ON T-HOLLOW-LIFITING MODULES

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Received: 17/4/2019 Accepted: 17/7/2019

Abstract
Let $M$ be an $R$-module, and let $T$ be a submodule of $M$. A submodule $K$ is called $T$-Small submodule $(K \ll_{T} M)$ if for every submodule $X$ of $M$ such that $T \subseteq K + X$ implies that $T \subseteq X$. In our work we give the definition of $T$-coclosed submodule and $T$-hollow-lifiting modules with many properties.

Keywords: T-small submodule, T-coessential submodule, T-coclosed submodule, T-hollow, T-lifiting module.

Introduction
Throughout this paper $R$ is commutative ring with identity and unitary $R$-modules, a submodule $N$ of $M$ is small denoted by $N \ll M$ if for any submodule $X$ of $M$, $N + X = M$ implies that $X = M$. Small submodule were generalized by many researchers [1, 2, 3]. In a previous work [4], the authors introduced the concept of $T$-small submodule, that a submodule $K$ of $M$ is $T$=small, $T \subseteq K + X$ implies that $T \subseteq X$.

In another article [5], H. Al Redeeni introduced the concept of $T$-hollow module and $T$-lifiting module. Also, $T$-coessential submodule was given the if $A, B$ submodule of $M$ such that $A \subseteq B$, $A$ is $T$-coessential of $B$ ($A \subseteq_{T-ce} B$) if $\frac{B}{A} \ll_{T+} \frac{M}{A}$. In the present work, we develop the properties of this concept.

In section one we introduce the $T$-coclosed submodule of $M$ and we investigate the basic properties of it.

In section two we introduce $T$-hollow-lifiting module: an $R$-module $M$ is called $T$-hollow- lifiting if for every submodule $N$ of $M$ with $\frac{M}{N}$ is $T$-hollow, then there exists a direct summand $K$ of $M$ such that $T \subseteq_{T-ce} N$. We give the basic properties and the relation between these modules with other concepts.

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**S, T-coclosed submodule:**
Let $R$ be a ring and $T$ be a submodule of an $R$-module $M$ and $A, B$ submodules of $M$ such that $A \subseteq B$, $A$ is called $T$-coessential of $B$ in $M$, briefly $(A \subseteq_{T-ce} B)$ if $\frac{B}{A} \ll_{T,A} M$ $A$ $[3]$.

**Lemma 1.1:[3]**
(1) If $T$ and $A$ are two submodules of a module $M$, then $A \ll T$ if and only if $O \subseteq_{T-ce} B$ in $M$.
(2) If $A$, $B$ and $T$ are submodules of a module $M$ such that $A \subseteq B$. Then $A \subseteq_{T-ce} B$ if whenever $T \subseteq B + X$ implies that $T \subseteq A + X$, for every submodule $X$ of $M$.
(3) If $A$, $B$, $C$ and $T$ are submodules of an $R$-module $M$ such that $A \subseteq B \subseteq C \subseteq M$. Then $B \subseteq_{T-ce} C$ in $M$ if $\frac{B}{A} \ll_{T,A} C \subseteq M$ $A$ $A$.
(4) Let $M$ and $N$ be two $R$-modules such that $T \subseteq M$ and $f: M \rightarrow N$ be an epimorphism. If $A \subseteq_{T-ce} B$ in $M$, then $f(A) \subseteq_{f(T)-ce} f(B)$ in $N$.

Now, we prove the following proposition:

**Proposition (1.2):**
Let $N$ and $M$ be $R$-modules such that $T \subseteq N$ and let $f: M \rightarrow N$ be an epimorphism. If $C \subseteq D \subseteq N$, then $C \subseteq_{T-ce} D$ iff $f^{-1}(C) \subseteq f^{-1}(T)-ce f^{-1}(D)$.

**Proof:**
(1) Assume that $C \subseteq_{T-ce} D$ and let $k$ be a submodule of $M$ such that $f^{-1}(T) + f^{-1}(C) \subseteq f^{-1}(D) + K$ and hence $T \subseteq D + f(K)$. Therefore $\frac{T+C}{C} \subseteq \frac{D+f(K)}{C}$. Thus $\frac{T+C}{C} \subseteq \frac{D}{C} + \frac{f(K)}{C}$ and then $\frac{T+C}{C} \subseteq \frac{f(k)}{C}$ therefore $T \subseteq f(k)$, so $f^{-1}(K) \subseteq K$. Hence $\frac{f^{-1}(T)+f^{-1}(C)}{f^{-1}(C)} \subseteq \frac{K}{f^{-1}(C)}$, then $\frac{f^{-1}(T)+f^{-1}(C)}{f^{-1}(C)} \ll \frac{f^{-1}(T)+f^{-1}(C)}{f^{-1}(C)}$ thus $f^{-1}(C) \subseteq_{f(T)-ce} f^{-1}(D)$. $\implies$ Clear by (Lemma 1.1).

**Definition (1.3):**
Let $A$ be a submodule of an $R$-module $M$. A submodule $L$ of $M$ is called $T$-coclosed in $M$ (denoted by $L \subseteq_{T-closed}$ $M$ $L$ if $L$ has a proper submodule $K \subseteq_{T-closed} L$ i.e if $K \subseteq_{T-closed} L$ then $K = L$.

**Remarks and Examples (1.4):**
1- If $T = M$ and $A \subseteq B$ be a submodule of $M$, then $A$ is $T$-coessential of $B$ if and only if $A$ is coessential of $B$. So $A$ is $T$-coclosed if and only if $A$ is coclosed in $M$.
2- Consider $Z_6$ as $Z$-modules. Let $T = \{0, 3\}$, $A = \{0\}$ and $B = \{0, 2, 4\}$.

\[ A \subseteq_{T-closed} B \text{ since } \frac{B}{A} \ll_{T,A} \frac{Z}{A} \text{ [3] but } \{0\} + \{0, 2, 4\}, \text{ thus } B \text{ is not } T- \text{coclosed in } Z_6. \]

But $A$ is coclosed in $B$.
3- Consider $Z_4$ as $Z$-module. Let $T = \{0, 2\}$, $B = \{0, 2\}$. Now, if $A = \{0\}$, then $B = \{0\} \subseteq_{T, Z_4}$ therefore $B \subseteq_{T-closed} Z_4$.

**Proposition (1.5):**
Let $T$ be a submodule of an $R$-module $M$ and $L$ be $T$-coclosed of $M$. Then $\frac{L}{K} \text{ is } T-coclosed in } \frac{M}{K}$ for every submodule $K$ of $M$.

**Proof:**
Suppose that there is a proper submodule $N$ of $L$ such that $\frac{N}{K} \subseteq \frac{L}{K}$ is $T$-coessential in $\frac{M}{K}$ then $\frac{L/K}{N/K} \ll_{T,A} \frac{M/K}{N/K}$ thus $\frac{L}{N} \ll_{T,A} \frac{M}{N}$, since $N \subseteq L$, hence $N \subseteq L$ and this is a contradiction ($N$ is proper).

Also since $L$ is $T$-coclosed, therefore $\frac{L}{K} \text{ is } T-K \text{coclosed in } \frac{M}{K}$.

**Proposition 1.6:** Let $K$ and $L$ be submodules of an $R$-module $M$ such that $T \subseteq L$. $K \ll \text{ Land } \frac{L}{K} \subseteq_{T-closed} \frac{M}{K}$ Then $L$ is $T$-coclosed in $M$.

**Proof:**
Let $N < L$ such that $N \subseteq_{T-closed} L$ and $N + K \subseteq L$, thus $\frac{N+K}{K} \subseteq \frac{L}{K}$. Hence $\frac{N+K}{K} \subseteq_{T-closed} \frac{L}{K}$ by (Lemma 1.1). But $\frac{L}{K} \text{coclosed then } + K = L$, thus $N = L$ (since $N \ll L$) therefore $L$ is $T$-coclosed.
Proposition 1.7: Let $T$ be a submodule of an $R$-module $M$ and $f : M \to N$ be an epimorphism such that $\ker f \ll_T M$. If $L \subseteq_T M$, then $f(L) \subseteq_T f(T)$. 

Proof: Let $A \subseteq f(L)$ such that $A \subseteq_T f(T)$, then $K = f^{-1}(A)$. Then by (prop. (1.2)) we have $K \subseteq_T L + \ker f$. But $\ker f \subseteq K = \ker f + K \cap L$ and since $\ker f \ll_T M$, then $L \cap K \subseteq_T \ker f$. Thus $L \subseteq_T K$. Therefore, $L \subseteq_T f^{-1}(A)$. Then $f(L) \subseteq A$, therefore $A = f(L)$.

S2: $T$-(hollow-lifting) module

Recall that a module $M$ is called hollow lifting for every submodule $N$ of $M$ with $M_N$ is hollow, there exists a direct summand $K$ of $M$ such that $K \subseteq_T N$ [5] we introduce the following concept.

Definition (2.1): Let $T$ be a submodule of an $R$-module $M$. $M$ is called $T$-(hollow-lifting) if for every submodule $N$ of $M$ with $M_N$ is hollow, there exists a direct summand $K$ of $M$ such that $K \subseteq_T N$. 

Remarks and Examples (2.2):

1. For non-zero module $M$ if $T = M$, then $M$ is $T$-(hollow-lifting) if and only if $M$ is hollow lifting.
2. Consider $Z_4$ as $Z$-module and $T = \{0, 3\}$ then $Z_4$ as $Z$-module is $T$-(hollow-lifting) module.
3. Every nonzero module $M$ is 0-(hollow-lifting) module.
4. It is clear that every module having no $T$-hollow factor modules is $T$-(hollow-lifting) module.

Proposition (2.3): Every $T$-(lifting) module is $T$-(hollow-lifting).

Proof: Let $T \subseteq M$, then for every submodule $N$ of $M$, there exists a direct summand $D$ and $H \ll_T M$ such that $N = D + H$. Now if $M_N$ is hollow to show $D \subseteq_T N$ i.e. $N_D \ll_T n + b$, then $T + D_D \subseteq_T N + b$ where $D \subseteq B \subseteq M$. Then $T + D_D = \frac{T + D}{D} \subseteq \frac{N + b}{D}$ thus $T \subseteq H + D$. But $\ll_T M$. Then $T \subseteq B$ and hence $D \subseteq_T N$.

Note: The converse of the above is not true, i.e $T$-(hollow-lifting) module needs not to be $T$-lifting.

Let $T$ and $N$ be submodules of $M$ such that $T \subseteq N$ and $M$ is indecomposable $R$-module which has no hollow factor module then $M$ is $T$-(hollow-lifting). To show that $M$ is not $T$-lifting suppose $M$ is $T$-lifting, and $N < M$, then there exists $K \leq T \oplus M$ such that $K \subseteq_T N$. Thus $M = K \oplus R$ where $R \leq M$. But if $K$ is indecomposable then $K = 0$ therefore $N \ll_T M$ and hence $M$ is $T$-hollow contradiction.

Proposition (2.4): Let $T$ be submodule of indecomposable $R$-module $M$, If $M$ is $T$-hollow lifting module then $M_T$ is hollow or has no $T$-hollow factor module.

Proof: Suppose $M$ has $T$-hollow factor module then there exists a paper submodule $N$ of $M$ such that $M_N$ is hollow, $M$ is $T$-hollow lifting, then there exists a direct summand $K$ of $M$ such that $K \subseteq_T N$, but $M$ is indecomposable then $K = \{0\}$ hence $N \ll_T M$, thus $M$ is $T$-hollow [5,2.2.8].

Proposition (2.5): Let $M$ be $T$-(hollow-lifting) module and $N, K$ be submodules of $M$ such that $M_K$ is $T_K$ hollow and $T \subseteq N + K$ then there exists a direct summand $A$ of $M$ such that $T \subseteq N + A$ and $A \subseteq_T K$ in $M$.

Proof: Let $N, K$ and $T$ be submodules of $M$ such that $M_N$ is $T_N$ hollow. Since $M$ is $T$-(hollow-lifting) then there exists a direct summand $A$ of $M$ such that $A \subseteq_T N + K$. Now $T \subseteq N + K$ then $\frac{T + A}{A} \subseteq \frac{N + A}{A}$, but $\frac{K}{A} \ll_T \frac{M}{A}$. Thus $\frac{T + A}{A} \subseteq \frac{N + A}{A}$ and hence $T \subseteq N + A$.

Proposition (2.6): Let $M$ be $T$-(hollow-lifting) module, then every $T$-coclosed submodule $K$ of $M$ with $M_K$ is $T_K$ hollow is a direct summand of $M$.

Proof: Suppose $M$ is $T$-(hollow-lifting) module and let $K$ be $T$-coclosed submodule in $M$ such that $N \subseteq_T K$ in $M$ but $K$ is $T$-coclosed so $K = N$ then $K$ is a direct summand of $M$. 

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References