DOI: 10.24996/ijs.2026.67.5.28

ISSN: 0067-2904

# On a class of analytic multivalent functions involving generalized Bernardi-Libera - Livingston integral operator

**Zainab H. Mahmood[1], Reem O. Rasheed [2], Kassim A. Jassim[3]**

[1]Department of Physics, College of Science, University of Baghdad, Baghdad, Iraq
[2]Department of Mathematics, College of Education Tuzkhurmatu, Tikrit University, Iraq
[3]Department of Mathematics/ College of Science / University of Baghdad/ Baghdad/Iraq

Received: 18/9/2023 Accepted: 13/5/2025 Published: 30/5/2026

**Abstract**

In this paper, a class of analytic and multivalent functions and its various properties is introduced, by using Generalized Bernardi-Libera-Livingston of Integral Operator which is defined on the open unite disk U. Coefficient bounds, distortion theorem, closure and radii of star likeness, convexity are obtained.

**Keywords**: Multivalent function, Operator, Convexity, Analytic function, Bernardi-Libera-Livingston Integral.

# حول فئة من الدوال التحليلية متعددة التكافؤ التي تتضمن عامل التكامل المعمم بيرناردي – ليبرا– ليفينكستون

**زينب هادي محمود[1] , ريم عمران رشيد[2] , قاسم عبدالحميد جاسم[3]**

[1]قسم علوم الفيزياء, كلية العلوم, جامعة بغداد, بغداد, العراق
[2]قسم الرياضيات – كلية تربية طوزخورماتو–جامعة تكريت – العراق
[3]قسم الرياضيات, كلية العلوم , جامعة بغداد, بغداد, العراق

**الخلاصه**

تم تقديم فئة جديدة من الدوال التحليلية متعددة التكافؤ وخصائصها المختلفة , باستخدام عامل التكامل المعمم بيرناردي – ليبرا – ليفينكستون المحدد على قرص الوحدة المفتوح U. وقد حصلنا على بعض الخصائص المهمة للفئات الفرعية وقمنا بفحص مثل متراجحة المعاملات ، نظريات النمو والتشويه ، خاصية الانغلاق و خاصية انصاف الاقطار النجمية والتحدبية .

## 1.Introduction

Let $\mathcal{A}$(s) be the class which contains the functions $f(z)$ of the form

$$f(z) = z^s - \sum_{n=k+s}^{\infty} a_n z^n \qquad (1)$$

$$a_n \geq 0 \text{ and } k, s \in \mathbb{N},$$

which are p-valent and regular in $\mathbb{U}$.

This introduces Bernardi-Libera-Livingston Operator:

*Email: zainab_hd@yahoo.com

$$\mathcal{F}_s^{\lambda} f(z)=\frac{\lambda+s}{z^{\lambda}}\int_0^z t^{\lambda-1} f(t)\,dt,\quad (\lambda>-s; z\in\mathbb{U}).$$

Simplifying we get

$$\mathcal{F}_s^{\lambda} f(z)=z^s-\sum_{n=k+s}^{\infty}\frac{\lambda+s}{\lambda+n}a_n z^n,$$

$$\left(\mathcal{F}_s^{\lambda} f(z)\right)^{(q)}=\frac{s!}{q!\,(s-q)!}z^{s-q}-\sum_{n=k+s}^{\infty}\frac{n!}{q!\,(n-q)!}\left(\frac{\lambda+s}{\lambda+n}\right)a_n z^{n-q}.$$

By using the operator $\mathcal{F}_s^{\lambda} f(z)$, we define a new subclass $\mathfrak{RH}_{k,q}^{s}(\lambda,b,\delta)$ if it satisfies the relation

$$\mathcal{R}e\left\{\frac{1}{b}\left(\frac{\delta z(\mathcal{F}_s^{\lambda} f(z))^{(q+1)}+\lambda z^2(\mathcal{F}_s^{\lambda} f(z))^{(q+2)}}{\lambda z(\mathcal{F}_s^{\lambda} f(z))^{(q+1)}+(\delta-\lambda)(\mathcal{F}_s^{\lambda} f(z))^{(q)}}-(s-q)\right)\right\}>\alpha \tag{2}$$

$s\in\mathbb{N}, q\in\mathbb{N}\cup\{0\}, z\in\mathbb{U}, s>max(q,-\lambda), b\in\mathbb{C}/\{0\}, \lambda\geq 0, 0<\delta\leq 1$ and $0\leq\alpha<1$. Also, $f(z)$ is in the subclass $\mathfrak{RS}_{k,q}^{s}(\lambda,b,\delta)$ if $zf'(z)\in\mathfrak{RH}_{k,q}^{s}(\lambda,b,\delta)$. Now, main results are introduced, as follows with the restriction $0\leq\lambda\leq 1$.
Many authors in [1-7] studied different classes of analytic multivalent functions.

**Theorem 1.1:** Let $f(z)\in\mathcal{A}(k)$ and given by (1). If

$$\sum_{n=k+s}^{\infty}\left(\frac{n!}{q!\,(n-q)!}\left(\frac{\lambda+s}{\lambda+n}\right)[\lambda(n-1-q)+\delta]|b|\right)a_n.$$
$$\leq 2\alpha|b|\left(\frac{s!}{q!(s-q)!}\right)[\lambda(s-1-q)+\delta] \tag{3}$$

where $q\in\mathbb{N}\cup\{0\}, s\in\mathbb{N}, z\in\mathbb{U}, s>max(q,-\lambda), b\in\mathbb{C}\cup\{0\}, \lambda\geq 0, 0<\delta\leq 1$ and $0\leq\alpha<1$. Then $f(z)\in\mathfrak{RH}_{k,q}^{s}(\lambda,b,\delta)$.

**Proof:** We will show that $f(z)\in\mathfrak{RH}_{k,q}^{s}(\lambda,b,\delta)$. Using the fact that $\mathcal{R}e(w)\geq\alpha$, if and only if
$|1-\alpha+w|\geq|1+\alpha-w|$. It suffices to see that

$$|A(z)+(1-\alpha)B(z)|-|A(z)+(1-\alpha)B(z)|\geq 0, \tag{4}$$

where

$A(z)=\delta z(\mathcal{F}_s^{\lambda} f(z))^{(q+1)}+\lambda z^2(\mathcal{F}_s^{\lambda} f(z))^{(q+2)}-(s-q)\left[\lambda z(\mathcal{F}_s^{\lambda} f(z))^{(q+1)}+(\delta-\lambda)(\mathcal{F}_s^{\lambda} f(z))^{(q)}\right]$

and

$$B(z)=b\left(\lambda z(\mathcal{F}_s^{\lambda} f(z))^{(q+1)}+(\delta-\lambda)(\mathcal{F}_s^{\lambda} f(z))^{(q)}\right).$$

Consider $|A(z)+(1-\alpha)B(z)|$

$$=\left|\delta z(\mathcal{F}_s^{\lambda} f(z))^{(q+1)}+\lambda z^2(\mathcal{F}_s^{\lambda} f(z))^{(q+2)}-(s-q)\left[\lambda z\left(\mathcal{F}_s^{\lambda} f(z)\right)^{(q+1)}+(\delta-\lambda)\left(\mathcal{F}_s^{\lambda} f(z)\right)^{(q)}\right]+(1-\alpha)b\left(\lambda z(\mathcal{F}_s^{\lambda} f(z))^{(q+1)}+(\delta-\lambda)(\mathcal{F}_s^{\lambda} f(z))^{(q)}\right)\right|$$

$$=\left|\sum_{n=k+s}^{\infty}\frac{n!}{q!(s-q)!}\left(\frac{\lambda+s}{\lambda+n}\right)[\lambda(n-1-q)+\delta][-n+q+s-q]a_n\, z^{n-q}+(1-\alpha)b\left(\left(\frac{s!}{q!(s-q)!}\right)[\lambda(s-1-q)+\delta]z^{n-q}-\sum_{n=k+s}^{\infty}\frac{n!}{q!(n-q)!}\left(\frac{\lambda+s}{\lambda+n}\right)[\lambda(n-1-q)+\delta]\,a_n\, z^{n-q}\right)\right|. \tag{5}$$

For sure Re($z$)<|$z$|, by letting $z\rightarrow$ 1- through real axis, (5) gives

$$\left|\sum_{n=k+s}^{\infty}\frac{n!}{q!\,(n-q)!}\left(\frac{\lambda+s}{\lambda+n}\right)[\lambda(n-1-q)+\delta][n-s]a_n + b(1-\alpha)\left([\lambda(s-1-q)+\delta]\left(\frac{s!}{q!\,(s-q)!}\right) - \sum_{n=k+s}^{\infty}\frac{n!}{q!\,(n-q)!}\left(\frac{\lambda+s}{\lambda+n}\right)[\lambda(n-1-q)+\delta]a_n\right)\right|$$

$$\sum_{n=k+s}^{\infty}\frac{n!}{q!\,(n-q)!}\left(\frac{\lambda+s}{\lambda+n}\right)[\lambda(n-1-q)+\delta]([n-s]-(1-\alpha)b)a_n \geq -b(1-\alpha)\,[\lambda(s-1-q)+\delta]\left(\frac{s!}{q!\,(s-q)!}\right). \quad (6)$$

Consider $|A(z)-B(z)(1+\alpha)|$

$$= \left|\delta z(\mathcal{F}_s^{\lambda}f(z))^{(q+1)} + \lambda z^2(\mathcal{F}_s^{\lambda}f(z))^{(q+2)} - (s-q)\left[\lambda z\left(\mathcal{F}_s^{\lambda}f(z)\right)^{(q+1)} + (\delta-\lambda)\left(\mathcal{F}_s^{\lambda}f(z)\right)^{(q)}\right] - (1+\alpha)b\left(\lambda z(\mathcal{F}_s^{\lambda}f(z))^{(q+1)} + (\delta-\lambda)(\mathcal{F}_s^{\lambda}f(z))^{(q)}\right)\right|$$

$$= \left|\sum_{n=k+s}^{\infty}\frac{n!}{q!(n-q)!}\left(\frac{\lambda+s}{\lambda+n}\right)[\lambda(n-1-q)+\delta][-n+q+s-q]a_n\,z^{n-q} - (1+\alpha)b\left(\left(\frac{s!}{q!(s-q)!}\right)[\lambda(s-1-q)+\delta]z^{n-q} - \sum_{n=k+s}^{\infty}\frac{n!}{q!(n-q)!}\left(\frac{\lambda+s}{\lambda+n}\right)[\lambda(n-1-q)+\delta]\,a_n\,z^{n-q}\right)\right|. \quad (7)$$

For sure Re($z$)<|$z$|, by letting $z\rightarrow$ 1- through real axis, (7) gives

$$\left|\sum_{n=k+s}^{\infty}\frac{n!}{q!\,(n-q)!}\left(\frac{\lambda+s}{\lambda+n}\right)[\lambda(n-1-q)+\delta][n-s]a_n - b(1+\alpha)\left([\lambda(s-1-q)+\delta]\left(\frac{s!}{q!\,(s-q)!}\right) - \sum_{n=k+s}^{\infty}\frac{n!}{q!\,(n-q)!}\left(\frac{\lambda+s}{\lambda+n}\right)[\lambda(n-1-q)+\delta]a_n\right)\right|$$

$$\sum_{n=k+s}^{\infty}\frac{n!}{q!\,(n-q)!}\left(\frac{\lambda+s}{\lambda+n}\right)[\lambda(n-1-q)+\delta]([n-s]+(1+\alpha)b)a_n \geq b(1+\alpha)\,[\lambda(s-1-q)+\delta]\left(\frac{s!}{q!\,(s-q)!}\right). \quad (8)$$

Consider $|A(z)+(1-\alpha)B(z)| - |A(z)-(1+\alpha)B(z)| \geq 0$ from (6) and (8), we obtain

$$\sum_{n=k+s}^{\infty}\frac{n!}{q!\,(n-q)!}\left(\frac{\lambda+s}{\lambda+n}\right)[\lambda(n-1-q)+\delta](-b\ \ )a_n - 2\alpha b\left(\frac{s!}{q!\,(s-q)!}\right)[\lambda(p-1-q)+\delta] \geq 0.$$

So, we have

$$\sum_{n=k+s}^{\infty} \frac{n!}{q!\,(n-q)!}\left(\frac{\lambda+s}{\lambda+n}\right)[\lambda(n-1-q)+\delta]\,|b|a_n$$
$$\geq 2\alpha b\left(\frac{s!}{q!\,(s-q)!}\right)[\lambda(p-1-q)+\delta].$$

Hence, we get

$$\sum_{n=k+s}^{\infty} \left(\frac{n!}{q!\,(n-q)!}\left(\frac{\lambda+s}{\lambda+n}\right)[\lambda(n-1-q)+\delta]|b|\right)a_n$$
$$\leq 2\alpha|b|\left(\frac{s!}{q!\,(s-q)!}\right)[\lambda(s-1-q)+\delta].$$

The maximum modulus principle implies that, $f(z) \in \mathfrak{RH}_{k,q}^{s}(\lambda, b, \delta)$.

**Corollary 1.2:** If $f(z) \in \mathfrak{RH}_{k,\mathrm{q}}^{s}(\lambda, b, \delta)$ then

$$a_n \leq \frac{|b|2\alpha\frac{s!}{q!(s-q)!}[\lambda(s-1-q)+\delta]}{\left(\frac{n!}{q!(n-q)!}\left(\frac{\lambda+s}{\lambda+n}\right)[\lambda(n-1-q)+\delta]|b|\right)}.$$

**Theorem 1.3:** A function $f(z) \in \mathcal{A}(\mathrm{k})$ and given by (1) is in $\mathfrak{RH}_{k,q}^{s}(\lambda, b, \delta)$ if and only if

$$\sum_{n=k+s}^{\infty} n\left(\frac{n!}{q!(n-q)!}\left(\frac{\lambda+s}{\lambda+n}\right)[\lambda(n-1-q)+\delta]|b|\right)a_n \leq 2\alpha s|b|\left(\frac{s!}{q!(s-q)!}\right)[\lambda(s-1-q)+\delta]. \quad (9)$$

**Proof:** Let $f(z) \in \mathfrak{RS}_{k,\mathrm{q}}^{s}(\lambda, b, \delta)$ therefore, $zf'(z) \in \mathfrak{RH}_{k,q}^{s}(\lambda, b, \delta)$.
Let $g(z) = zf'(z)$. Then $g(z) \in \mathfrak{RH}_{k,\mathrm{q}}^{s}(\lambda, b, \delta)$ therefore,

$$\mathcal{R}e\left\{\frac{1}{b}\left(\frac{\delta z(\mathcal{F}_s^{\lambda}f(z))^{(q+1)} + \lambda z^2(\mathcal{F}_s^{\lambda}f(z))^{(q+2)}}{\lambda z(\mathcal{F}_s^{\lambda}f(z))^{(q+1)} + (\delta-\lambda)(\mathcal{F}_s^{\lambda}f(z))^{(q)}} - (s-q)\right)\right\} > \alpha \quad (10)$$

$$\delta z(\mathcal{F}_s^{\lambda}f(z))^{(q+1)} + \lambda z^2(\mathcal{F}_s^{\lambda}f(z))^{(q+2)}$$
$$= \left(\frac{s!}{q!\,(s-q)!}\right)s(s-q)\left[\delta + \lambda(s-1-q)z^{s-q} - \sum_{n=k+s}^{\infty}\frac{n!}{q!\,(n-q)!}\left(\frac{\lambda+s}{\lambda+n}\right)n(n-q)[\delta + \lambda(s-1-q)]a_n z^{n-q}\right].$$

Now consider

$$\lambda z(\mathcal{F}_s^{\lambda}f(z))^{(q+1)} + (\delta-\lambda)\left(\mathcal{F}_s^{\lambda}f(z)\right)^{(q)}$$
$$= s\left(\frac{s!}{q!\,(s-q)!}\right)[\lambda(s-q)+\delta-\lambda]z^{s-q} - \sum_{n=k+s}^{\infty}\frac{n!}{q!\,(n-q)!}\left(\frac{\lambda+s}{\lambda+n}\right)n[\lambda(n-q)+\delta-\lambda]a_n z^{n-q}.$$

From (10) we have

$$\mathcal{R}e\left\{\frac{1}{b}\left(\frac{\sum_{n=k+s}^{\infty}\frac{n!}{q!\,(n-q)!}\left(\frac{\lambda+s}{\lambda+n}\right)[\lambda(n-1-q)+\delta]n[-n+m]a_n z^{n-q}}{\left(\frac{s!}{q!\,(s-q)!}\right)m[\lambda(s-1-q)+\delta]z^{s-q} - \sum_{n=k+s}^{\infty}\frac{n!}{q!\,(n-q)!}\left(\frac{\lambda+s}{\lambda+n}\right)n[\lambda(n-1-q)+\delta]a_n z^{n-q}}\right)\right\} > \alpha. \quad (11)$$

We know that Re($z$) < | $z$|, $z \to$ 1- through real axis, we learn from (11)

$$\frac{1}{|b|}\left[\frac{\sum_{n=k+s}^{\infty}\frac{n!}{q!\,(n-q)!}\left(\frac{ꝺ+s}{ꝺ+n}\right)[ꝺ(n-q-1)+\delta][-n+m]a_n}{\left(\frac{s!}{q!\,(s-q)!}\right)m[ꝺ(s-1-q)+\delta]-\sum_{n=k+s}^{\infty}\frac{n!}{q!\,(n-q)!}\left(\frac{ꝺ+s}{ꝺ+n}\right)n[ꝺ(n-1-q)+\delta]a_n}\right]$$

$\geq \alpha.$

Simplifying yields

$$\sum_{n=k+s}^{\infty}\left(\frac{n!}{q!\,(n-q)!}\left(\frac{ꝺ+s}{ꝺ+n}\right)[ꝺ(n-1-q)+\delta]|b|\right)a_n \geq 2\alpha s|b|\left(\frac{s!}{q!\,(s-q)!}\right)[ꝺ(s-1-q)+\delta].$$

Hence, we get (9)

From (9), set $|z|=1$ to show the converse.

$$\mathcal{R}e\left\{\left(\frac{\delta z(\mathcal{F}_s^{ꝺ}f(z))^{(q+1)}+ꝺz^2(\mathcal{F}_s^{ꝺ}f(z))^{(q+2)}}{ꝺz(\mathcal{F}_s^{ꝺ}f(z))^{(q+1)}+(\delta-ꝺ)(\mathcal{F}_s^{ꝺ}f(z))^{(q)}}-(s-q)\right)\right\}$$

$$=\mathcal{R}e\left\{\left(\frac{\sum_{n=k+s}^{\infty}\frac{n!}{q!\,(n-q)!}\left(\frac{ꝺ+s}{ꝺ+n}\right)n[ꝺ(n-1-q)+\delta][n-s]\,a_nz^{n-q}}{s\left(\frac{s!}{q!\,(s-q)!}\right)[ꝺ(s-1-q)+\delta]z^{s-q}-\sum_{n=k+s}^{\infty}\frac{n!}{q!\,(n-q)!}\left(\frac{ꝺ+s}{ꝺ+n}\right)n[ꝺ(n-1-q)+\delta]a_nz^{n-q}}\right)\right\}$$

$$\leq 2\alpha|b|\frac{s\frac{n!}{q!\,(n-q)!}[ꝺ(s-1-q)+\delta]z^{n-q}-\sum_{n=k+s}^{\infty}\frac{n!}{q!\,(n-q)!}\left(\frac{ꝺ+s}{ꝺ+n}\right)n[ꝺ(n-1-q)+\delta]a_nz^{n-q}}{s\frac{n!}{q!\,(n-q)!}[ꝺ(n-1-q)+\delta]z^{n-q}-\sum_{n=k+s}^{\infty}\frac{n!}{q!\,(n-q)!}\left(\frac{ꝺ+s}{ꝺ+n}\right)n[ꝺ(n-1-q)+\delta]a_nz^{n-q}}.$$

$$=2\alpha|b|$$

Hence, by the maximum modulus principle, $g(z)\in\Re\mathfrak{H}_{\boldsymbol{k,q}}^{\boldsymbol{s}}(ꝺ,b,\delta)$.

**Corollary 1.4:** $f(z)\in\Re\mathfrak{S}_{\boldsymbol{k,q}}^{\boldsymbol{s}}(ꝺ,b,\delta)$ then

$$a_n\leq\frac{2|b|\alpha s[ꝺ(s-1-q)+\delta]\left(\frac{s!}{q!\,(s-q)!}\right)}{n\left(\frac{n!}{q!\,(n-q)!}\left(\frac{ꝺ+s}{ꝺ+n}\right)[ꝺ(n-1-q)+\delta]|b|\right)},n\geq k+s.$$

**Theorem 1.5:** If $f(z)\in\Re\mathfrak{H}_{\boldsymbol{k,q}}^{\boldsymbol{s}}(ꝺ,b,\delta)$, then

$$|z|^s-|z|^{k+s}\frac{2\alpha|b|\left(\frac{s!}{q!\,(s-q)!}\right)[ꝺ(s-1-q)+\delta]}{\left(\frac{(k+s)!}{q!\,(k+s-q)!}\left(\frac{ꝺ+s}{ꝺ+k+s}\right)[ꝺ(k+s-1-q)+\delta]|b|\right)}\leq|f(z)|$$

$$\leq |z|^s+|z|^{k+s}\frac{2\alpha|b|\left(\frac{s!}{q!\,(s-q)!}\right)[ꝺ(s-1-q)+\delta]}{\left(\frac{(k+s)!}{q!\,(k+s-q)!}\left(\frac{ꝺ+s}{ꝺ+k+s}\right)[ꝺ(k+s-1-q)+\delta]|b|\right)}.$$

With equality hold for

$$f(z)=z^s-z^{k+s}\frac{2\alpha|b|\left(\frac{s!}{q!\,(s-q)!}\right)[ꝺ(s-1-q)+\delta]}{\left(\frac{(k+s)!}{q!\,(k+s-q)!}\left(\frac{ꝺ+s}{ꝺ+k+s}\right)[ꝺ(k+s-1-q)+\delta]|b|\right)}$$

**Proof:** $f(z) \in \Re\mathfrak{S}_{k,q}^{s}(\lambda, b, \delta)$
Therefore, from (1)

$$\sum_{n=k+s}^{\infty} a_n \leq \frac{2|b|\alpha s[\lambda(s-1-q)+\delta]\left(\frac{s!}{q!\,(s-q)!}\right)}{\left(\frac{n!}{q!\,(n-q)!}\left(\frac{\lambda+s}{\lambda+n}\right)[\lambda(n-1-q)+\delta]|b|\right)}$$

$$|f(z)| \geq |z|^s - \sum_{n=k+s}^{\infty} |a_n||z|^n$$
$$\geq |z|^s - |z|^{k+s}\sum_{n=k+s}^{\infty}|a_n|$$
$$\geq |z|^s - |z|^{k+s}\frac{2\alpha|b|\left(\frac{s!}{q!\,(s-q)!}\right)[\lambda(s-1-q)+\delta]}{\left(\frac{(k+s)!}{q!\,(k+s-q)!}\left(\frac{\lambda+s}{\lambda+k+s}\right)[\lambda(k+s-1-q)+\delta]|b|\right)}.$$

Similarly,

$$|f(z)| \leq |z|^s - |z|^{k+s}\frac{2\alpha|b|\left(\frac{s!}{q!\,(s-q)!}\right)[\lambda(s-1-q)+\delta]}{\left(\frac{(k+s)!}{(k+s-q)!\,q!}\left(\frac{\lambda+s}{\lambda+k+s}\right)[\lambda(k+s-1-q)+\delta]|b|\right)}.$$

With equality hold for

$$|f(z)| = |z|^s - |z|^{k+s}\frac{2\alpha|b|\left(\frac{s!}{q!\,(s-q)!}\right)[\lambda(s-1-q)+\delta]}{\left(\frac{(k+s)!}{q!\,(k+s-q)!}\left(\frac{\lambda+s}{\lambda+k+s}\right)[\lambda(k+s-1-q)+\delta]|b|\right)}.$$

Therefore, we get required result.

**Theorem 1.6:** If $f(z) \in \Re\mathfrak{S}_{k,q}^{s}(\lambda, b, \delta)$ then

$$|z|^s - |z|^{k+s}\frac{2|b|\alpha s[\lambda(s-1-q)+\delta]\left(\frac{s!}{q!\,(s-q)!}\right)}{(k+s)\left(\frac{(s+k)!}{q!\,(k+s-q)!}\left(\frac{s+\lambda}{\lambda+k+s}\right)[\lambda(k+s-1-q)+\delta]|b|\right)} \leq |f(z)|$$
$$\leq |z|^s + |z|^{k+s}\frac{2\alpha s|b|\left(\frac{s!}{q!\,(s-q)!}\right)[\lambda(s-1-q)+\delta]}{(k+s)\left(\frac{(k+s)!}{q!\,(k+s-q)!}\left(\frac{\lambda+s}{\lambda+k+s}\right)[\lambda(k+s-1-q)+\delta]|b|\right)}.$$

With equality hold for

$$f(z) = |z|^s - |z|^{k+s}\frac{2\alpha s|b|\left(\frac{s!}{q!\,(s-q)!}\right)[\lambda(s-1-q)+\delta]}{(k+s)\left(\frac{(k+s)!}{q!\,(k+s-q)!}\left(\frac{\lambda+s}{\lambda+k+s}\right)[\lambda(k+s-1-q)+\delta]|b|\right)}.$$

**Proof:** Let $f(z) \in \Re\mathfrak{S}_{k,q}^{s}(\lambda, b, \delta)$. Therefore,

$$\sum_{n=k+s}^{\infty} a_n \leq \frac{2\alpha s|b|\left(\frac{s!}{q!\,(s-q)!}\right)[\lambda(s-1-q)+\delta]}{n\left(\frac{n!}{q!\,(n-q)!}\left(\frac{\lambda+s}{\lambda+n}\right)[\lambda(n-1-q)+\delta]|b|\right)}.$$

We have

$$|f(z)| \geq |z|^s - \sum_{n=k+s}^{\infty} |a_n||z|^n$$
$$\geq |z|^s - |z|^{k+s} \sum_{n=k+s}^{\infty} |a_n|$$
$$\geq |z|^s - |z|^{k+s} \frac{2\alpha\, s|b|\left(\frac{s!}{q!\,(s-q)!}\right)[\lambda(s-1-q)+\delta]}{(k+s)\left(\frac{(k+s)!}{q!\,(k+s-q)!}\left(\frac{\lambda+s}{\lambda+k+s}\right)[\lambda(k+s-1-q)+\delta]|b|\right)}.$$

Similarly,

$$|f(z)| \leq |z|^s + |z|^{k+s} \frac{2\alpha s|b|\left(\frac{s!}{q!\,(s-q)!}\right)[\lambda(s-1-q)+\delta]}{(k+s)\left(\frac{(k+s)!}{q!\,(k-q+s)!}\left(\frac{\lambda+s}{\lambda+k+s}\right)[\lambda(k-1-q+s)+\delta]|b|\right)}.$$

Therefore, we get the required result.

**Theorem 1.7:** If $f(z) \in \mathfrak{R}\mathfrak{H}_{k,q}^{s}(\lambda, b, \delta)$ then

$$s|z|^{s-1} + |z|^{k+s-1} \frac{2\alpha|b|\left(\frac{s!}{q!\,(s-q)!}\right)(k+s)[\lambda(s-1-q)+\delta]}{\left(\frac{(k+s)!}{(k+s-q)!\,q!}\left(\frac{\lambda+s}{\lambda+k+s}\right)[\lambda(k+s-1-q)+\delta]|b|\right)} \leq |f'(z)|$$
$$\leq s|z|^{s-1} + |z|^{k+s-1} \frac{2\alpha|b|\left(\frac{s!}{q!\,(s-q)!}\right)(k+p)[\lambda(s-1-q)+\delta]}{\left(\frac{(k+s)!}{q!\,(k+s-q)!}\left(\frac{\lambda+s}{\lambda+k+s}\right)[\lambda(k+s-1-q)+\delta]|b|\right)}.$$

**Proof:** Let $f(z) \in \mathfrak{R}\mathfrak{H}_{k,q}^{s}(\lambda, b, \delta)$ therefore, from (1)

$$\sum_{n=k+s}^{\infty} a_n \leq \frac{2|b|\alpha\left(\frac{s!}{q!\,(s-q)!}\right)[\lambda(s-1-q)+\delta]}{\left(\frac{n!}{q!\,(n-q)!}\left(\frac{\lambda+s}{\lambda+n}\right)[\lambda(n-1-q)+\delta]|b|\right)}$$

$$|f'(z)| \geq s|z|^{s-1} - \sum_{n=k+s}^{\infty} |a_n| n |z|^{n-1} \geq s|z|^{s-1} - |z|^{k+s-1}(k+s)\sum_{n=k+s}^{\infty} |a_n|$$
$$\geq s|z|^{s-1} - |z|^{k+s-1} \frac{2\alpha|b|\left(\frac{s!}{q!\,(s-q)!}\right)(k+s)[\lambda(s-1-q)+\delta]}{\left(\frac{(k+s)!}{q!\,(k+s-q)!}\left(\frac{\lambda+s}{\lambda+k+s}\right)[\lambda(k+s-1-q)+\delta]|b|\right)}.$$

Similarly,

$$|f'(z)| \leq s|z|^{s-1} - |z|^{k+s-1}\frac{2\alpha|b|\left(\frac{s!}{q!\,(s-q)!}\right)(k+s)[\lambda(s-1-q)+\delta]}{\left(\frac{(k+s)!}{q!\,(k+s-q)!}\left(\frac{\lambda+s}{\lambda+k+s}\right)[\lambda(k+s-1-q)+\delta]|b|\right)}.$$

Therefore, we get the required result

**Theorem 1.8:** If $f(z) \in \Re\mathfrak{S}_{k,q}^{s}(\lambda, b, \delta)$ then

$$s|z|^{s-1} + |z|^{k+s-1}\frac{2\alpha s|b|\left(\frac{s!}{q!\,(s-q)!}\right)[\lambda(s-1-q)+\delta]}{\left(\frac{(k+s)!}{(k+s-q)!\,q!}\left(\frac{\lambda+s}{\lambda+k+s}\right)[\lambda(k+s-1-q)+\delta]|b|\right)} \leq |f'(z)|$$

$$\leq s|z|^{s-1} + |z|^{k+s-1}\frac{2\alpha s|b|\left(\frac{s!}{q!\,(s-q)!}\right)[\lambda(s-1-q)+\delta]}{\left(\frac{(k+p)!}{q!\,(k+s-q)!}\left(\frac{\lambda+s}{\lambda+k+s}\right)[\lambda(k+s-1-q)+\delta]|b|\right)}.$$

**Proof:** Let $f(z) \in \Re\mathfrak{S}_{k,q}^{s}(\lambda, b, \delta)$

Therefore, from Theorem 1.1

$$\sum_{n=k+s}^{\infty} a_n \leq \frac{2\alpha s|b|\left(\frac{s!}{q!\,(s-q)!}\right)[\lambda(s-1-q)+\delta]}{n\left(\frac{n!}{q!\,(n-q)!}\left(\frac{\lambda+s}{\lambda+n}\right)[\lambda(n-1-q)+\delta]|b|\right)}$$

$$|f'(z)| \geq s|z|^{s-1} - \sum_{n=k+s}^{\infty}|a_n|n|z|^{n-1} \geq s|z|^{s-1} - |z|^{k+s-1}(k+s)\sum_{n=k+s}^{\infty}|a_n|$$

$$\geq s|z|^{s-1} - |z|^{k+s-1}\frac{2\alpha s|b|\left(\frac{s!}{q!\,(k+s-q)!}\right)(k+s)[\lambda(s-1-q)+\delta]}{\left(\frac{(k+s)!}{q!\,(k+s-q)!}\left(\frac{\lambda+s}{\lambda+k+s}\right)[\lambda(k+s-1-q)+\delta]|b|\right)}.$$

Similarly,

$$|f'(z)| \leq s|z|^{s-1} + |z|^{k+s-1}\frac{2|b|\alpha s\left(\frac{s!}{q!\,(s-q)!}\right)(k+s)[\lambda(s-1-q)+\delta]}{\left(\frac{(k+s)!}{q!\,(k+s-q)!}\left(\frac{\lambda+s}{\lambda+k+s}\right)[\lambda(k+s-1-q)+\delta]|b|\right)}.$$

Therefore, we get the required result.

**Theorem1.9:** If $f(z) \in \Re\mathfrak{H}_{k,q}^{s}(\lambda, b, \delta)$**,** then $h \in \mathrm{K}(\propto)$ in $|\boldsymbol{z}| < r_{\mathbf{1}}(s, k, q, \lambda, b, \delta, \propto)$ where

$$r_1 = \inf_n\left(\left(\frac{(s-\propto)\frac{n!}{q!\,(n-q)!}\left(\frac{\lambda+s}{\lambda+n}\right)[\lambda(n-1-q)+\delta]|b|}{2\alpha n|b|\left(\frac{s!}{q!\,(s-q)!}\right)[\lambda(n-1-q)+\delta]}\right)^{\frac{1}{n-s}}\right).$$

**Proof:** We need to show that $\left|\frac{f'(z)}{z^{s-1}} - s\right| < s-\propto$

$$\left|\frac{f'(z)}{z^{s-1}} - s\right| \leq \sum_{n=k+s}^{\infty} n|a_n|\,|z|^{n-s} \leq s-\propto. \quad (12)$$

From (1) we have

$$\sum_{n=k+s}^{\infty} a_n \leq \frac{2\alpha|b|\left(\frac{s!}{q!\,(s-q)!}\right)[\lambda(s-1-q)+\delta]}{\left(\frac{n!}{q!\,(n-q)!}\left(\frac{\lambda+s}{\lambda+n}\right)[\lambda(n-1-q)+\delta]|b|\right)}.$$

That is

$$\sum_{n=k+s}^{\infty}\left(\frac{\left(\frac{n!}{q!\,(n-q)!}\left(\frac{ꝺ+s}{ꝺ+n}\right)[ꝺ(n-1-q)+\delta]|b|\right)}{2|b|\alpha\left(\frac{s!}{q!\,(s-q)!}\right)[ꝺ(s-1-q)+\delta]}\right)a_n \le 1.$$

Observe that (12) is true if

$$\frac{n}{(s-\propto)}|z|^{n-s} \le \frac{\left(\frac{n!}{q!\,(n-q)!}\left(\frac{ꝺ+s}{ꝺ+n}\right)|b|[ꝺ(n-1-q)+\delta]\right)}{2|b|\alpha\left(\frac{s!}{q!\,(s-q)!}\right)[ꝺ(s-1-q)+\delta]}.$$

Therefore,

$$|z| \le \left(\frac{(s-\propto)\left(\frac{n!}{q!\,(n-q)!}\left(\frac{ꝺ+s}{ꝺ+n}\right)[ꝺ(n-1-q)+\delta]|b|\right)}{2\alpha n|b|\left(\frac{s!}{q!\,(s-q)!}\right)[ꝺ(s-1-q)+\delta]}\right)^{\frac{1}{n-s}},$$

where ( $s \neq n, s, n \in \mathbb{N}$ ).

**Theorem 1.10:** If $f(z) \in \Re\mathfrak{H}^{s}_{k,q}(ꝺ, b, \delta)$, then $h \in \mathcal{S}^*(\propto)$ in $|z| < r_2(s, k, q, ꝺ, b, \delta, \propto)$ where

$$r_2 = \inf_{n}\left(\left(\frac{(s-\propto)\left(\frac{n!}{q!\,(n-q)!}\left(\frac{ꝺ+s}{ꝺ+n}\right)[ꝺ(n-1-q)+\delta]|b|\right)}{(n-\propto)2\alpha|b|\left(\frac{s!}{q!\,(s-q)!}\right)[ꝺ(s-1-q)+\delta]}\right)^{\frac{1}{n-s}}\right).$$

**Proof:** We must show that

$$\left|\frac{f'(z)}{z^{s-1}} - s\right| < s-\propto.$$

We have

$$\left|\frac{zf'(z)}{f(z)} - s\right| = \left|\frac{-\sum_{n=k+s}^{\infty}(n-s)a_n z^n}{z^s - \sum_{n=k+s}^{\infty} a_n z^n}\right| \le \frac{\sum_{n=k+s}^{\infty}(n-s)|a_n|\,|z|^{n-s}}{1-\sum_{n=k+s}^{\infty}|a_n|\,|z|^{n-s}} \le s - \propto. \qquad (13)$$

Hence (13) hold true if

$$\sum_{n=k+s}^{\infty}\frac{(n-\propto)}{(s-\propto)}|a_n|\,|z|^{n-s} \le 1. \qquad (14)$$

From (3) we have

$$\sum_{n=k+s}^{\infty}\frac{\left(\frac{n!}{q!\,(n-q)!}\left(\frac{ꝺ+s}{ꝺ+n}\right)[ꝺ(n-1-q)+\delta]|b|\right)}{2\alpha|b|\left(\frac{s!}{q!\,(s-q)!}\right)[ꝺ(s-1-q)+\delta]}a_n \le 1. \qquad (15)$$

Hence by using (14) and (15) we get

$$\frac{(n-\propto)}{(s-\propto)}|z|^{n-s} \le \frac{\left(\frac{n!}{q!\,(n-q)!}\left(\frac{ꝺ+s}{ꝺ+n}\right)[ꝺ(n-1-q)+\delta]|b|\right)}{2\alpha|b|\left(\frac{s!}{q!\,(s-q)!}\right)[ꝺ(s-1-q)+\delta]}$$

$$|z|^{n-s} \le \frac{(s-\propto)\left(\frac{n!}{(n-q)!\,q!}\left(\frac{ꝺ+s}{ꝺ+n}\right)[ꝺ(n-1-q)+\delta]|b|\right)}{(n-\propto)2\alpha|b|\left(\frac{s!}{(s-q)!\,q!}\right)[ꝺ(s-1-q)+\delta]}$$

$$|z| \le \left(\frac{(s-\propto)\left(\frac{n!}{q!\,(n-q)!}\left(\frac{ꝺ+s}{ꝺ+n}\right)[ꝺ(n-1-q)+\delta]|b|\right)}{(n-\propto)2\alpha|b|\left(\frac{s!}{q!\,(s-q)!}\right)[ꝺ(s-1-q)+\delta]}\right)^{\frac{1}{n-s}},$$

where ( $s \neq n, s, n \in \mathbb{N}$ ).

**Theorem 1.11:** If $f(z) \in \Re\mathfrak{H}_{k,q}^{s}(\lambda, b, \delta)$, then $h \in \boldsymbol{C}(\propto)$ in $|\boldsymbol{z}| < r_3(s, k, q, \lambda, b, \delta, \propto)$ where

$$r_3 = \inf_n \left( \left( \frac{s(s-\propto)\left(\frac{n!}{q!\,(n-q)!}\left(\frac{\lambda+s}{\lambda+n}\right)[\lambda(n-1-q)+\delta]|b|\right)}{n(n-\propto)2\alpha|b|\left(\frac{s!}{q!\,(s-q)!}\right)[\lambda(n-1-q)+\delta]} \right)^{\frac{1}{n-s}} \right)$$

**Proof:** $f$ is convex if $zf'$ is starlike. We must show that

$$\left|\frac{f'(z)}{z^{s-1}} - s\right| < s-\propto.$$

That is

$$\sum_{n=k+s}^{\infty} \frac{n(n-\propto)}{s(s-\propto)} |a_n|\,|z|^{n-s} \le 1. \tag{16}$$

From (3) we have

$$\sum_{n=k+s}^{\infty} \frac{\left(\frac{n!}{q!\,(n-q)!}\left(\frac{\lambda+s}{\lambda+n}\right)[\lambda(n-1-q)+\delta]|b|\right)}{2\alpha|b|\left(\frac{s!}{q!\,(s-q)!}\right)[\lambda(s-1-q)+\delta]} a_n \le 1. \tag{17}$$

Hence, by using (16) and (17) we get

$$\frac{n(n-\propto)}{s(s-\propto)}|z|^{n-s} \le \frac{\left(\frac{n!}{q!\,(n-q)!}\left(\frac{\lambda+s}{\lambda+n}\right)[\lambda(n-1-q)+\delta]|b|\right)}{2\alpha|b|\left(\frac{s!}{q!\,(s-q)!}\right)[\lambda(s-1-q)+\delta]}$$

$$|z| \le \left( \frac{s(s-\propto)\left(\frac{n!}{q!\,(n-q)!}\left(\frac{\lambda+s}{\lambda+n}\right)[\lambda(n-1-q)+\delta]|b|\right)}{(n-\propto)2\alpha|b|\left(\frac{s!}{q!\,(n-q)!}\right)[\lambda(s-1-q)+\delta]} \right)^{\frac{1}{n-s}},$$

where ( $s \neq n, s, n \in \mathbb{N}$ ).

**Theorem 1.12:** Let $f_1(z) = z^s$ and

$$f_n(z) = z^s - \frac{\left(\frac{n!}{q!\,(n-q)!}\left(\frac{\lambda+s}{\lambda+n}\right)[\lambda(n-1-q)+\delta]|b|\right)}{2\alpha|b|\left(\frac{s!}{q!\,(s-q)!}\right)[\lambda(s-1-q)+\delta]} \text{ for } n \ge k+s.$$

Then if $f(z) \in \Re\mathfrak{H}_{k,q}^{s}(\lambda, b, \delta)$ if and only if $f(z)$ can be expresses in the form $f(z) = \lambda_1 f_1(z) + \sum_{n=k+s}^{\infty} \lambda_n f_n(z)$ where $\lambda_n \ge 0$ and $\lambda_1 + \sum_{n=k+s}^{\infty} \lambda_n = 1$.

## 2. Conclusions

We obtained some properties of our subclass which is related to Generalized Bernardi-Libera-Livingston of Integral Operator and investigated coefficient bounds and established distortion theorem, radii of star likeness, convexity and closure theorem.

## REFERENCES

**[1]** M. K. Aouf, and J. Dziok, "Inclusion and neighborhood properties of certain subclasses of analytic and multivalent functions," *European Journal of Pure and Applied Mathematics*, vol. 2, no. 4, pp. 544-553, 2009.

**[2]** W. G. Atshan, "Subclass of meromorphic functions with positive coefficients defined by Ruscheweyh derivative. II," *Surveys in Mathematics and its Applications* vol. 3, no.1, pp. 67-77, 2008.

**[3]** N. E. Cho, and H. M. Srivastava, "Argument estimates of certain analytic functions defined by a class of multiplier transformations," *Mathematical and Computer Modelling*, vol. 37, no. 2, pp. 39-49, 2003.

**[4]** A. W. Goodman, "Univalent functions and nonanalytic curves," *Proceedings of the American Mathematical Society* vol. 8, no. 3, pp. 598-601, 1957.

**[5]** K. A. Jassim, Z. H. Mahmood, and B. N. Shihab, "On the Characteristic Properties for Subclasses of the Higher Derivative of Analytic Functions," *Iraqi Journal of Science*, vol. 61, no. 12, pp. 3372-3378, 2020.

**[6]** R. K. Raina and H. M. Srivastava, "Inclusion and neighborhood properties of some analytic and multivalent functions," *Journal of Inequalities in Pure and Applied. Mathematics,* vol.7, no.1, pp. 1443-5756, 2006.

**[7]** H. M. Srivastava, S. S. Eker, and B. Şeker, "Inclusion and neighborhood properties for certain classes of multivalent analytic functions of complex order associated with the convolution structure," *Journal of Applied mathematics and computation*, vol. 212, no.1, pp.66-71, 2009.