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A Class of Meromorphic Multivalent Functions with Positive Coefficients Defined by Struve Function

Suha J. Hamaad [1]*, Abdul Rahman S. Juma [2], Hassan H. Ebrahim [3]

[1]*Department of Mathematics, College of Education for Pure Sciences –University of Tikrit , Tikrit, Iraq*
[2]*Department of Mathematics, College of Education for Pure Sciences –University of Anbar, Ramadi, Iraq*
[3] *Department of Mathematics, College of Computer Science and Mathematics–University of Tikrit , Tikrit, Iraq*

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Abstract

The main goal of this work is to obtain new geometric properties for meromorphic multivalent functions in the punctured disk $(UD)^* = UD\backslash\{0\} = \{z: z \in \mathbb{C}\ and\ 0 < |z| < 1\}$, so we presented a new class $\mathcal{J}_{c,p}^{\zeta,a}(\beta,\delta,\sigma)$ by using Hadamard product. In this paper we obtained many important properties such as coefficient estimates, radius of convexity, distortion theorem, convex combination and closure theorem. In addition, we derived the integral transforms and Hadamard product for functions belonging to the class $\mathcal{J}_{c,p}^{\zeta,a}(\beta,\delta,\sigma)$.

Keywords: meromorphic functions**,** Struve functions, Hadamard product, convex function, convolution.

فئة من الدوال متعددة التكافؤ ذات المعاملات الموجبة المحددة بواسطة دالة ستروف

سهى جمعه حماد[1]*, عبد الرحمن سلمان جمعه[2], حسن حسين ابراهيم[3]

[1]القسم الرياضيات, كلية التربية للعلوم الصرفة , جامعة تكريت, صلاح الدين, العراق
[2]القسم الرياضيات, كلية التربية للعلوم الصرفة, جامعة الانبار , الانبار , العراق
[3]القسم الرياضيات, كلية علوم الحاسوب و الرياضيات , جامعة تكريت, صلاح الدين, العراق

الخلاصة

الهدف الرئيسي من هذا العمل هو الحصول على خصائص هندسية جديدة للدوال متعددة التكافؤ الميرومورفية $\mathcal{J}_{c,p}^{\zeta,a}(\beta,\delta,\sigma)$ (الدوال التحليلية باستثناء عدد منته من النقاط) في القرص المثقوب، لذلك قدمنا فئة جديدة
باستخدام الالتفاف. في هذا البحث حصلنا على العديد من الخواص مثل تقدير المعاملات ، نصف قطر التحدب، نظرية التشوه، التركيبة المحدبة و نظرية الانغلاق. بالإضافة الى ذلك اشتقينا تحويلات التكامل ومنتج هادامارد للدوال التي تنتمي للفئة $\mathcal{J}_{c,p}^{\zeta,a}(\beta,\delta,\sigma)$.

*Email:suhajumaa1987@tu.edu.iq

1. Introduction

Let $\sum(p,\imath)$ be the class of all meromorphic functions of the form:

$$\chi(z) = z^{-p} + \sum_{\zeta=\imath}^{\infty} a_\zeta z^{\zeta-p} \quad (p,\imath \in N = \{1,2,3,\dots\}), \tag{1}$$

which are analytic and p −valent in the punctured disk $(UD)^* = UD\backslash\{0\} = \{z: z \in \mathbb{C}\ and\ 0 < |z| < 1\}$.

A function $\chi(z) \in \sum(p,\imath)$ is said to be the meromorphic p −valent starlike function of order ω if

$$Re\left\{-\frac{z\chi'(z)}{\chi(z)}\right\} > \omega \qquad (0 \le \omega < p),$$

for all $z \in (UD)^*$. This class is represented by the symbol $MS_p^*(\omega)$ [1-4].

A function $\chi(z) \in \sum(p,\imath)$ is said to be the meromorphic p −valent convex function of order ω if

$$Re\left\{-\left(1 + \frac{z\chi''(z)}{\chi'(z)}\right)\right\} > \omega, \qquad (0 \le \omega < p)$$

for all $z \in (UD)^*$. This class is represented by the symbol $MC_p(\omega)$, [1, 2], [5].

In this work we will use the Struve function that was presented and used in many previous works see [6] which is defined as follows

$$\acute{\check{U}}_{r,b,c}(z) = 2^r\sqrt{\pi}\Gamma\left(r + \frac{b+2}{2}\right) z^{\frac{-r-1}{2}} \sum_{\zeta=0}^{\infty} \frac{(-c)^\zeta(\sqrt{z}/2)^{2\zeta+r+1}}{\Gamma(\zeta+3/2)\Gamma(r+\zeta+(b+2)/2)} = \sum_{\zeta=0}^{\infty} \frac{(-c/4)^\zeta}{(3/2)_\zeta(a)_\zeta} z^\zeta, \qquad (z \in UD)$$

and

$$\acute{\check{U}}_{a,c}(z) = z\acute{\check{U}}_{r,b,c}(z) = z + \sum_{\zeta=1}^{\infty} \frac{(-c/4)^\zeta}{(3/2)_\zeta(a)_\zeta} z^{\zeta+1}, \text{ [7-9]}$$

where $r,b,c \in \mathbb{C}, a = r + (b+2)/2 \neq 0,-1,-2,\dots$ and $(a)_\zeta$ is the Pochhammer symbol (or shifted factorial) expressed terms of the gamma function, by

$$(a)_\zeta = \frac{\Gamma(a+\zeta)}{\Gamma(a)} = \begin{cases} 1; & (\zeta = 0) \\ a(a+1)(a+2)\dots(a+\zeta-1) & (\zeta \in N = \{1,2,3,\dots\}). \end{cases}$$

The Struve functions has undergone many complexities and investigation see [10]. Struve functions have application in surface-wave and water-wave issues, unstable aerodynamics resistive MHD instability theory and optical direction. Struve functions have lately appeared in a number of particle systems [11]. By setting

$$\delta_{a,c,p}(z) = z^{-p} + \sum_{\zeta=\imath}^{\infty} \frac{(-c/4)^\zeta}{(3/2)_\zeta(a)_\zeta} z^{\zeta-p}.$$

By using the convolution (or Hadamard product), we defined a new operator

$$\check{S}_{c,p}^{\zeta,a}: \sum(p,\imath) \to \sum(p,\imath)$$

which is defined as follows

$$\check{S}_{c,p}^{\zeta,a}\chi(z) = \delta_{a,c,p}(z) * \chi(z) = z^{-p} + \sum_{\zeta=\imath}^{\infty} \frac{(-c/4)^\zeta}{(3/2)_\zeta(a)_\zeta} a_\zeta z^{\zeta-p} \quad (z \in UD). \tag{2}$$

Definition 1.1: A function $\chi \in \sum(p,\imath)$ is said to be in the class $\mathcal{J}_{c,p}^{\zeta,a}(\beta,\delta,б)$ if it satisfies the following condition

$$\left|\frac{z^{p+1}\left(\check{S}_{c,p}^{\zeta,a}\chi(z)\right)' + p}{(3\beta-1)z^{p+1}\left(\check{S}_{c,p}^{\zeta,a}\chi(z)\right)' + (3\beta\delta - p)}\right| < б \ (z \in (UD)^*), \tag{3}$$

where $0 < б \le 1,\ \frac{1}{3} \le \beta \le 1,\ 0 \le \delta < p$.

2. Coefficiente estimates

Theorem 2.1: A function $\chi(z)$ defined by in Eq.(1) is in the class $\mathcal{J}_{c,p}^{\zeta,a}(\beta,\delta,б)$ if and only if

$$\sum_{\zeta=i}^{\infty} (\zeta-p)\frac{(-c/4)^{\zeta}}{(3/2)_{\zeta}(a)_{\zeta}}(1+3б\beta-б)\ a_{\zeta}\ \leq 3б\beta(p-\delta), \qquad (4)$$

for $0<б\leq 1,\ \frac{1}{3}\leq\beta\leq 1,\ 0\leq\delta\ <p, p\in N.$

Proof: Suppose Eq. (4) verified. We get

$$\left|z^{p+1}\left(\check{S}_{c,p}^{\zeta,a}\chi(z)\right)'+p\right|-б\left|(3\beta-1)z^{p+1}\left(\check{S}_{c,p}^{\zeta,a}\chi(z)\right)'+(3\beta\delta-p)\right|<0.$$

Provided

$$\left|\sum_{\zeta=i}^{\infty} (\zeta-p)\frac{(-c/4)^{\zeta}}{(3/2)_{\zeta}(a)_{\zeta}}\ a_{\zeta}z^{\zeta}\ \right|-б\left|3\beta(p-\delta)-\sum_{\zeta=i}^{\infty}(3\beta-1)(\zeta-p)\frac{(-c/4)^{\zeta}}{(3/2)_{\zeta}(a)_{\zeta}}\ a_{\zeta}z^{\zeta}\ \right|<0. \qquad (5)$$

For $|z|<r=1$, the left side of the inequality (5) bounded from above by

$$\sum_{\zeta=i}^{\infty} (\zeta-p)\frac{(-c/4)^{\zeta}}{(3/2)_{\zeta}(a)_{\zeta}}\ a_{\zeta}r^{\zeta}-3б\beta(p-\delta)+\sum_{\zeta=i}^{\infty} б(3\beta-1)(\zeta-p)\frac{(-c/4)^{\zeta}}{(3/2)_{\zeta}(a)_{\zeta}}\ a_{\zeta}r^{\zeta}\ <$$
$$\sum_{\zeta=i}^{\infty} (\zeta-p)\frac{(-c/4)^{\zeta}}{(3/2)_{\zeta}(a)_{\zeta}}(1+3б\beta-б)\ a_{\zeta}-3б\beta(p-\delta)\leq 0.$$

Hence, $\chi(z)\in\ \mathcal{J}_{c,p}^{\zeta,a}(\beta,\delta,б)$.

Conversely, assume

$$\left|\frac{z^{p+1}\left(\check{S}_{c,p}^{\zeta,a}\chi(z)\right)'+p}{(3\beta-1)z^{p+1}\left(\check{S}_{c,p}^{\zeta,a}\chi(z)\right)'+(3\beta\delta-p)}\right|$$
$$=\left|\frac{\sum_{\zeta=i}^{\infty} (\zeta-p)\frac{(-c/4)^{\zeta}}{(3/2)_{\zeta}(a)_{\zeta}}\ a_{\zeta}z^{\zeta}}{3\beta(p-\delta)-\sum_{\zeta=i}^{\infty}(3\beta-1)(\zeta-p)\frac{(-c/4)^{\zeta}}{(3/2)_{\zeta}(a)_{\zeta}}\ a_{\zeta}z^{\zeta}}\right|<б, \quad (z\in(UD)^{*}).$$

The fact that $|Re(z)|<|z|$ for all z allows us to arrive at

$$Re\left\{\frac{\sum_{\zeta=i}^{\infty} (\zeta-p)\frac{(-c/4)^{\zeta}}{(3/2)_{\zeta}(a)_{\zeta}}\ a_{\zeta}z^{\zeta}}{3\beta(p-\delta)-\sum_{\zeta=i}^{\infty}(3\beta-1)(\zeta-p)\frac{(-c/4)^{\zeta}}{(3/2)_{\zeta}(a)_{\zeta}}\ a_{\zeta}z^{\zeta}}\right\}<б,\ (z\in(UD)^{*})\,. \qquad (6)$$

Choose z values on the real axis now such that $z^{p+1}\left(\check{S}_{c,p}^{\zeta,a}\chi(z)\right)'$ is real. Upon clearing the denominator in Eq. (6) and allowing $z\to 1^{-}$ to take on positive values, we get

$$\sum_{\zeta=i}^{\infty} (\zeta-p)\frac{(-c/4)^{\zeta}}{(3/2)_{\zeta}(a)_{\zeta}}(1+3б\beta-б)\ a_{\zeta}\leq 3б\beta(p-\delta).$$

We may simply establish the following coefficient estimates from Theorem 2.1.

Corollary 2.2: If the function $\chi(z)$ defined by Eq. (1) belongs to the class $\mathcal{J}_{c,p}^{\zeta,a}(\beta,\delta,б)$ then

$$a_{\zeta}\leq\frac{3б\beta(p-\delta)(3/2)_{\zeta}(a)_{\zeta}}{(-c/4)^{\zeta}(\zeta-p)\,(1+3б\beta-б)}, \qquad (\zeta\geq i\,,p,i\in N).$$

The conclusions are sharpe for the function

$$\chi(z)=\frac{1}{z^{p}}+\frac{3б\beta(p-\delta)(3/2)_{\zeta}(a)_{\zeta}}{(-c/4)^{\zeta}(\zeta-p)\,(1+3б\beta-б)}z^{\zeta-p}\,.$$

Putting $c=-4$ and $(a)_{\zeta}=\frac{1}{(3/2)_{\zeta}}$ and $\beta=б=1$ in Theorem 2.1, we have

Corollary 2.3: A function $\chi(z)$ defined by Eq. (1) is in the class $\mathcal{J}_{p}^{\zeta}(\delta)(0\leq\delta\ <p)$ if and only if

$$\sum_{\zeta=i}^{\infty}(\zeta-p)\,a_\zeta \le (p-\delta).$$

Remark 2.4: If $\chi(z) \in \mathcal{J}_{c,p}^{\zeta,a}\left(\frac{1}{3},\delta,1\right)$, then

$$a_\zeta \le \frac{(3/2)_\zeta (a)_\zeta\,(p-\delta)}{(-c/4)^\zeta(\zeta-p)}, \quad (\zeta \ge i\,, p, i \in N)$$

and equality holds for

$$\chi(z)=\frac{1}{z^p}+\frac{(3/2)_\zeta (a)_\zeta\,(p-\delta)}{(-c/4)^\zeta(\zeta-p)}z^{\zeta-p}.$$

3. Distortion Theorem

Theorem 3.1: If the function $\chi(z)$ is defined by Eq. (1) which is in the class $\mathcal{J}_{c,p}^{\zeta,a}(\beta,\delta,б)$, then for $0<|z|=\mathrm{r}<1$,

$$\frac{1}{\mathrm{r}^p}-\frac{3б\beta(p-\delta)}{p(1+3б\beta-б)}\frac{1}{\mathrm{r}^p}\le|\chi(z)|\le\frac{1}{\mathrm{r}^p}+\frac{3б\beta(p-\delta)}{p(1+3б\beta-б)}\frac{1}{\mathrm{r}^p}, \quad (7)$$

and

$$\frac{p}{\mathrm{r}^{p+1}}-\frac{3б\beta(p-\delta)}{(1+3б\beta-б)}\frac{1}{\mathrm{r}^{p+1}}\le|\chi'(z)|\le\frac{p}{\mathrm{r}^{p+1}}+\frac{3б\beta(p-\delta)}{(1+3б\beta-б)}\frac{1}{\mathrm{r}^{p+1}}. \quad (8)$$

The function $\chi(z)$ given in the following form

$$\chi(z)=\frac{1}{z^p}+\frac{3б\beta(p-\delta)}{p(1+3б\beta-б)}\frac{1}{z^p}, \quad (9)$$

is satisfied by the bounds in Eq. (7) and Eq. (8).

Proof*:* From Theorem 2.1, we have

$$p(1+3б\beta-б)\sum_{\zeta=i}^{\infty}a_\zeta \le \sum_{\zeta=i}^{\infty}(\zeta-p)\frac{(-c/4)^\zeta}{(3/2)_\zeta(a)_\zeta}(1+3б\beta-б)\;a_\zeta \le 3б\beta(p-\delta)$$

this means that

$$\sum_{\zeta=i}^{\infty}a_\zeta \le \frac{3б\beta(p-\delta)}{p(1+3б\beta-б)}.$$

Therefore, for $0<|z|=\mathrm{r}<1$,

$$|\chi(z)|\le\frac{1}{\mathrm{r}^p}+\sum_{\zeta=i}^{\infty}a_\zeta \mathrm{r}^{\zeta-p} \le\frac{1}{\mathrm{r}^p}+\frac{1}{\mathrm{r}^p}\sum_{\zeta=i}^{\infty}a_\zeta \quad (10)$$

$$\le\frac{1}{\mathrm{r}^p}+\frac{3б\beta(p-\delta)}{p(1+3б\beta-б)}\frac{1}{\mathrm{r}^p}$$

and

$$|\chi(z)|\ge\frac{1}{\mathrm{r}^p}-\sum_{\zeta=i}^{\infty}a_\zeta \mathrm{r}^{\zeta-p} \ge\frac{1}{\mathrm{r}^p}-\frac{1}{\mathrm{r}^p}\sum_{\zeta=i}^{\infty}a_\zeta \quad (11)$$

$$\ge\frac{1}{\mathrm{r}^p}-\frac{3б\beta(p-\delta)}{p(1+3б\beta-б)}\frac{1}{\mathrm{r}^p},$$

Eq.(10) and Eq. (11), yield Eq. (7). From Theorem 2.1, it also follows that

$$\sum_{\zeta=i}^{\infty}(\zeta-p)\;a_\zeta \le\frac{3б\beta(p-\delta)}{(1+3б\beta-б)}.$$

Hence,

$$|\chi'(z)| \leq \frac{p}{\mathrm{r}^{p+1}} + \sum_{\zeta=i}^{\infty} (\zeta - p)\ a_\zeta\, \mathrm{r}^{\zeta-p-1}$$

$$\leq \frac{p}{\mathrm{r}^{p+1}} + \frac{1}{\mathrm{r}^{p+1}} \sum_{\zeta=i}^{\infty} (\zeta - p)\ a_\zeta \tag{12}$$

$$\leq \frac{p}{\mathrm{r}^{p+1}} + \frac{3б\beta(p-\delta)}{(1+3б\beta - б)} \frac{1}{\mathrm{r}^{p+1}}$$

and

$$|\chi'(z)| \geq \frac{p}{\mathrm{r}^{p+1}} - \sum_{\zeta=i}^{\infty} (\zeta - p)\ a_\zeta\, \mathrm{r}^{\zeta-p-1}$$

$$\geq \frac{p}{\mathrm{r}^{p+1}} - \frac{1}{\mathrm{r}^{p+1}} \sum_{\zeta=i}^{\infty} (\zeta - p)\ a_\zeta \tag{13}$$

$$\geq \frac{p}{\mathrm{r}^{p+1}} - \frac{3б\beta(p-\delta)}{(1+3б\beta - б)} \frac{1}{\mathrm{r}^{p+1}}.$$

The function $\chi(z)$ defined in Eq. (9) can be observed as extremal for Theorem 3.1.

4. Radius of convexity

Theorem 4.1: If the function $\chi(z)$ defined by Eq. (1) belongs to the class $\mathcal{J}_{c,p}^{\zeta,a}(\beta, \delta, б)$, then $\chi(z)$ is meromorphically $p-$valent convex of order $\omega(0 \leq \omega < p)$ in $0 < |z| < \mathrm{r}(a, \zeta, c, p, \beta, \delta, б)$, where

$$\mathrm{r}(a, \zeta, c, p, \beta, \delta, б) = \inf_{\zeta} \left[\frac{p(p-\omega)\,(-c/4)^{\zeta}(1+3б\beta - б)}{3б\beta(p-\delta)(3/2)_\zeta (a)_\zeta (\zeta + p - \omega)} \right]^{\frac{1}{\zeta}} \tag{14}$$

$(\zeta \geq i\,, p, i \in N)$.

The concluded product is sharp.

Proof: To prove theorem we need to show that

$$\left| \frac{\left(z\chi'(z)\right)' + p\chi'(z)}{\chi'(z)} \right| \leq p - \omega \ \text{ for } 0 < |z| < \mathrm{r}(a, \zeta, c, p, \beta, \delta, б).$$

$$\left| \frac{\left(z\chi'(z)\right)' + p\chi'(z)}{\chi'(z)} \right| = \left| \frac{\sum_{\zeta=i}^{\infty} \zeta(\zeta-p) a_\zeta z^{\zeta-p-1}}{-pz^{-p-1} + \sum_{\zeta=i}^{\infty}(\zeta-p)a_\zeta z^{\zeta-p-1}} \right| \leq \frac{\sum_{\zeta=i}^{\infty} \zeta(\zeta-p)a_\zeta \mathrm{r}^{\zeta}}{p - \sum_{\zeta=i}^{\infty}(\zeta-p)a_\zeta \mathrm{r}^{\zeta}}$$

Hence,

$$\left| \frac{\left(z\chi'(z)\right)' + p\chi'(z)}{\chi'(z)} \right| \leq p - \omega \quad \text{if } \sum_{\zeta=i}^{\infty} \frac{(\zeta-p)(\zeta+p-\omega)}{p(p-\omega)} a_\zeta \mathrm{r}^{\zeta} \leq 1 \tag{15}$$

from Theorem 2.1, it follows that

$$\sum_{\zeta=i}^{\infty} \frac{(\zeta-p)(-c/4)^{\zeta}(1+3б\beta - б)}{3б\beta(p-\delta)(3/2)_\zeta (a)_\zeta}\ a_\zeta \leq 1. \tag{16}$$

Given Eq.(16) it follows that Eq. (15), is correct if

$$\frac{(\zeta-p)(\zeta+p-\omega)}{p(p-\omega)} \mathrm{r}^{\zeta} \leq \frac{(\zeta-p)(-c/4)^{\zeta}(1+3б\beta - б)}{3б\beta(p-\delta)(3/2)_\zeta (a)_\zeta}\ (\zeta \geq i\,, i \in N)$$

or

$$\mathrm{r}^{\zeta} \leq \frac{p(p-\omega)\,(-c/4)^{\zeta}(1+3\text{б}\beta-\text{б})}{3\text{б}\beta(p-\delta)(3/2)_{\zeta}(a)_{\zeta}(\zeta+p-\omega)}$$

in other words

$$\mathrm{r} \leq \left[\frac{p(p-\omega)\,(-c/4)^{\zeta}(1+3\text{б}\beta-\text{б})}{3\text{б}\beta(p-\delta)(3/2)_{\zeta}(a)_{\zeta}(\zeta+p-\omega)}\right]^{\frac{1}{\zeta}} (\zeta \geq i\,, p, i \in N).$$

5.Extreme points

Theorem 5.1: Let $\chi_{i-1}(z) = \frac{1}{z^p}$ and

$\chi_{\zeta-p}(z) = \frac{1}{z^p} + \frac{3\text{б}\beta(p-\delta)(3/2)_{\zeta}(a)_{\zeta}}{(-c/4)^{\zeta}(\zeta-p)(1+3\text{б}\beta-\text{б})} z^{\zeta-p}, \qquad (\zeta \geq i\,, p, i \in N).$

Then $\chi(z)$ belongs to the class $\mathcal{J}_{c,p}^{\zeta,a}(\beta,\delta,\text{б})$ if and only if it can be written in the form

$$\chi(z) = \sum_{\zeta=i-1}^{\infty} \text{ϻ}_{\zeta-p}\, \chi_{\zeta-p}(z).$$

Where $\text{ϻ}_{\zeta-p} \geq 0$ and $\sum_{\zeta=i-1}^{\infty} \text{ϻ}_{\zeta-p} = 1.$

Proof: Let

$\chi(z) = \sum_{\zeta=i-1}^{\infty} \text{ϻ}_{\zeta-p}\, \chi_{\zeta-p}(z)$, where $\text{ϻ}_{\zeta-p} \geq 0$ and $\sum_{\zeta=i-1}^{\infty} \text{ϻ}_{\zeta-p} = 1.$

Then,

$\chi(z) = \sum_{\zeta=i-1}^{\infty} \text{ϻ}_{\zeta-p}\, \chi_{\zeta-p}(z)$

$= \text{ϻ}_{i-1}\chi_{i-1}(z) + \sum_{\zeta=i}^{\infty} \text{ϻ}_{\zeta-p}\, \chi_{\zeta-p}(z)$

$= \left(1 - \sum_{\zeta=i}^{\infty} \text{ϻ}_{\zeta-p}\right)\frac{1}{z^p} + \sum_{\zeta=i}^{\infty} \text{ϻ}_{\zeta-p}\left(\frac{1}{z^p} + \frac{3\text{б}\beta(p-\delta)(3/2)_{\zeta}(a)_{\zeta}}{(-c/4)^{\zeta}(\zeta-p)(1+3\text{б}\beta-\text{б})} z^{\zeta-p}\right)$

$= \frac{1}{z^p} + \sum_{\zeta=i}^{\infty} \text{ϻ}_{\zeta-p} \frac{3\text{б}\beta(p-\delta)(3/2)_{\zeta}(a)_{\zeta}}{(-c/4)^{\zeta}(\zeta-p)(1+3\text{б}\beta-\text{б})} z^{\zeta-p}$

since

$\sum_{\zeta=i}^{\infty} \text{ϻ}_{\zeta-p} \frac{3\text{б}\beta(p-\delta)(3/2)_{\zeta}(a)_{\zeta}}{(-c/4)^{\zeta}(\zeta-p)(1+3\text{б}\beta-\text{б})} . \frac{(-c/4)^{\zeta}(\zeta-p)(1+3\text{б}\beta-\text{б})}{3\text{б}\beta(p-\delta)(3/2)_{\zeta}(a)_{\zeta}}$

$= \sum_{\zeta=i}^{\infty} \text{ϻ}_{\zeta-p} = 1 - \text{ϻ}_{i-1} \leq 1.$

This shows that $\chi(z) \in \mathcal{J}_{c,p}^{\zeta,a}(\beta,\delta,\text{б})$.

Conversely: suppose that $\chi(z) \in \mathcal{J}_{c,p}^{\zeta,a}(\beta,\delta,\text{б})$ and consider

$\text{ϻ}_{\zeta-p} = \sum_{\zeta=i}^{\infty} \frac{(-c/4)^{\zeta}(\zeta-p)(1+3\text{б}\beta-\text{б})}{3\text{б}\beta(p-\delta)(3/2)_{\zeta}(a)_{\zeta}} a_{\zeta}\,, (\zeta \geq i\,, p, i \in N)$

and $\text{ϻ}_{i-1} = 1 - \sum_{\zeta=i}^{\infty} \text{ϻ}_{\zeta-p}$.

Then,

$\sum_{\zeta=i-1}^{\infty} \text{ϻ}_{\zeta-p}\, \chi_{\zeta-p}(z) = \text{ϻ}_{i-1}\chi_{i-1}(z) + \sum_{\zeta=i}^{\infty} \text{ϻ}_{\zeta-p}\, \chi_{\zeta-p}(z)$

$= \left(1 - \sum_{\zeta=i}^{\infty} \text{ϻ}_{\zeta-p}\right)\frac{1}{z^p} + \sum_{\zeta=i}^{\infty} \frac{(-c/4)^{\zeta}(\zeta-p)(1+3\text{б}\beta-\text{б})}{3\text{б}\beta(p-\delta)(3/2)_{\zeta}(a)_{\zeta}} a_{\zeta}\left[\frac{1}{z^p} + \frac{3\text{б}\beta(p-\delta)(3/2)_{\zeta}(a)_{\zeta}}{(-c/4)^{\zeta}(\zeta-p)(1+3\text{б}\beta-\text{б})} z^{\zeta-p}\right]$

$= \frac{1}{z^p} - \sum_{\zeta=i}^{\infty} \text{ϻ}_{\zeta-p} \frac{1}{z^p} + \sum_{\zeta=i}^{\infty} \frac{(-c/4)^{\zeta}(\zeta-p)(1+3\text{б}\beta-\text{б})a_{\zeta}}{3\text{б}\beta(p-\delta)(3/2)_{\zeta}(a)_{\zeta}} \frac{1}{z^p} + \sum_{\zeta=i}^{\infty} a_{\zeta} z^{\zeta-p}$

$= \frac{1}{z^p} + \sum_{\zeta=i}^{\infty} a_{\zeta} z^{\zeta-p}$

$= \chi(z).$

6. Convex combination

Theorem 6.1: The class $\mathcal{J}_{c,p}^{\zeta,a}(\beta,\delta,\text{б})$ is closed under convex linear combination.

Proof: Let

$$\chi_j(z) = \frac{1}{z^p} + \sum_{\zeta=i}^{\infty} a_{\zeta,j} z^{\zeta-p} \ \left(a_{\zeta,j} \geq 0; j = 1,2\right) \tag{17}$$

belongs to the class $\mathcal{J}_{c,p}^{\zeta,a}(\beta,\delta,б)$. We need to show that the function $\hbar$ is defined as follows

$$\hbar(z) = (1-t)\chi_1(z) + t\chi_2(z) \quad (0 \leq t \leq 1)$$

also, it belongs to the class $\mathcal{J}_{c,p}^{\zeta,a}(\beta,\delta,б)$. Since

$$\hbar(z) = \frac{1}{z^p} + \sum_{\zeta=i}^{\infty}\left[(1-t)a_{\zeta,1} + ta_{\zeta,2}\right] z^{\zeta-p} \ (0 \leq t \leq 1)$$

and by Theorem 2.1, we have

$\sum_{\zeta=i}^{\infty} \frac{(\zeta-p)(-c/4)^{\zeta}}{(3/2)_{\zeta}(a)_{\zeta}}(1+3б\beta-б) \ \left[(1-t)a_{\zeta,1} + ta_{\zeta,2}\right] = (1-t)\sum_{\zeta=i}^{\infty} \frac{(\zeta-p)(-c/4)^{\zeta}}{(3/2)_{\zeta}(a)_{\zeta}}(1+3б\beta-б) + t\sum_{\zeta=i}^{\infty} \frac{(\zeta-p)(-c/4)^{\zeta}}{(3/2)_{\zeta}(a)_{\zeta}}(1+3б\beta-б)$

$\leq (1-t)3б\beta(p-\delta) + t\, 3б\beta(p-\delta) = 3б\beta(p-\delta).$

so $\hbar(z) \in \mathcal{J}_{c,p}^{\zeta,a}(\beta,\delta,б)$. Hence, the proof has been completed.

7. Closure Theorem

Theorem 7.1: Let $\chi_k \in \mathcal{J}_{c,p}^{\zeta,a}(\beta,\delta,б), k = 1,2,\ldots,t$ then

$$\Im(z) = \sum_{k=1}^{t} e_k\, \chi_k(z) \in \mathcal{J}_{c,p}^{\zeta,a}(\beta,\delta,б).$$

For $\chi_k(z) = \frac{1}{z^p} + \sum_{\zeta=i}^{\infty} a_{\zeta,k} z^{\zeta-p}$ where $\sum_{k=1}^{t} e_k = 1$.

Proof: We will rewrite

$\Im(z) = \sum_{k=1}^{t} e_k \left(\frac{1}{z^p} + \sum_{\zeta=i}^{\infty} a_{\zeta,k} z^{\zeta-p}\right) = \sum_{k=1}^{t} e_k \frac{1}{z^p} + \sum_{k=1}^{t}\sum_{\zeta=i}^{\infty} e_k\, a_{\zeta,k} z^{\zeta-p} = \frac{1}{z^p} + \sum_{\zeta=i}^{\infty}\sum_{k=1}^{t} e_k\, a_{\zeta,k} z^{\zeta-p}.$

Furthermore, since $\chi_k(z)(k = 1,2,\ldots,t\,)$ belongs to the class $\mathcal{J}_{c,p}^{\zeta,a}(\beta,\delta,б)$, then by Theorem 2.1, we get

$\sum_{\zeta=i}^{\infty} (\zeta-p)\frac{(-c/4)^{\zeta}}{(3/2)_{\zeta}(a)_{\zeta}}(1+3б\beta-б) \ a_{\zeta,k} \leq 3б\beta(p-\delta).$

So, it is sufficing to prove that

$$\sum_{\zeta=i}^{\infty} \frac{(\zeta-p)(-c/4)^{\zeta}}{(3/2)_{\zeta}(a)_{\zeta}}(1+3б\beta-б)\left(\sum_{k=1}^{t} e_k\, a_{\zeta,k}\right)$$

$$= \sum_{k=1}^{t} e_k \sum_{\zeta=i}^{\infty} \frac{(\zeta-p)(-c/4)^{\zeta}}{(3/2)_{\zeta}(a)_{\zeta}}(1+3б\beta-б) \ a_{\zeta,k} \leq \sum_{k=1}^{t} e_k \left[3б\beta(p-\delta)\right] = 3б\beta(p-\delta).$$

As required.

8.Integral transforms

Theorem 8.1: If $\chi(z)$ is in the class $\mathcal{J}_{c,p}^{\zeta,a}(\beta,\delta,б)$, then the integral transforms

$$\mathcal{F}_{\tau+p-1}(z) = \tau \int_0^1 \upsilon^{\tau+p-1}\, \chi(\upsilon z) d\upsilon, 0 < \tau < \infty$$

are in the class $\mathcal{J}_{p}^{\zeta}(\theta), 0 \leq \theta < p$, where

$$\theta = \theta(p, \beta, \delta, б, \tau) = \frac{p^2(1 + 3б\beta - б) - 3б\beta p(p - \delta)\tau}{p\tau(1 + 3б\beta - б)}.$$

the result above is the best possible for the function $\chi(z)$ given by Eq. (7).

Proof: Suppose $\chi(z) = \frac{1}{z^p} + \sum_{\zeta=i}^{\infty} a_\zeta z^{\zeta-p} \in \mathcal{J}_{c,p}^{\zeta,a}(\beta, \delta, б)$. Then we get

$$\mathcal{F}_{\tau+p-1}(z) = \tau \int_0^1 \upsilon^{\tau+p-1} \chi(\upsilon z) d\upsilon$$

$$= \frac{1}{z^p} + \sum_{\zeta=i}^{\infty} \frac{\tau}{\tau + \zeta} a_\zeta z^{\zeta-p}.$$

In view of Corollary 2.3, it is sufficient to show that

$$\sum_{\zeta=i}^{\infty} \frac{(\zeta - p)}{(p - \delta)} \cdot \frac{\tau}{\tau + \zeta} a_\zeta \leq 1.$$

Since $\chi(z) \in \mathcal{J}_{c,p}^{\zeta,a}(\beta, \delta, б)$, we have

$\sum_{\zeta=i}^{\infty} \frac{(\zeta-p)(-c/4)^\zeta(1+3б\beta-б)}{3б\beta(p-\delta)(3/2)_\zeta(a)_\zeta} a_\zeta \leq 1.$

Thus,

$$\frac{(\zeta - p)\tau}{(\tau + \zeta)(p - \theta)} \leq \frac{(\zeta - p)(-c/4)^\zeta(1 + 3б\beta - б)}{3б\beta(p - \delta)(3/2)_\zeta(a)_\zeta}$$

for each ζ,

$$\theta \leq \frac{p(\zeta - p)(-c/4)^\zeta(1 + 3б\beta - б) - 3б\beta(p - \delta)(3/2)_\zeta(a)_\zeta(\zeta - p)\tau}{(\zeta - p)(-c/4)^\zeta(1 + 3б\beta - б)(\tau + \zeta)}. \quad (18)$$

Since the function that appears on the right side of Eq. (18) is increasing for each ζ, we will set $\zeta = 0$ in Eq. (18), we have

$$\theta \leq \frac{p^2(1 + 3б\beta - б) - 3б\beta p(p - \delta)\tau}{p\tau(1 + 3б\beta - б)}.$$

9.Hadamard product properties

Definition9.1: For the functions

$$\chi_j(z) = z^{-p} + \sum_{\zeta=i}^{\infty} a_{\zeta,j} z^{\zeta-p} \quad (p, i \in N = \{1,2,3, \dots\}, j = 1,2), \quad (19)$$

the Hadamard Product or (convolution) of the functions χ_1 and χ_2 denoted by $\chi_1 * \chi_2$, that is

$$(\chi_1 * \chi_2)(z) = z^{-p} + \sum_{\zeta=i}^{\infty} a_{\zeta,1} a_{\zeta,2} z^{\zeta-p}.$$

Theorem 9.2: Let the functions χ_j $(j = 1,2)$ defined by Eq. (19) be in the class $\mathcal{J}_{c,p}^{\zeta,a}(\beta, \delta, б)$. Then $(\chi_1 * \chi_2) \in \mathcal{J}_{c,p}^{\zeta,a}(\beta, \delta, б)$, where

$$\xi = p + \frac{3б\beta(p - \delta)^2}{p(1 + 3б\beta - б)}.$$

The result is sharp for the function $\chi_j(z)$ $(j = 1,2)$ given by

$$\chi_j(z) = \frac{1}{z^p} + \frac{3б\beta(p - \delta)}{p(1 + 3б\beta - б)} \frac{1}{z^p} (j = 1,2, p \in N).$$

Proof: In order to prove this theorem we will use the schild and Silverman technique that was used recently in [12], to find largest ξ such that

$$\sum_{\zeta=i}^{\infty} \frac{(\zeta-p)(-c/4)^{\zeta}(1+3б\beta-б)}{3б\beta(p-\xi)(3/2)_{\zeta}(a)_{\zeta}}\ a_{\zeta,1}a_{\zeta,2} \leq 1,$$

for $\chi_j \in \mathcal{J}_{c,p}^{\zeta,a}(\beta,\xi,б),(j=1,2)$. Since $\chi_j \in \mathcal{J}_{c,p}^{\zeta,a}(\beta,\delta,б),(j=1,2)$, we will easy get that

$$\sum_{\zeta=i}^{\infty} \frac{(\zeta-p)(-c/4)^{\zeta}(1+3б\beta-б)}{3б\beta(p-\delta)(3/2)_{\zeta}(a)_{\zeta}}\ a_{\zeta,j} \leq 1\ (j=1,2).$$

So, by applying the Cauchy-Schwarz inequality, we have

$$\sum_{\zeta=i}^{\infty} \frac{(\zeta-p)(-c/4)^{\zeta}(1+3б\beta-б)}{3б\beta(p-\delta)(3/2)_{\zeta}(a)_{\zeta}}\ \sqrt{a_{\zeta,1}a_{\zeta,2}} \leq 1. \quad (20)$$

This means that we need to show that

$\frac{a_{\zeta,1}a_{\zeta,2}}{(p-\xi)} \leq \frac{\sqrt{a_{\zeta,1}a_{\zeta,2}}}{(p-\delta)}\ (\zeta \geq i, p \in N).$

More simplify, that

$$\sqrt{a_{\zeta,1}a_{\zeta,2}} \leq \frac{(p-\xi)}{(p-\delta)}\ (\zeta \geq i, p \in N).$$

Then and by the inequality (20) it suffices to show that

$$\frac{3б\beta(p-\delta)(3/2)_{\zeta}(a)_{\zeta}}{(\zeta-p)(-c/4)^{\zeta}(1+3б\beta-б)} \leq \frac{(p-\xi)}{(p-\delta)}\ (\zeta \geq i). \quad (21).$$

From Eq. (21) it follows that

$$\xi \leq p - \frac{3б\beta(p-\delta)^2(3/2)_{\zeta}(a)_{\zeta}}{(\zeta-p)(-c/4)^{\zeta}(1+3б\beta-б)} (\zeta \geq i).$$

Now, if we define the function Q as follows

$$Q(\zeta) = p - \frac{3б\beta(p-\delta)^2(3/2)_{\zeta}(a)_{\zeta}}{(\zeta-p)(-c/4)^{\zeta}(1+3б\beta-б)} (\zeta \geq i).$$

But

$Q(\zeta+1) - Q(\zeta) = \frac{3б\beta(p-\delta)^2(3/2)_{\zeta}(a)_{\zeta}}{(\zeta-p)(-c/4)^{\zeta}(1+3б\beta-б)}\left[\frac{(c/4)+(3/2+\zeta)(a+\zeta)(\zeta-p)}{(\zeta+1-p)(c/4)}\right] > 0,$

for $0 < б \leq 1$, $\frac{1}{3} \leq \beta \leq 1$, $0 \leq \delta < p$. Therefore, the function $Q(\zeta)$ is an increasing for each ζ. Hence, we conclude that

$$\xi \leq Q(0) = p + \frac{3б\beta(p-\delta)^2}{p(1+3б\beta-б)},$$

this finishes the proof of Theorem.

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References

[1] I. Al-Shbeil, J. Gong, S. Ray, S. Khan, N. Khan, and H. Alaqad, "The Properties of Meromorphic Multivalent q-Starlike Functions in the Janowski Domain," *Fractal and Fractional,* vol. 7, p. 438, 2023.

[2] C.-M. Yan, R. Srivastava, and J.-L. Liu, "Properties of certain subclass of meromorphic multivalent functions associated with q-difference operator," *Symmetry,* vol. 13, p. 1035, 2021.

[3] O. Aljuboori and K. A. Jassim, "New Subclasses of Meromorphically Uniformly of Multivalent Functions with Positive and Fixed Second Coefficients," *Iraqi Journal of Science*, pp. 365-372, 2023.

[4] M. I. Hameed, M. H. Ali, and B. N. Shihab, "A certain Subclass of Meromorphically Multivalent Q-Starlike Functions Involving Higher-Order Q-Derivatives," *Iraqi Journal of Science,* pp. 251-258, 2022.

[5] A. A. Hussein and K. A. H. Jassim, "On A Certain Class of Meromorphic Multivalent Functions Defined by Fractional Calculus Operator," *Iraqi Journal of Science,* pp. 2685-2696, 2019.

[6] P. Gochhayat and A. Prajapati, "Applications of third order differential subordination and superordination involving generalized Struve function," *arXiv preprint arXiv:1809.06512*, 2018.

[7] H. Orhan and N. Yagmur, "Geometric properties of generalized Struve functions," Ann Alexandru loan Cuza Univ-Math, vol. 10, 2014.

[8] H. Srivastava, G. Murugusundaramoorthy, and T. J. J. A. C. M. Janani, "Uniformly starlike functions
and uniformly convex functions associated with the Struve function,", *J. Appl. Computat. Math*, vol. 3, pp. 1-4, 2014.

[9] U. D. Muhey, H. M. Srivastava, R. J. H. J. o. M. Mohsan, "Univalence of certain integral operators
involving generalized Struve functions," *Hacettepe Journal of Mathematics and Statistics*, vol. 47, pp. 821-833, 2018.

[10] R. M. Ali, S. R. Mondal, and K. S. Nisar, "Monotonicity properties of the generalized Struve functions," *J. Korean Math. Soc,* vol. 54, pp. 575-598, 2017.

[11] J. A. A. Mosiun and S. A. Halim, "Properties of functions involving Struve function," *Mathematics,* vol. 6, p. 240, 2018.

[12] M. Aouf and B. Frasin, "Properties of some families of meromorphic multivalent functions involving certain linear operator," *Filomat*, vol. 24, pp. 35-54, 2010.