambda_{0M_2} = u_7(u_2+u_5) - u_1u_2$

So, either

$[\lambda^2 - \mathrm{tr(A)}\lambda + \mathrm{Det(A)}] = 0, where$

$A = \begin{bmatrix} -(u_2+u_5) & u_1 \\ u_2 & -u_7 \end{bmatrix},$

which gives the first two eigenvalues $\lambda_{0M_1}$ and $\lambda_{0M_2}$ are negative provided that

$$u_7(u_2+u_5) > u_1u_2. \qquad (4.1b)$$

Or

$-(u_{11}+u_{12}) - \lambda = 0$, which gives

$\lambda_{0M_3} = -(u_{11}+u_{12}) < 0.$

Therefore, $E_0$ is stable under condition (4.1b),otherwise $E_0$ is unstable.

### 4.2 Stability of $E_1$

At $E_1$ the JM becomes

$$\bar{J} = J(E_1) = \begin{bmatrix} \bar{J}_{11} & \bar{J}_{12} & \bar{J}_{13} \\ \bar{J}_{21} & \bar{J}_{22} & \bar{J}_{23} \\ 0 & 0 & \bar{J}_{33} \end{bmatrix}, \qquad (4.2a)$$

where:

$\bar{J}_{11} = -(u_2+u_5)$ ,$\bar{J}_{12} = u_1(1-2\bar{M}_2)$,$\bar{J}_{13} = -u_1L\bar{M}_2(1-\bar{M}_2) - \frac{u_3\bar{M}_1}{u_4+\bar{M}_1} < 0$

$\bar{J}_{21} = u_2$ ,$\bar{J}_{22} = -u_7 < 0$ ,$\bar{J}_{23} = -u_6\bar{M}_2$ ,$\bar{J}_{33} = \frac{u_8\bar{M}_1}{u_4+\bar{M}_1} + (u_9-u_{10})\bar{M}_2 - (u_{11}+u_{12})$ .

Then the (CEs) of $\bar{J}$ is given by:

$[\lambda^2 - \mathrm{tr(B)}\,\lambda + \mathrm{Det(B)}]\left[\frac{u_8\bar{M}_1}{u_4+\bar{M}_1} + (u_9-u_{10})\bar{M}_2 - (u_{11}+u_{12}) - \lambda\right] = 0,$

where:

$tr(\mathrm{B}) = \lambda_{1M_1} + \lambda_{1M_2} = \bar{J}_{11} + \bar{J}_{22} = -(u_2+u_5+u_7) < 0,$

$Det(\mathrm{B}) = \lambda_{0M_1}.\lambda_{0M_2} = (\bar{J}_{11}.\bar{J}_{22}) - \bar{J}_{12}\bar{J}_{21} = u_7(u_2+u_5) - u_1u_2(1-2\bar{M}_2).$

So, either

$[\lambda^2 - \mathrm{tr(B)}\,\lambda + \mathrm{Det(B)}] = 0$, where

$\mathrm{B} = \begin{bmatrix} \bar{J}_{11} & \bar{J}_{12} \\ \bar{J}_{21} & \bar{J}_{22} \end{bmatrix},$

Which gives the first two eigenvalues $\lambda_{1M_1}$ and $\lambda_{1M_2}$ which are negative provided that

$$\bar{M}_2 < \frac{1}{2}. \qquad (4.2b)$$

Or

$\frac{u_8\bar{M}_1}{u_4+\bar{M}_1} + (u_9-u_{10})\bar{M}_2 - (u_{11}+u_{12}) - \lambda = 0$, which gives

$\lambda_{1M_3} = \frac{u_8\bar{M}_1}{u_4+\bar{M}_1} + (u_9-u_{10})\bar{M}_2 - (u_{11}+u_{12}) < 0.$

Therefore, $E_1$ is stable if we add to condition (4.2b), the next conditions should be held:

$$u_9 < u_{10}\ , \qquad (4.2c)$$

$$\frac{u_8\bar{M}_1}{u_4+\bar{M}_1} < (u_{11}+u_{12}) - (u_9-u_{10})\bar{M}_2\ . \qquad (4.2d)$$

Otherwise $E_1$ is unstable .

### 4.3 Stability of $E_2$

At $E_2$ the JM is:

$$J^* = J(E_2) = \left[J_{ij}^*\right]_{3\times 3} \ , \tag{4.3a}$$

where

$J_{11}^* = -(u_2 + u_5) - \dfrac{u_3 u_4 M_3^*}{(u_3 + M_1^*)^2} < 0 \ , J_{12}^* = \dfrac{u_1(1 - 2M_2^*)}{1 + LM_3^*} \ ,$

$J_{13}^* = \dfrac{-u_1 L M_2^*(1 - M_2^*)}{(1 + LM_3^*)^2} - \dfrac{u_3 M_1^*}{u_4 + M_1^*} < 0, J_{21}^* = u_2 \ , J_{22}^* = -u_6 M_3^* - u_7 < 0 \ , J_{23}^* = -u_6 M_2^*,$

$J_{31}^* = \dfrac{u_4 u_8 M_3^*}{(u_4 + M_1^*)^2} \ , J_{32}^* = (u_9 - u_{10})M_3^* \ , J_{33}^* = \dfrac{u_8 M_1^*}{u_4 + M_1^*} + (u_9 - u_{10})M_2^* - (u_{11} + u_{12}) \ .$

Then the (CEs ) of $J(E_2)$ is given by:

$$\lambda^3 + L_1 \lambda^2 + L_2 \lambda + L_3 = 0, \tag{4.3b}$$

where:

$L_1 = -(J_{11}^* + J_{22}^* + J_{33}^*),$

$L_2 = -[J_{23}^* J_{32}^* + J_{12}^* J_{21}^* + J_{13}^* J_{31}^* - J_{11}^*(J_{22}^* + J_{33}^*) - J_{22}^* J_{33}^*],$

$L_3 = -[J_{33}^*(J_{11}^* J_{22}^* - J_{12}^* J_{21}^*) + J_{32}^*(J_{13}^* J_{21}^* - J_{11}^* J_{23}^*) + J_{31}^*(J_{12}^* J_{21}^* - J_{13}^* J_{22}^*)].$

Now by "Routh Hurwitz criterion the roots of eq. $(4.5b)$, have negative real parts, iff $L_1 > 0,\ L_3 > 0,$ and $\Delta = (L_1 L_2 - L_3)L_3 > 0.$

$Now, L_i > 0, i = 1,3$ , if we add to condition ( 3.3h) that the next conditions hold:

$$\frac{u_8 M_1^*}{u_4 + M_1^*} + (u_9 - u_{10})M_2^* < (u_{11} + u_{12}), \tag{4.3c}$$

$$M_2^* > \frac{1}{2}, \tag{4.3d}$$

$j_{23} j_{31} < j_{21} j_{33}.$

Further,

$\Delta = (C_1 - C_2)[-(J_{33}^*(J_{11}^* J_{22}^* - J_{12}^* J_{21}^* + J_{32}^*(J_{13}^* J_{21}^* - J_{11}^* J_{23}^*) + J_{31}^*(J_{12}^* J_{21}^* - J_{13}^* J_{22}^*)], where$

$C_1 = (J_{11}^* + J_{22}^* + J_{33}^*)(J_{23}^* J_{32}^* + J_{12}^* J_{21}^* + J_{13}^* J_{31}^* - J_{11}^*(J_{22}^* + J_{33}^*) - J_{22}^* J_{33}^*), and$

$C_2 = -(J_{33}^*(J_{11}^* J_{22}^* - J_{12}^* J_{21}^* + J_{32}^*(J_{13}^* J_{21}^* - J_{11}^* J_{23}^*) + J_{31}^*(J_{12}^* J_{21}^* - J_{13}^* J_{22}^*)$

Hence $\Delta > 0$, under conditions (4.3c) and (4.3d), with:

$$C_1 > C_2 \ . \tag{4.3e}$$

Then, $E_2$ is LS, otherwise $E_2$ is unstable.

## 5. Global stability analysis (GSA):

In this section the GSA of model (1) are studied.

**Theorem (5.1):**

The EP $E_0 = (0,0,0)$ is GAS in the subregion of $Int. R_+^3$ if :

$$M_2 > 1. \tag{5.1a}$$

**Proof:** Consider the following function

$Z_1(M_1, M_2, M_3) = M_1(t) + M_2(t) + M_3(t).$

$Z_1(M_1, M_2, M_3) \epsilon\ C^1(R_+^3, R)$, and $Z_1(E_0) = 0$, and $Z_1(M_1, M_2, M_3) > 0$ , $\forall (M_1, M_2, M_3) \neq E_0$ , by differentiating $Z_1$ w. r. t time $t$, yields

$$\frac{dZ_1}{dt} = \frac{u_1 M_2(1 - M_2)}{1 + LM_3} - (u_3 - u_8)\frac{M_1 M_3}{u_4 + M_1} - (u_6 - u_9)M_2 M_3 - u_5 M_1 - u_7 M_2 - (u_{11} + u_{12})M_3.$$

Therefore;

$$\frac{dZ_1}{dt} < u_1 M_2(1 - M_2) - (u_3 - u_8)\frac{M_1 M_3}{u_4 + M_1} - (u_6 - u_9)M_2 M_3 - u_5 M_1 - u_7 M_2$$

$-(u_{11}+u_{12})M_3$.

Now, according to the natural facts, $u_3 > u_8, u_6 > u_9$ and $condition\ (5.1a)$ we get:
$\frac{dZ_1}{dt} < -u_5M_1 - u_7M_2 - (u_{11}+u_{12})M_3$.
So, $\frac{dZ_1}{dt} < 0$ . Hence $E_0$ is GAS.

**Theorem (5.2) :**
The EP $E_2 = (\bar{M}_1, \bar{M}_2, 0)$ of system ( 1 ) is GAS in the subregion $Int.R_+^3$ if :

$$M_1 < \bar{M}_1, \quad (5.2a)$$
$$M_2 < \bar{M}_2 < 1, \quad (5.2b)$$
$$\bar{V}_1 > \bar{V}_2, \quad (5.2c)$$

$where$:

$$\bar{V}_1 = (\bar{M}_1 - M_1)\frac{u_1\bar{M}_2(1-\bar{M}_2)}{\bar{M}_1} + \frac{u_{2\bar{M}_1}}{M_2\bar{M}_2}(M_2-\bar{M})^2,$$
$$\bar{V}_2 = \frac{u_3\bar{M}_1M_3}{u_4+M_1} + u_6\bar{M}_2M_3.$$

**Proof:** Define the function:
$Z_2(M_1,M_2,M_3) = (M_1 - \bar{M}_1 - \bar{M}_1\, ln\frac{M_1}{\bar{M}_1}) + (M_2 - \bar{M}_2 - \bar{M}_2\, ln\frac{M_2}{\bar{M}_2}) + M_3$.
$Z_2(M_1,M_2,M_3)\epsilon\, C^1(R_+^3, R)$, $Z_2(E_1) = 0$, and $Z_2(M_1,M_2,M_3) > 0$ ; $\forall(M_1,M_2,M_3) \neq E_1$ , by differentiating $Z_2$ w.r.t time $t$, and simplify algebraic calculations yields

$$\frac{dZ_2}{dt} < (M_1 - \bar{M}_1)\frac{u_1\bar{M}_2(1-\bar{M}_2)}{\bar{M}_1} - \frac{u_{2\bar{M}_1}}{M_2\bar{M}_2}(M_2-\bar{M})^2 - (u_3-u_8)\frac{M_1M_3}{u_4+M_1} - (u_6-u_9)M_2M_3 - (u_{11}+u_{12})M_3 - (M_1-\bar{M}_1)\frac{u_1\bar{M}_2(1-\bar{M}_2)}{\bar{M}_1(1+LM_3)} + \frac{u_3\bar{M}_1M_3}{u_4+M_1} + u_6\bar{M}_2M_3.$$

Now, by the natural facts, $u_3 > u_8, u_6 > u_9$ and conditions (5.2a)-(5.2b), we get:
$\frac{dZ_2}{dt} < (M_1 - \bar{M}_1)\frac{u_1\bar{M}_2(1-\bar{M}_2)}{\bar{M}_1} - \frac{u_{2\bar{M}_1}}{M_2\bar{M}_2}(M_2-\bar{M})^2 + \frac{u_3\bar{M}_1M_3}{u_4+M_1} + u_6\bar{M}_2M_3$
$= -\bar{V}_1 + \bar{V}_2$.
So, $\frac{dZ_2}{dt} < 0$ under condition (5.2c). Hence $E_1$ is GAS.

**Theorem (5.3) :**
The EP $E_3 = (M_1^*, M_2^*, M_3^*)$of model ( 1 ) is GAS in the subregion of $Int.R_+^3$ that if:

$$M_1 < M_1^*, \quad (5.3a)$$
$$M_2 < M_2^* < \frac{1}{2}, \quad (5.3b)$$
$$M_3 > M_3^*, \quad (5.3c)$$
$$V_1^* > V_2^*, \quad (5.3d)$$

$where$:

$$V_1^* = -(M_1 - M_1^*)\left(\frac{u_1M_2(1-M_2)}{M_1(1+LM_3)}\right) + \frac{u_2M_1^*}{M_2M_2^*}(M_2-M_2^*)^2 + u_{10}(M_2-M_2^*)(M_3-M_3^*),$$

$V_2^* = -(M_1 - M_1^*)\left(\frac{u_1 {M_2}^*(1-{M_2}^*)}{M_1^*(1+L{M_3}^*)}\right) + (u_3 - u_8)\frac{M_1^* M_3}{u_3+M_1} + (u_3 - u_8)\frac{M_1 M_3^*}{u_3+M_1^*} + (u_6 - u_9){M_2}^* M_3 + (u_6 - u_9)M_2 {M_3}^* - u_{10}(M_2 - M_2^*)(M_3 - M_3^*) + \frac{u_2}{M_2}(M_1 - M_1^*)(M_2 - M_2^*)$.

**Proof:** Define the function:

$$Z_3(M_1, M_2, M_3) = \left(M_1 - M_1^* - {M_1}^* \ln\frac{M_1}{{M_1}^*}\right) + \left(M_2 - {M_2}^* - {M_2}^* \ln\frac{M_2}{{M_2}^*}\right) + (M_3 - {M_3}^* - {M_3}^* \ln\frac{M_3}{{M_3}^*}) .$$

$Z_3(M_1, M_2, M_3) \epsilon\, C^1(R_+^3, R)$, $Z_3(E_2) = 0$, and $Z_3(M_1, M_2, M_3) > 0$ ;
$\forall (M_1, M_2, M_3) \neq E_2$ , by differentiating $Z_3$ w.r.t time $t$ yields

$$\begin{aligned}\frac{dZ_3}{dt} = &-(M_1 - M_1^*)\left(\frac{u_1 M_2(1-M_2)}{M_1(1+LM_3)}\right) + \frac{u_2 M_1^*}{M_2 M_2^*}(M_2 - M_2^*)^2 + u_{10}(M_2 - M_2^*)(M_3 - M_3^*)\\ &-(M_1 - M_1^*)\left(\frac{u_1 {M_2}^*(1-{M_2}^*)}{M_1^*(1+L{M_3}^*)}\right) + (u_3 - u_8)\frac{M_1^* M_3}{u_3+M_1} + (u_3 - u_8)\frac{M_1 M_3^*}{u_3+M_1^*}\\ &+(u_6 - u_9){M_2}^* M_3 + (u_6 - u_9)M_2 {M_3}^* - u_{10}(M_2 - M_2^*)(M_3 - M_3^*)\\ &+\frac{u_2}{M_2}(M_1 - M_1^*)(M_2 - M_2^*) - (u_3 - u_8)\frac{M_1 M_3}{u_3+M_1} - (u_3 - u_8)\frac{M_1^* M_3^*}{u_3+M_1^*}\\ &-(u_6 - u_9)M_2 M_3 - (u_6 - u_9)M_2^* M_3^*.\end{aligned}$$

Now, according to the natural facts, $u_3 > u_8, u_6 > u_9$.

$$\begin{aligned}\frac{dZ_3}{dt} < &-(M_1 - M_1^*)\left(\frac{u_1 M_2(1-M_2)}{M_1(1+LM_3)}\right) + \frac{u_2 M_1^*}{M_2 M_2^*}(M_2 - M_2^*)^2 + u_{10}(M_2 - M_2^*)(M_3 - M_3^*)\\ &-(M_1 - M_1^*)\left(\frac{u_1 {M_2}^*(1-{M_2}^*)}{M_1^*(1+L{M_3}^*)}\right) + (u_3 - u_8)\frac{M_1^* M_3}{u_3+M_1} + (u_3 - u_8)\frac{M_1 M_3^*}{u_3+M_1^*}\\ &+(u_6 - u_9){M_2}^* M_3 + (u_6 - u_9)M_2 {M_3}^* - u_{10}(M_2 - M_2^*)(M_3 - M_3^*)\\ &+\frac{u_2}{M_2}(M_1 - M_1^*)(M_2 - M_2^*).\end{aligned}$$

Again by the natural facts, $u_3 > u_8, u_6 > u_9$ and conditions (5.3a)-(5.3c), we get:
$\frac{dZ_3}{dt} < -V_1^* + V_2^*$.

Therefore ,$\frac{dZ_3}{dt} < 0$ under condition (5.3d). Hence $E_2$ is GAS.

## 6. Numerical simulation:

In this section, the behavior of the system (1) are studied numerically to confirm the our analytical results, for the set of parameters:

$$\left.\begin{array}{c} u_1 = 0.3\,, L = 0.1\,, u_2 = 0.3\,, u_3 = 0.4\,, u_4 = 0.4\,, u_5 = 0.1\,, u_6 = 0.4, u_7 = 0.1 \\ u_8 = 0.3, u_9 = 0.3, u_{10} = 0.1, u_{11} = 0.1, u_{12} = 0.01. \end{array}\right\} \quad (6.1)$$

with the starting point (0.5,1,1).

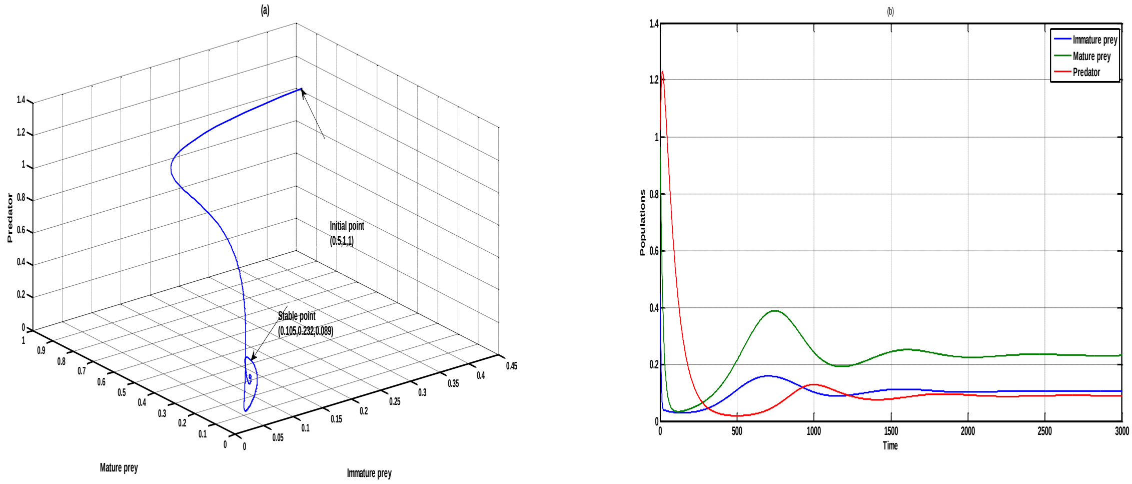


**Figure. (6.1 )** (a) The solution approaches to $E_2 = (0.105,0.232,0.089)$ (b) The time series of the attractor in Figure. (6.1a).

Now, by changing one factor each time, to see the effect of each parameter on behavior of the solution, the results were summarized in the table 2.

**Table 2:** Numerical behavior of the system (1)

| Range of parameter | The stable point | The bifurcation point |
|---|---|---|
| $0.1 < u_1 \leq 0.133$<br>$0.133 < u_1 < 0.19$<br>$0.19 \leq u_1 < 1$ | $E_0$<br>$E_1$<br>$E_2$ | $u_1 = 0.133$<br>$u_1 = 0.19$ |
| $0.1 < L \leq 0.3$ | $E_2$ | |
| $0.1 < u_2 \leq 2$ | $E_2$ | |
| $0.4 < u_3 \leq 2$ | $E_2$ | |
| $0.1 < u_4 \leq 0.42$<br>$0.42 < u_4 \leq 2$ | $E_2$.<br>$E_1$ | $u_4 = 0.42$ |
| $0.1 \leq u_5 \leq 0.335$<br>$0.335 \leq u_5 < 0.601$<br>$0.601 \leq u_5 \leq 1$ | $E_2$<br>$E_1$<br>$E_0$ | $u_5 = 0.335$<br>$u_5 = 0.601$ |
| $0.4 \leq u_6 \leq 2$ | $E_4$ | |
| $0.1 \leq u_7 < 0.178$<br>$0.178 \leq u_7 \leq 0.225$<br>$0.225 < u_7 \leq 1$ | $E_2$<br>$E_1$<br>$E_0$ | $u_7 = 0.178$<br>$u_7 = 0.225$ |
| $0.1 < u_8 < 0.143$<br>$0.143 < u_8 \leq 0.3$ | $E_1$<br>$E_2$ | $u_8 = 0.134$ |
| $0.1 < u_9 \leq 0.109$<br>$0.109 < u_9 \leq 0.3$ | $E_1$<br>$E_2$ | $u_9 = 0.42$ |
| $0.1 < u_{10} \leq 0.290$<br>$0.109 < u_{10} \leq 0.3$ | $E_2$<br>$E_1$ | $u_{10} = 0.290$ |
| $0.1 < u_{11} \leq 0.20$<br>$0.20 < u_{11} \leq 1$ | $E_2$<br>$E_1$ | $u_{11} = 0.20$ |
| $0.1 < u_{12} \leq 0.20$<br>$0.20 < u_{12} \leq 1$ | $E_2$<br>$E_1$ | $u_{12} = 0.20$ |

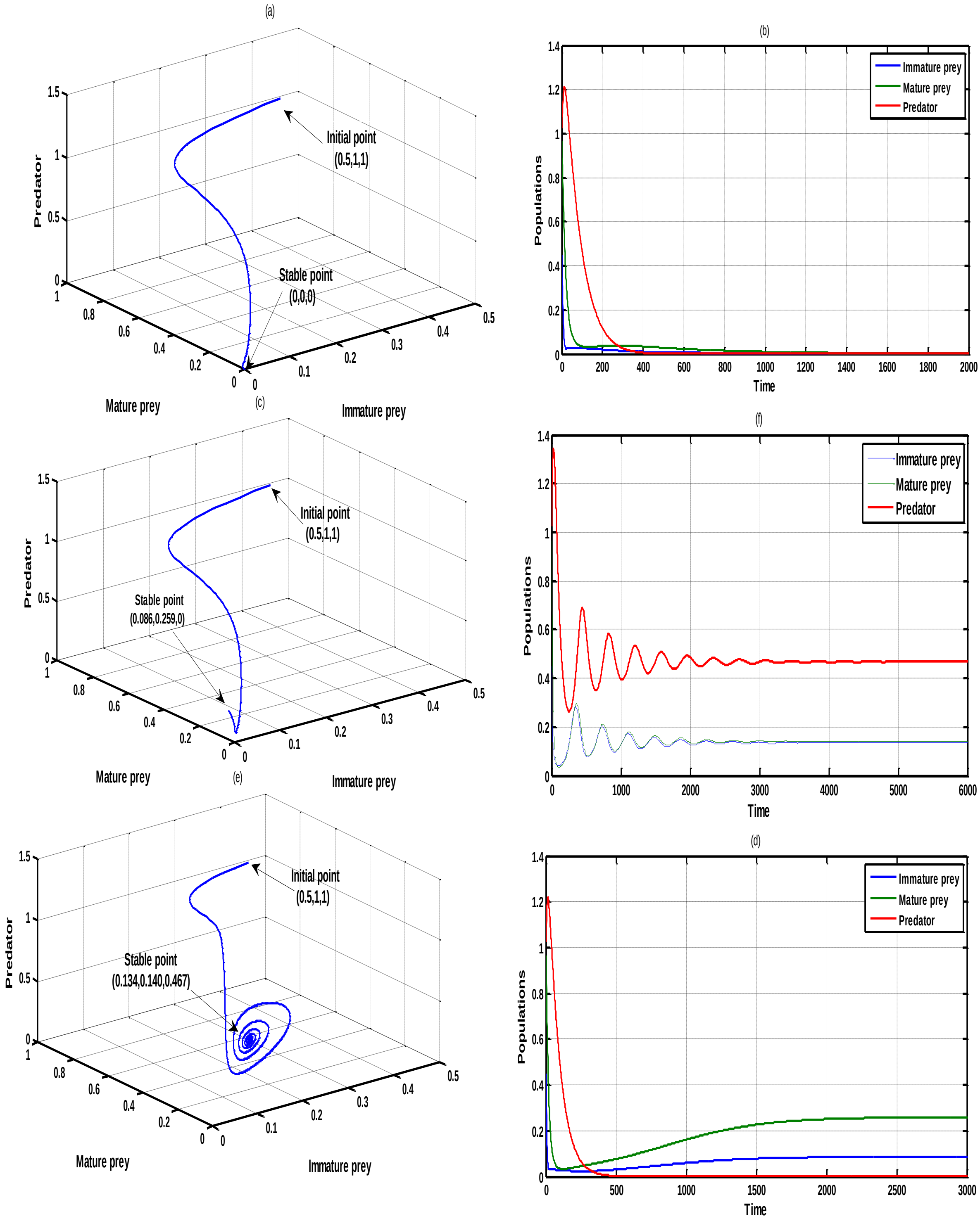


**Figure. (6.2 ):** The solution of model (1): (a)-(b) the solution approaches to $E_0 = (0,0,0)$ for typical value $u_1 = 0.1$, (c)-(d) the solution approaches to $E_1 = (0.086,0.259,0)$ for typical value $u_1 = 0.18$, (e)-(f) the solution approaches to $E_2 = (0.134,0.140,0.467)$ for typical value $u_1 = 1$.

## 7.Conclusions

An ecological model with stage-structure prey with fear and anti-predator involving a harvesting on all species has been examined. The local stability and global stability of this model are studied .Also, the global dynamics is studied numerically to know the effect of the parameters on the dynamics of the system, mainly the influence of anti-predation and terror, and the effects can be summarized as follows:

**1-** System (1) does not have periodic solutions in Int. $R_+^3$ .

**2-** The parameters $u_{i,} i = 1,4,5,7,8,9,10,11,12$ played an important role on the dynamical behavior of model(1).

**3-** It is noticed that the behavior of the system (1) does not change, if , $u_2$ , $u_3$and $u_6$ are varied.


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