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ON T-HOLLOW-LIFITING MODULES

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Abstract

Let M be an R-module, and let T be a submodule of M. A submodule K is called T-Small submodule ($K \ll_T M$) if for every submodule X of M such that $T \subseteq K + X$ implies that $T \subseteq X$. In our work we give the definition of T-coclosed submodule and T-hollow-lifiting modules with many properties.

Keywords: T-small submodule, T-coessential submodule, T-coclosed submodule, T-hollow, T-lifiting module.

المقاسات الرفع المجوفة من النمط T

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الخلاصه

لتكن Mمقاسا معرف على الحلقة Rوليكن Tمقاسا جزئيا من M . المقاس الجزئي K يدعى مقاس جزئي صغير من النمط T اذا كان لكل مقاس جزئي X من M بحيث ان $T \subseteq K + X$ يؤدي الى ان $T \subseteq X$.في هذا العمل سنعطي تعريف المقاس الجزئي المغلق المضاد من النمط -T ومقاسات الرفع المجوفه من النمط-T مع العديد من الخصائص.

Introduction

Throughout this paper R is commutative ring with identity and unitary R-modules, a submodule N of M is small denoted by $N \ll M$ if for any submodule X of M, $N + X = M$ implies that $X = M$. Small submodule were generalized by many researchers [1, 2, 3].In a previous work [4], the authors introduced the concept of T-small submodule, that a submodule K of M is T=small, $T \subseteq K + X$ implies that $T \subseteq X$.

In another article [5], H. Al Redeeni introduced the concept of T-hollow module and T-lifiting module. Also, T-coessential submodule was given the if A, B submodule of M such that $A \subseteq B$, A is T-coessential of B ($A \subseteq_{T-ce} B$) if $\frac{B}{A} \ll_{\frac{T+A}{A}} \frac{M}{A}$. In the present work, we develop the properties of this concept.

In section one we introduce the T-coclosed submodule of $M$and we investigate the basic properties of it.

In section two we introduce T-hollow-lifiting module: an R-module M is called T-hollow- lifiting if for every submodule N of M with $\frac{M}{N}$ is $\frac{T}{N}$-hollow, then there exists a direct summand K of M such that $T \subseteq_{T-ce} N$. We give the basic properties and the relation between these modules with other concepts.

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S_1. T-coclosed submodule:

Let R be a ring and T be a submodule of an R-module M and A, B submodules of M such that $A \subseteq B$, A is called T-coessential of B in M, brifly ($A \subseteq_{T-ce} B$) if $\frac{B}{A} \ll_{\frac{T+A}{A}} \frac{M}{A}$ [3].

Lemma 1.1:[3]

(1) If T and A are two submoduls of a module M, then $A \ll_T M$ if and only if $O \subseteq_{T-ce} B$ in M.

(2) If A, B and T are submodules of a module M such that $A \subseteq B$. Then $A \subseteq_{T-ce} B$ if whenever $T \subseteq B + X$ implies that $T \subseteq A + X$, for every submodule X of M.

(3) If A, B , C and T are submodules of on R-module M such that $A \subseteq B \subseteq C \subseteq M$. Then $B \subseteq_{T-ce} C$ in M iff $\frac{B}{A} \ll_{\frac{T+A}{A}-ce} \frac{C}{A}$ in $\frac{M}{A}$.

(4) Let M and N be two R-modules such that $T \leq M$ and $f: M \to N$ be an epimorphism. If $A \subseteq_{T-ce} B$ in M, then $f(A) \subseteq_{f(T)-ce} f(B)$ in N.

Now, we prove the following propositions:-

Proposition (1.2):

Let N and M be R-modules such that $T \leq N$ and let $f: M \to N$ be an epimorphism. If $C \subseteq D \subseteq N$, then $C \subseteq_{T-ce} D$ iff $f^{-1}(C) \subseteq_{f^{-1}(T)-ce} f^{-1}(D)$.

Proof:

$\Longrightarrow$) Assume that $C \subseteq_{T-ce} D$ and let $\frac{k}{f^{-1}(C)}$ be a submodule of $\frac{M}{f^{-1}(C)}$ such that $\frac{f^{-1}(T)+f^{-1}(C)}{f^{-1}(C)} \subseteq \frac{f^{-1}(D)}{f^{-1}(C)} + \frac{K}{f^{-1}(C)}$. Then $f^{-1}(T) \subseteq f^{-1}(T) + f^{-1}(C) \subseteq f^{-1}(D) + K$ and hence $T \subseteq D + f(K)$. Therefore $\frac{T+C}{C} \subseteq \frac{D+f(K)}{C}$. Thus $\frac{T+C}{C} \subseteq \frac{D}{C} \subseteq \frac{f(K)}{C}$ and then $\frac{T+C}{C} \subseteq \frac{f(K)}{C}$ therefore $T \subseteq f(k)$, so $f^{-1}(K) \subseteq K$. Hence $\frac{f^{-1}(T)+f^{-1}(C)}{f^{-1}(C)} \subseteq \frac{K}{f^{-1}(C)}$, then $\frac{f^{-1}(D)}{f^{-1}(C)} \ll_{\frac{f^{-1}(T)+f^{-1}(C)}{f^{-1}(C)}} \frac{N}{f^{-1}(C)}$ thus $f^{-1}(C) \subseteq_{f(T)-ce} f^{-1}(D)$. $\Longleftarrow$) Clear by (Lemma 1.1).

Now, we introduce the following definition of T-coclosed submodule.

Definition(1.3):

Let T be a submodule of an R-module M. Asubmodule L of M is called T-coclosed in M (denoted by) $L \subseteq_{T-ce} M$ if L has a proper submodule $K \subseteq_{T-ce} L$ i.e if $K \subseteq_{T-ce} L$ then $K = L$.

Remarks and Examples (1.4):

1- If $T = M$ and $A \subseteq B$ be a submodule of M, then A is T-coessenal of B if and only if A is coessential of B. So A is T-coclosed if and only if A is coclosed in M.

2- Consider Z_6 as Z-modules. Let $T = \{\overline{0}, \overline{3}\}$, $A = \{0\}$ and $B = \{\overline{0}, \overline{2}, \overline{4}\}$.

$A \subseteq_{T-ce} B$ since $\frac{B}{A} \ll_T \frac{Z}{A}$ [3] but $\{\overline{0}\} + \{\overline{0}, \overline{2}, \overline{4}\}$, thus B is not T- coclosed in Z_6 , but A is coclosed in B.

3- Consider Z_4 as Z-module. Let $T = \{\overline{0}, \overline{2}\}$, $B = \{\overline{0}, \overline{2}\}$. Now, if $A = \{0\}$, then $\frac{B}{\{0\}} \cong \{\overline{0}, \overline{2}\}$[ ] if $A = \{\overline{0}, \overline{2}\} = \frac{B}{B} = \{0\} \ll_T Z_4$, therefore $B \subseteq_{T-ce} Z_4$.

Proposition (1.5):-

Let T be a submodule of an R-module M and L be T-coclosed of M. Then $\frac{L}{K}$ is $\frac{T}{K}$-coclosed in $\frac{M}{K}$ for every submodule K of M.

Proof: Suppose that there is a proper submodule N of L such that $\frac{N}{K} \subseteq \frac{L}{K}$ is $\frac{T}{K}$-coesential in $\frac{M}{K}$ then $\frac{L/K}{N/K} \ll_{\frac{\frac{T}{K}+\frac{N}{K}}{N/K}} \frac{M/K}{N/K}$ thus $\frac{L}{N} \ll_{\frac{T+N}{N}} \frac{M}{N}$, since $N \subset L$, hence $N \subseteq L$ and this is a contradiction(N is proper). Also since L is T-coclosed, therefore $\frac{L}{K}$ is $\frac{T}{K}$-coclosed in $\frac{M}{K}$.

Proposition 1.6: Let , K and L be submodules of an R-module M such that $T \leq L$, $K \ll$ Land $\frac{L}{K} \subseteq_{T-ce} \frac{M}{K}$. Then L is T-coclosed in M.

Proof: Let $N < L$ such that $N \subseteq_{T-ce} L$ and $N + K \subseteq L$,thus $\frac{N+K}{K} \subseteq \frac{L}{K}$,Hence $\frac{N+K}{K} \subset_{T-ce} \frac{L}{K}$by (Lemma 1.1). But $\frac{L}{K}$ is $\frac{T}{K}$-coclosed then $+K = L$, thus $N = L$ (since $N \ll L$) therefore L is T-coclosed.

Proposition 1.7: Let T be a submodule of an R-module M and $f: M \to N$be an epimorphism such that $\ker f \ll_T M$. If $L \subset_{T-ce} M$, then $f(L) \subseteq_{f(T)-cc} N$.

Proof: Let $A \subseteq f(L)$ such that $A \subseteq_{f(T)-ce} f(L)$,let $K = f^{-1}(A)$ then by (prop. (1.2)) we have $K \subseteq_{T-ce} L + Ker\, f$. But $Ker\, f \subseteq K = \ker f + K \cap L$ and since $\ker f \ll_T M$, then $L \cap K \subseteq_{T-ce} \ker f + L \cap K$ by (lemma 1.1). Therefore $L \cap K \subseteq_{T-ce} L$. But L is T-coclosed in M, thus $L \cap K = L$ and hence $L = K$ and $L \subseteq f^{-1}(A)$. Then $f(L) \subset A$ therefore $A = f(L)$.

$S_{2.}$ *T*-(hollow-lifting) module

Recall that a module M is called hollow- lifting for every submodule N of M with $\frac{M}{N}$ is hollow, there exists a direct summand K of M such that $K \subseteq_{ce} N$ [6] we introduce the following concept.

Definition (2.1):- Let T be a submodule of an R-module M. M is called T-(hollow-lifting) if for every N of M with $\frac{M}{N}$ is $\frac{T}{N}$-hollow; there exists a direct summand K of M such that $K \subseteq_{T-ce} N$.

Remarks and Examples (2.2):

1- For non-zero module M if $T = M$, then M is M-(hollow-lifting) if and only if $M$is hollow lifting thus Z_4 as Z- module is Z_4 –(hollow-lifting) module

2- Consider Z_6 as Z-module and $T = \{\overline{0}, \overline{3}\}$ then Z_6 as Z-module is T-(hollow-lifting) module.

3- Every nonzero module M is 0-(hollow-lifting) module.

4- It is clear that every module having no T-hollow factor modules is T-(hollow-lifting) module.

An R module is called T-lifting module if for every submodule X of M, there exists a direct summand D of $M$and $H \ll_T M$ such that $X = D + H$ [5].

Proposition (2.3):- Every T-lifting module is T-(hollow-lifting).

Proof: Let $T \leq M$, then for every submodule N of M, there exists a direct summand D and $H \ll_T M$ such that $N = D + H$. Now if $\frac{M}{N}$ is $\frac{T}{N}$-hollow to show $D \subseteq_{T-ce} N$ i.e $\frac{N}{D} \ll_{\frac{T+D}{D}} \frac{M}{D}$, let $\frac{T+D}{D} \subseteq \frac{N}{D} + \frac{B}{D}$, where $D \subseteq B \subseteq M$. Then $\frac{T+D}{D} \subseteq \frac{N}{D} + \frac{B}{D}$ thus $T \subseteq H + D$. But $\ll_T M$, Then $T \subseteq B$ and hence $D \subseteq_{T-ce} N$.

Note: The converse of the above is not true, i.e T-(hollow-lifting) module needs not to be T-lifting.

Let T and N be submodules of Msuch that $T \subseteq N$ and M be indecomposable R-module which has no hollow factor module then M is T-(hollow-lifting). To show that $M$is not T-lifting suppose M is T-lifting, and $N < M$, then there exists $K \leq \oplus M$ such that $K \subseteq_{T-ce} N$, Thus $M = K \oplus \acute{K}$ where $\acute{K} \leq M$. But is indecomposable thus $K = 0$ therefore $N \ll_T M$ and hence M is T-hollow contradaction.

Proposition (2.4): Let T be submodule of indecomposable R-module M, If M is T-hollow lifting module then $M$is T-hollow or has no T-hollow factor module.

Proof: Suppose M has T-hollow factor module then there exists a paper submodule N of M such that $\frac{M}{N}$ is $\frac{T}{N}$-hollow, M is T-hollow lifting, then there exists a direct summand K of M such that $\subseteq_{T-ce} N$. but M is indecomposable then $K = \{0\}$ hence $N \ll_T M$, thus M is T-hollow [5,2.2.8].

Proposition (2.5): Let M be T-(hollow-lifting) module and N, K be submodules of M such that $\frac{M}{K}$ is $\frac{T}{K}$-hollow and $T \subseteq N + K$ then there exists a direct summand A of M such that $T \subseteq N + A$ and $A \subseteq_{T-ce} K$ in M.

Proof : Let N, K and T be submodules of M such that $\frac{M}{K}$ is $\frac{T}{K}$-hollow. Since M is T-(hollow-lifting) then there exists a direct summand A of M such that $A \subseteq_{T-ce} K$ in M. Now $T \subseteq N + K$ then $\frac{T+A}{A} \subseteq \frac{N+K+A}{A} = \frac{K}{A} + \frac{N+A}{A}$, but $\frac{K}{A} \ll_{\frac{T+A}{A}} \frac{M}{A}$.Thus $\frac{T+A}{A} \subseteq \frac{N+A}{A}$ and hence $T \subseteq N + A$.

Proposition (2.6): Let M be T-(hollow-lifting) module, then every T-coclosed submodule K of M with $\frac{M}{K}$ is $\frac{T}{K}$-hollow is a direct summand of M.

Proof: Suppose M is T-(hollow-lifting) module and let K be T-coclosed submodule in M such that $N \subseteq_{T-ce} K$ in M but K is T-coclosed so $K = N$ then K is a direct summand of M.

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