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Some Properties of gw −Prime Submodules and the Effect of Localization on Their Structures

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Abstract
In this paper, gw −prime modules are studied and investigated. Some properties of this type of submodules are proved. Some conditions are given under which every submodule of a gw −prime submodule is maximal and some conditions are given which make every proper submodule of gw −prime modules as a prime submodule. It is proved that multiplication modules in which gw −prime submodules are finitely generated are Noetherian. In addition, the effects of localization on gw −prime submodules are studied and some results concerning the localization of gw −prime submodules are proved.

Keywords: Maximal submodules, prime submodules, gw −prime submodules, multiplicative systems and localization

بعض خواص المقاسات الجزئية الاولية من النمط (gw) و تأثير التحميل على بناها

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الخلاصة
في هذا البحث تمت دراسة المقاسات الاولية من النمط (gw) حيث تمت البرهنة على بعض خواصها و اعطيت بعض الشروط عند توفرها يصبح كل مقاس جزئي من مقاس اولي من النمط (gw) مقاسا جزئيا اعظما و كذلك اعطيت شروطا اخرى و التي تجعل كل مقاس جزئي فعلي من مقاس اولي من النمط (gw) مقاسا جزئيا اوليا. تمت البرهنة على ان المقاسات الضربية و التي تكون فيها المقاسات الجزئية الاولية من النمط (gw) منتهية التولد هى مقاسات نويتيرية. بالاضافة الى ذلك لقد تمت دراسة تأثير التمحيل على المقاسات الجزئية الاولية من النمط (gw) حيث تمت البرهنة على بعض النتائج التي تتعلق بتمحيل المقاسات الجزئية الاولية من النمط (gw).

1. Introduction

In [1-3], several authors studied and investigated weakly prime submodules, almost prime submodules, weakly S −prime submodules and gave many properties and characterizations

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of them in multiplication modules. In [4], Z. Bilgin, K. H. Oral, and Ü. Tekir, defined a new type of weakly prime submodules which they called gw −prime submodules and they proved some of their properties. In [5], H. A. Said and A. K. Jabbar, studied the localization of gw −prime submodules and proved some of their properties.

In this paper, we continue the studding of this type of submodules and some of their properties and look for the effects of localization on this type of submodule.

Let M be an R −module and N be a proper submodule of M. An R −module M is gw −prime if $abK = 0$, for $a, b \in R$, K a submodule of M, then $a^2K = 0$ or $b^2K = 0$ and N is called a gw −prime submodule if M/N is a gw −prime R −module [4] equivalently, N is a gw −prime submodule if for each $a, b \in R$ and each submodule K of M, the inclusion $abK \subseteq N$ implies $a^2K \subseteq N$ or $b^2K \subseteq N$ [4]. N is irreducible if whenever L and K are submodules of M with $N = L \cap K$, then either $N = L$ or $N = K$ [6] and it is primary if $a \in R$ and $x \in M$ such that $ax \in N$, then $x \in N$ or $a^nM \subseteq N$ for some $n \in \mathbb{Z}_+$ [7-10] and N is semiprime if $a \in R$ and $x \in M$ with $a^2x \in N$, then $ax \in N$ [11-13]. An R −module M is Noetherian if it satisfies $(a.c.c)$ for submodules [14] and N is prime if $r \in R$, $m \in M$ with $rm \in N$, then either $m \in N$ or $rM \subseteq N$ [15,16], and M is prime if the zero submodule is prime [17, 18]. As well as M is called faithful if $Ann(M) = 0$, that is, $(0: M) = 0$ [19, 20]. An R −module M is called a multiplication module, if N is a submodule of M, then there exists an ideal A of R such that $N = AM$ [21-23]. If M is a multiplication R −module and K, L submodules of M, then $K = AM$ and $L = BM$ for some ideals A, B of R, then KL is defined as $KL = ABM$ [11]. If S is a multiplicatively system in R, then $M_S = \{\frac{m}{s}: m \in M, s \in S\}$ is an R_S −module, where $\frac{x}{s} + \frac{y}{t} = \frac{tx+sy}{st}$, $\frac{x}{s}\frac{y}{t} = \frac{xy}{st}$, for $\frac{x}{s}, \frac{y}{t} \in M_S$ and $R_S = \{\frac{r}{s}: r \in R, s \in S\}$ (or $S^{-1}R$ [6]), where $\frac{a}{s} + \frac{b}{t} = \frac{ta+sb}{st}$ and $\frac{a}{s}\frac{b}{t} = \frac{ab}{st}$, for $\frac{a}{s}, \frac{b}{t} \in R_S$. If P is prime, then $R \setminus P$ is a multiplicative system and the ring $R_P = \{\frac{a}{p}: a \in R, p \notin P\}$ is local, called the localization of R at $R \setminus P$. Let R be a commutative ring with identity. A proper ideal A of R is called prime if $a, b \in R$ such that $ab \in A$, then $a \in A$ or $b \in A$ and A is called semiprime if $a^2 \in A$, then $a \in A$. In this paper, R is a commutative ring with identity $1 \neq 0$ and M is a left R −module unless otherwise stated.

2. Some properties of gw −prime submodules

In this section $N, L,$ and K are submodules of M. It is clear that, a prime ideal is semiprime but the converse is not true in general.

Example 2.1. The ideal $< 6 >$ in the ring $\mathbb{Z}$ is semiprime but not prime. Since $2.3 = 6 \in < 6 >$ but $2 \notin < 6 >$ and $3 \notin < 6 >$, so that $< 6 >$ is not prime. Next, let for $a \in \mathbb{Z}$ we have $a^2 \in < 6 >$, then $a^2 = 6k$ for some $k \in \mathbb{Z}$, then we get $2|a$ and $3|a$, so that $a = 2m = 3n$ for some $m, n \in \mathbb{Z}$. Now, $2|2m$, but $2m = 3n$, so that $2|3n$ and as $2 \nmid 3$, we get $2|n$, so we get $n = 2t$ for some $t \in \mathbb{Z}$, then $a = 3n = 3.2t = 6t \in < 6 >$, so that $< 6 >$ is semiprime but not prime.

In the following result, some conditions are given which make a certain type of semiprime ideals in commutative rings as prime ideals.

Proposition 2.2. Let N be gw −prime with $K \nsubseteq N$. If $(N: K)$ is semiprime, then $(N: K)$ is prime.

Proof. If $(N:K) = R$, then $1 \in (N:K)$, so we get $1K \subseteq N$, that is $K \subseteq N$ which is a contradiction. Hence, $(N:K) \neq R$. Let $ab \in (N:K)$ for $a, b \in R$, then $abK \subseteq N$ and as N is gw −prime, we get $a^2K \subseteq N$ or $b^2K \subseteq N$, that gives $a^2 \in (N:K)$ or $b^2 \in (N:K)$ and as $(N:K)$ is semiprime, we get $a \in (N:K)$ or $b \in (N:K)$. Hence, $(N:K)$ is prime.

Remark 2.3. If $x \in M$, then $(N:x) = (N:< x >)$. To prove this, let $r \in (N:x)$, then $rx \in N$. If $y \in < x >$ is any element, then $y = sx$ for some $s \in R$ and then $ry = rsx = srx \in N$, that means $r < x > \subseteq N$, so that $r \in (N:< x >)$. Hence, $(N:x) \subseteq (N:< x >)$. Next, let $y \in (N:< x >)$, then $y < x > \subseteq N$, so that $yx \in y < x > \subseteq N$, that gives $y \in (N:x)$, so that $(N:< x >) \subseteq (N:x)$. Hence, $(N:x) = (N:< x >)$.

Now, we give the following corollary:

Corollary 2.4. Let N be gw −prime. If $y \in M \setminus N$ and $(N:y)$ is semiprime, then $(N:y)$ is prime.
Proof. Since, $y \notin N$, so that $< y > \nsubseteq N$. As $(N:y) = (N:< y >)$, we get $(N:< y >)$ is semiprime. Hence, by Proposition 2.2, $(N:< y >)$ is prime, so that from Remark 2.3, we get $(N:y)$ is prime.

Remarks 2.5 (1). The gw −prime submodule is a generalization of a prime submodule. Let N be a prime submodule of M. Let $a, b \in R$ and K a submodule of M with $abK \subseteq N$. As N is prime, we get $aM \subseteq N$ or $bK \subseteq N$. As, $K \subseteq M$, we get that $aK \subseteq N$ or $bK \subseteq N$, which implies that $a^2K = aaK \subseteq aN \subseteq N$ or $b^2K = bbK \subseteq bK \subseteq N$. Hence, N is gw −prime.
(2) The gw −prime module is a generalization of a prime module. If M is a prime R −module, then the zero submodule of M is prime and hence it is gw −prime, so that M is gw −prime.

In below we give two examples, one for a gw −prime submodule which is not prime and the other for a gw −prime module which is not a prime module.

Examples 2.6 (1). Take the $\mathbb{Z}_8$ −module $\mathbb{Z}_8$. The submodule $A = \{\overline{0}, \overline{4}\}$ of $\mathbb{Z}_8$ is gw −prime but not prime. Let K be a submodule of $\mathbb{Z}_8$, with $\overline{r}\,\overline{s}\,K \subseteq A$, where $\overline{r}, \overline{s} \in \mathbb{Z}_8$. If $\overline{r} = \overline{0}$ or $\overline{s} = \overline{0}$, then clearly $\overline{r}.\overline{s}K = \{\overline{0}\}$ for all submodules K of A, so that $(\overline{r})^2K = \{\overline{0}\} \subseteq A$ and $(\overline{s})^2K = \{\overline{0}\} \subseteq A$ for all submodules K of $\mathbb{Z}_8$, so that we discuss the case when $\overline{r} \neq \overline{0}$ and $\overline{s} \neq \overline{0}$. Let for $\overline{0} \neq \overline{r} \in \mathbb{Z}_8$ and $\overline{0} \neq \overline{s} \in \mathbb{Z}_8$ and a submodule K of $\mathbb{Z}_8$ we have $\overline{r}.\overline{s}K \subseteq \{\overline{0}, \overline{4}\} = A$. The $\mathbb{Z}_8$ −module $\mathbb{Z}_8$ contains only four submodules which are $K_1 = \{\overline{0}\}$, $K_2 = \{\overline{0}, \overline{4}\}$, $K_3 = \{\overline{0}, \overline{2}, \overline{4}, \overline{6}\}$, $K_4 = \mathbb{Z}_8$.

(i) For the submodule $K_1 = \{\overline{0}\}$, for any $\overline{r}, \overline{s} \in \mathbb{Z}_8$, we have $\overline{r}.\overline{s}K_1 = \{\overline{0}\} \subseteq \{\overline{0}, \overline{4}\} = A$ and clearly $(\overline{r})^2K_1 = \{\overline{0}\} \subseteq \{\overline{0}, \overline{4}\} = A$ and $(\overline{s})^2K_1 = \{\overline{0}\} \subseteq \{\overline{0}, \overline{4}\} = A$.

(ii) For the submodule $K_2 = \{\overline{0}, \overline{4}\}$, since K_2 is a submodule of $\mathbb{Z}_8$, then for $\overline{r}, \overline{s} \in \mathbb{Z}_8$, $\overline{r}.\overline{s}K_2 \subseteq K_2 = \{\overline{0}, \overline{4}\} = A$, so for all $\overline{r}, \overline{s} \in \mathbb{Z}_8$, $(\overline{r})^2K_2 \subseteq K_2 = \{\overline{0}, \overline{4}\} = A$ and
$(\overline{s})^2K_2 \subseteq K_2 = \{\overline{0}, \overline{4}\} = A$.

(iii) We discuss the case for the submodules K_3, K_4 in the following table.

$K_3 = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$		$K_4 = \mathbb{Z}_8$	
$\bar{1}.\bar{1}K_3 \nsubseteq A = \{\bar{0}, \bar{4}\}$	-	$\bar{1}.\bar{1}\mathbb{Z}_8 \nsubseteq A = \{\bar{0}, \bar{4}\}$	-
$\bar{1}.\bar{2}K_3 \subseteq A = \{\bar{0}, \bar{4}\}$	$(\bar{2})^2 K_3 \subseteq A = \{\bar{0}, \bar{4}\}$	$\bar{1}.\bar{2}\mathbb{Z}_8 \nsubseteq A = \{\bar{0}, \bar{4}\}$	-
$\bar{1}.\bar{3}K_3 \nsubseteq A = \{\bar{0}, \bar{4}\}$	-	$\bar{1}.\bar{3}\mathbb{Z}_8 \nsubseteq A = \{\bar{0}, \bar{4}\}$	-
$\bar{1}.\bar{4}K_3 \subseteq A = \{\bar{0}, \bar{4}\}$	$(\bar{4})^2 K_3 \subseteq A = \{\bar{0}, \bar{4}\}$	$\bar{1}.\bar{4}\mathbb{Z}_8 \subseteq A = \{\bar{0}, \bar{4}\}$	$(\bar{4})^2 \mathbb{Z}_8 \subseteq A = \{\bar{0}, \bar{4}\}$
$\bar{1}.\bar{5}K_3 \nsubseteq A = \{\bar{0}, \bar{4}\}$	-	$\bar{1}.\bar{5}\mathbb{Z}_8 \nsubseteq A = \{\bar{0}, \bar{4}\}$	-
$\bar{1}.\bar{6}K_3 \subseteq A = \{\bar{0}, \bar{4}\}$	$(\bar{6})^2 K_3 \subseteq A = \{\bar{0}, \bar{4}\}$	$\bar{1}.\bar{6}\mathbb{Z}_8 \nsubseteq A = \{\bar{0}, \bar{4}\}$	-
$\bar{1}.\bar{7}K_3 \nsubseteq A = \{\bar{0}, \bar{4}\}$	-	$\bar{1}.\bar{7}\mathbb{Z}_8 \nsubseteq A = \{\bar{0}, \bar{4}\}$	-
$\bar{2}.\bar{2}K_3 \subseteq A = \{\bar{0}, \bar{4}\}$	$(\bar{2})^2 K_3 \subseteq A = \{\bar{0}, \bar{4}\}$	$\bar{2}.\bar{2}\mathbb{Z}_8 \subseteq A = \{\bar{0}, \bar{4}\}$	$(\bar{2})^2 \mathbb{Z}_8 \subseteq A = \{\bar{0}, \bar{4}\}$
$\bar{2}.\bar{3}K_3 \subseteq A = \{\bar{0}, \bar{4}\}$	$(\bar{2})^2 K_3 \subseteq A = \{\bar{0}, \bar{4}\}$	$\bar{2}.\bar{3}\mathbb{Z}_8 \nsubseteq A = \{\bar{0}, \bar{4}\}$	-
$\bar{2}.\bar{4}K_3 \subseteq A = \{\bar{0}, \bar{4}\}$	$(\bar{2})^2 K_3 \subseteq A = \{\bar{0}, \bar{4}\}$	$\bar{2}.\bar{4}\mathbb{Z}_8 \subseteq A = \{\bar{0}, \bar{4}\}$	$(\bar{2})^2 \mathbb{Z}_8 \subseteq A = \{\bar{0}, \bar{4}\}$
$\bar{2}.\bar{5}K_3 \subseteq A = \{\bar{0}, \bar{4}\}$	$(\bar{2})^2 K_3 \subseteq A = \{\bar{0}, \bar{4}\}$	$\bar{2}.\bar{5}\mathbb{Z}_8 \nsubseteq A = \{\bar{0}, \bar{4}\}$	-
$\bar{2}.\bar{6}K_3 \subseteq A = \{\bar{0}, \bar{4}\}$	$(\bar{2})^2 K_3 \subseteq A = \{\bar{0}, \bar{4}\}$	$\bar{2}.\bar{6}\mathbb{Z}_8 \subseteq A = \{\bar{0}, \bar{4}\}$	$(\bar{2})^2 \mathbb{Z}_8 \subseteq A = \{\bar{0}, \bar{4}\}$
$\bar{2}.\bar{7}K_3 \subseteq A = \{\bar{0}, \bar{4}\}$	$(\bar{2})^2 K_3 \subseteq A = \{\bar{0}, \bar{4}\}$	$\bar{2}.\bar{7}\mathbb{Z}_8 \nsubseteq A = \{\bar{0}, \bar{4}\}$	-
$\bar{3}.\bar{3}K_3 \nsubseteq A = \{\bar{0}, \bar{4}\}$	-	$\bar{3}.\bar{3}\mathbb{Z}_8 \nsubseteq A = \{\bar{0}, \bar{4}\}$	-
$\bar{3}.\bar{4}K_3 \subseteq A = \{\bar{0}, \bar{4}\}$	$(\bar{4})^2 K_3 \subseteq A = \{\bar{0}, \bar{4}\}$	$\bar{3}.\bar{4}\mathbb{Z}_8 \subseteq A = \{\bar{0}, \bar{4}\}$	$(\bar{4})^2 \mathbb{Z}_8 \subseteq A = \{\bar{0}, \bar{4}\}$
$\bar{3}.\bar{5}K_3 \nsubseteq A = \{\bar{0}, \bar{4}\}$	-	$\bar{3}.\bar{5}\mathbb{Z}_8 \nsubseteq A = \{\bar{0}, \bar{4}\}$	-
$\bar{3}.\bar{6}K_3 \subseteq A = \{\bar{0}, \bar{4}\}$	$(\bar{6})^2 K_3 \subseteq A = \{\bar{0}, \bar{4}\}$	$\bar{3}.\bar{6}\mathbb{Z}_8 \nsubseteq A = \{\bar{0}, \bar{4}\}$	-
$\bar{3}.\bar{7}K_3 \nsubseteq A = \{\bar{0}, \bar{4}\}$	-	$\bar{3}.\bar{7}\mathbb{Z}_8 \nsubseteq A = \{\bar{0}, \bar{4}\}$	-
$\bar{4}.\bar{4}K_3 \subseteq A = \{\bar{0}, \bar{4}\}$	$(\bar{4})^2 K_3 \subseteq A = \{\bar{0}, \bar{4}\}$	$\bar{4}.\bar{4}\mathbb{Z}_8 \subseteq A = \{\bar{0}, \bar{4}\}$	$(\bar{4})^2 \mathbb{Z}_8 \subseteq A = \{\bar{0}, \bar{4}\}$

$\overline{4}.\overline{5}K_3 \subseteq A = \{\overline{0},\overline{4}\}$	$(\overline{4})^2K_3 \subseteq A = \{\overline{0},\overline{4}\}$	$\overline{4}.\overline{5}\mathbb{Z}_8 \subseteq A = \{\overline{0},\overline{4}\}$	$(\overline{4})^2\mathbb{Z}_8 \subseteq A = \{\overline{0},\overline{4}\}$
$\overline{4}.\overline{6}K_3 \subseteq A = \{\overline{0},\overline{4}\}$	$(\overline{4})^2K_3 \subseteq A = \{\overline{0},\overline{4}\}$	$\overline{4}.\overline{6}\mathbb{Z}_8 \subseteq A = \{\overline{0},\overline{4}\}$	$(\overline{4})^2\mathbb{Z}_8 \subseteq A = \{\overline{0},\overline{4}\}$
$\overline{4}.\overline{7}K_3 \subseteq A = \{\overline{0},\overline{4}\}$	$(\overline{4})^2K_3 \subseteq A = \{\overline{0},\overline{4}\}$	$\overline{4}.\overline{7}\mathbb{Z}_8 \subseteq A = \{\overline{0},\overline{4}\}$	$(\overline{4})^2\mathbb{Z}_8 \subseteq A = \{\overline{0},\overline{4}\}$
$\overline{5}.\overline{5}K_3 \nsubseteq A = \{\overline{0},\overline{4}\}$	-	$\overline{5}.\overline{5}\mathbb{Z}_8 \nsubseteq A = \{\overline{0},\overline{4}\}$	-
$\overline{5}.\overline{6}K_3 \subseteq A = \{\overline{0},\overline{4}\}$	$(\overline{6})^2K_3 \subseteq A = \{\overline{0},\overline{4}\}$	$\overline{5}.\overline{6}\mathbb{Z}_8 \nsubseteq A = \{\overline{0},\overline{4}\}$	-
$\overline{5}.\overline{7}K_3 \nsubseteq A = \{\overline{0},\overline{4}\}$	-	$\overline{5}.\overline{7}\mathbb{Z}_8 \nsubseteq A = \{\overline{0},\overline{4}\}$	-
$\overline{6}.\overline{6}K_3 \subseteq A = \{\overline{0},\overline{4}\}$	$(\overline{6})^2K_3 \subseteq A = \{\overline{0},\overline{4}\}$	$\overline{6}.\overline{6}\mathbb{Z}_8 \subseteq A = \{\overline{0},\overline{4}\}$	$(\overline{6})^2\mathbb{Z}_8 \subseteq A = \{\overline{0},\overline{4}\}$
$\overline{6}.\overline{7}K_3 \subseteq A = \{\overline{0},\overline{4}\}$	$(\overline{6})^2K_3 \subseteq A = \{\overline{0},\overline{4}\}$	$\overline{6}.\overline{7}\mathbb{Z}_8 \nsubseteq A = \{\overline{0},\overline{4}\}$	-
$\overline{7}.\overline{7}K_3 \nsubseteq A = \{\overline{0},\overline{4}\}$	-	$\overline{7}.\overline{7}\mathbb{Z}_8 \nsubseteq A = \{\overline{0},\overline{4}\}$	-

Hence, $A = \{\overline{0},\overline{4}\}$ is a gw −prime submodule of $\mathbb{Z}_8$. Next, $\overline{2}.\overline{2} = \overline{4} \in \{\overline{0},\overline{4}\} = A$, while we have $\overline{2} \notin A$ and $\overline{2}\mathbb{Z}_8 = \{\overline{0},\overline{2},\overline{4},\overline{8}\} \nsubseteq \{\overline{0},\overline{4}\}$, so that $A = \{\overline{0},\overline{4}\}$ is not prime in $\mathbb{Z}_8$. Hence, $A = \{\overline{0},\overline{4}\}$ is gw −prime but not prime.

(2) Take the $\mathbb{Z}_4$ −module $\mathbb{Z}_4$. The submodule $A = \{\overline{0}\}$ of $\mathbb{Z}_4$ is gw −prime but not prime. Let K be a submodule of $\mathbb{Z}_4$, $\overline{r}\,\overline{s}\,K = \{\overline{0}\}$, where $\overline{r},\overline{s} \in \mathbb{Z}_4$.
If $\overline{r} = \overline{0}$ or $\overline{s} = \overline{0}$, then clearly $\overline{r}.\overline{s}\,K = \{\overline{0}\}$ for all submodules K of A, so that $(\overline{r})^2K = \{\overline{0}\}$ or $(\overline{s})^2K = \{\overline{0}\}$ for all submodules K of $\mathbb{Z}_4$, so that we discuss the case when $\overline{r} \neq \overline{0}$ and $\overline{s} \neq \overline{0}$.
Let $\overline{0} \neq \overline{r} \in \mathbb{Z}_4$ and $\overline{0} \neq \overline{s} \in \mathbb{Z}_4$ and a submodule K of $\mathbb{Z}_4$ we have $\overline{r}.\overline{s}\,K \subseteq \{\overline{0}\}$. The $\mathbb{Z}_4$ −module $\mathbb{Z}_4$ contains only three submodules which are $K_1 = \{\overline{0}\}$, $K_2 = \{\overline{0},\overline{2}\}$, $K_3 = \mathbb{Z}_4$.
(i) For the submodule $K_1 = \{\overline{0}\}$, for $\overline{r},\overline{s} \in \mathbb{Z}_4$, $\overline{r}.\overline{s}\,K_1 = \{\overline{0}\}$, clearly $(\overline{r})^2K_1 = \{\overline{0}\}$ and $(\overline{s})^2K_1 = \{\overline{0}\}$.
(ii) We discuss the case for the submodules K_2 and K_3 in the following table.

$K_2 = \{\overline{0},\overline{2}\}$		$K_3 = \mathbb{Z}_4$	
$\overline{1}.\overline{1}\,K_2 \neq \{\overline{0}\}$	-	$\overline{1}.\overline{1}\,\mathbb{Z}_4 \neq \{\overline{0}\}$	-
$\overline{1}.\overline{2}\,K_2 = \{\overline{0}\}$	$(\overline{2})^2K_2 = \{\overline{0}\}$	$\overline{1}.\overline{2}\,\mathbb{Z}_4 \neq \{\overline{0}\}$	-
$\overline{1}.\overline{3}\,K_2 \neq \{\overline{0}\}$	-	$\overline{1}.\overline{3}\,\mathbb{Z}_4 \neq \{\overline{0}\}$	-
$\overline{2}.\overline{2}\,K_2 = \{\overline{0}\}$	$(\overline{2})^2K_2 = \{\overline{0}\}$	$\overline{2}.\overline{2}\,\mathbb{Z}_4 = \{\overline{0}\}$	$(\overline{2})^2\mathbb{Z}_4 = \{\overline{0}\}$
$\overline{2}.\overline{3}\,K_2 = \{\overline{0}\}$	$(\overline{2})^2K_2 = \{\overline{0}\}$	$\overline{2}.\overline{3}\,\mathbb{Z}_4 \neq \{\overline{0}\}$	-
$\overline{3}.\overline{3}\,K_2 \neq \{\overline{0}\}$	-	$\overline{3}.\overline{3}\,\mathbb{Z}_4 \neq \{\overline{0}\}$	-

Hence, $\mathbb{Z}_4$ is a gw −prime $\mathbb{Z}_4$ −module. Next, $\overline{2}.\overline{2} = \overline{0} \in \{\overline{0}\}$ but we have $\overline{2} \notin \{\overline{0}\}$ and $\overline{2}\mathbb{Z}_4 = \{\overline{0},\overline{2}\} \nsubseteq \{\overline{0}\}$, so that $\{\overline{0}\}$ is not prime in $\mathbb{Z}_4$, therefore, $\mathbb{Z}_4$ is not a prime $\mathbb{Z}_4$ −module.

From Example 2.5 (2), we deduce that a proper submodule of a gw −prime module need not be prime.

Now, we provide some conditions under which a proper submodule of a gw −prime module is prime.

Theorem 2.7. If M is gw −prime in which the zero submodule is primary and $(0{:}M)$ is semiprime and if for every $a \in M$, there is a homo. $f{:}M \to R$, $f(a)a = a$, then every proper submodule of M is prime.

Proof. Let K be any proper submodule of M. To show K is a prime submodule. Let $r \in R$ and $m \in M$, we have $rm \in K$ and $m \notin K$. First, we prove that $< r > M \cap K \subseteq < r > K$. Suppose that $a \in < r > M \cap K$, then $a \in < r > M$ and $a \in K$. As $a \in M$, by the given condition, there is an R −morphism $f{:}M \to R$ such that $f(a)a = a$. Now, as $a \in < r > M$, we get $a = \sum_{i=1}^{k} r_i m_i$, where $r_i \in < r >$ and $m_i \in M$. Then, $a = f(a)a = f\left(\sum_{i=1}^{k} r_i m_i\right)a = \left(\sum_{i=1}^{k} r_i f(m_i)\right)a \in < r > K$ (since, $< r >$ is an ideal of R, $r_i \in < r >$, $f(m_i) \in R$ and $a \in K$), so that $< r > M \cap K \subseteq < r > K$. Now, since, $rm \in < r > M$ and $rm \in K$, we get that $rm \in < r > M \cap K \subseteq < r > K$. Hence, we get $rm = tz$ for some $t \in < r >$ and $z \in K$. Then $t = ur$ for some $u \in R$. Hence, $rm = tz = urz = ruz$, then $r(m - uz) = 0 \in \{0\}$. As the zero submodule of M is primary, $m - zu \in \{0\}$ or $r^n M = \{0\}$ for some $n \in \mathbb{Z}_+$. If $m - uz \in \{0\}$, then $m - uz = 0$, so that $m = uz \in K$, which is a contradiction, so that we get $r^n M = 0$ and then $r^n \in (0{:}M)$ and as $(0{:}M)$ is semiprime, $r \in (0{:}M)$, that is $rM = 0 \subseteq K$. Hence, K is prime.

It is known that, if M is nonzero and every proper submodule of M is finitely generated, then M is Noetherian, so that nonzero multiplication modules in which proper submodules are finitely generated are Noetherian. Now, we give the following result in which we restrict the condition just for gw −prime submodules, that is we prove that nonzero multiplication modules in which every gw −prime submodule is finitely generated are Noetherian.

Theorem 2.8. If M is a non-zero multiplication R −module and every gw −prime submodule is finitely generated, then M is Noetherian.

Proof. As $M \neq 0$, by [24, Theorem 2.5], M contains a maximal submodule, say K, so that K is prime and hence by Remark 2.5 (1), K is gw −prime. Hence, from the given condition in the statement, we get K is finitely generated, so that $K = Rk_1 + Rk_2 + \cdots + Rk_n$, for $1 \leq i \leq n$ and $k_i \in K$. Next, since $K \neq M$, so that there exists $m \in M$ and $m \notin K$. Hence, we get $K \subset K + Rm \subseteq M$ and as K is maximal, we get $M = K + Rm = Rk_1 + Rk_2 + \cdots + Rk_n + Rm$, so that M is finitely generated. If M is not Noetherian, then $P = \{H{:}H$ is not a finitely generated submodule of $M\} \neq \emptyset$, so by Zorn's Lemma, P contains a maximal element, say F. That is, F is a submodule of M and it is not finitely generated and it is maximal with respect to this property. As M is finitely generated, so that $M \notin P$. Now, suppose that $M = Rm_1 + Rm_2 + \cdots + Rm_t$, where $m_i \in M$ for $1 \leq i \leq t$. Since, M is multiplication, so that $F = AM$, where $A = (F{:}M)$. To show A is prime. If $A = R$, then $F = AM = RM = M$, which gives that $M \in P$ (since $F \in P$), this is a contradiction. Hence, $A \neq R$. Suppose that $x, y \in R$, with $xy \in A$ but $x \notin A$ and $y \notin A$, that is $x \notin (F{:}M)$ and $y \notin (F{:}M)$. Hence, we get $xM \nsubseteq F$ and $yM \nsubseteq F$, so that there exists $a \in M$ such that $xa \notin F$, and then $F \subset F + xM$. As F is a maximal element in P, we get $F + xM \notin P$, so that $F + xM$ is finitely generated. Hence, $F + xM = \sum_{i=1}^{l} Rd_i$, where $d_i = f_i + xm_i$, for $f_i \in F$ and $m_i \in M$. Next, $x(F + yM) = xF + xyM \subseteq F + AM = F + F = F$, so that $F + yM \subseteq (F{:}x)$. That is, $F \subset F + yM \subseteq (F{:}x)$, so that $(F{:}x) \notin P$. Hence, $(F{:}x)$ is finitely generated, so that $(F{:}x) = \sum_{i=1}^{j} Rh_i$. Now, let $f \in$

F, then $f \in F + xM = \sum_{i=1}^{l} Rd_i = \sum_{i=1}^{l} Rf_i + \sum_{i=1}^{l} Rxm_i$. It follows that the set $\{f_1, f_2, \dots, f_l, xm_1, xm_2, \dots, xm_j\}$ is a generator set for F, that means F is finitely generated, which is a contradiction. Hence, A is a prime ideal of R. Now, if $g \in ann(M)$, then $gM = \{0\} \subseteq F$, so that $g \in (F: M) = A$, so that $ann(M) \subseteq A$ and since F is a proper submodule of M (since if $F = M$, then as M is finitely generated, so that F is finitely generated, this gives that $F \notin P$ which is a contradiction). Then by [24, Corollary 2.11], we get AM is prime, that is, F is prime and hence by Remark 2.5 (1), F is gw −prime and thus by the given condition we get F is finitely generated and this gives $F \notin P$ which is again a contradiction. Hence, we get that M is Noetherian.

Corollary 2.9. If every gw −prime ideal in R is finitely generated, then R is a Noetherian ring.
Proof. By considering R as an R −module, and since gw −prime submodules are gw −prime ideals, so by Theorem 2.8, the result follows.

Remark 2.10. If M is cyclic and faithful, then $M \cong R$.

If we drop the property of being M is cyclic in Remark 2.10 by the conditions that M to be multiplication and R to be semilocal, then still we get $M \cong R$.

Proposition 2.11. If M a faithful multiplication module M over a semilocal ring R, then $M \cong R$.
Proof. By Theorem 2.8, M is Noetherian and as M is faithful and multiplication, we get R is Noetherian. Next, as R is semilocal and M is multiplication, we get, M is cyclic [25] and hence by Remark 2.10, we get $M \cong R$.

If we drop the property of being M is cyclic in Remark 2.10 by the conditions that M to be multiplication and there exists a module homomorphism from M to R, then still we get $M \cong R$.

Proposition 2.12. Let M be a faithful multiplication R −module. If $f: M \to R$ is an epimorphism, then $M \cong R$.
Proof. $\ker f = AM$ for some ideal A of R. Now, we have $0 = f(\ker f) = f(AM) = Af(M) = AM = \ker f$, so that f is one to one and thus f is an isomorphism. Hence, $M \cong R$.

Proposition 2.13. Let M be a faithful multiplication R −module. If every gw −prime submodule of M is cyclic and there is a nonzero divisor $a \in R$ such that Ra is a maximal ideal of R, then $M \cong R$.
Proof. As Ra is maximal, we get RaM is a maximal submodule of M [24], so that it is prime and hence RaM is a gw −prime submodule of M, so that by the given condition we get RaM is cyclic, so let $RaM = Rx$ for some $x \in M$. Let $y \in M \backslash Rx$, then $Rx \subset Rx + Ry \subseteq M$ and as Rx is maximal, we get $M = Rx + Ry$. Now, $ay = 1.ay \in RaM = Rx$, so that $ay = bx$ for some $b \in R$. If possible, suppose that $b \in Ra$, then $b = ra$ for some $r \in R$. Then $ay = bx = rax$, that gives $a(y - rx) = 0$ and as a is a nonzero divisor, we get $y - rx = 0$, so that $y = rx \in Rx$, which is a contradiction. Hence, $b \notin Ra$, then we get $Ra \subset Ra + Rb \subseteq R$ and as Ra is a maximal ideal of R, we get $R = Ra + Rb$. Now, as $1 \in R$, we get $1 = ca + db$ for some $c, d \in R$. Now, let $m \in M = Rx + Ry$, then we have $m = ex + fy$ for some $e, f \in R$. Next, $m = 1.m = (ca + db)(ex + fy) = caex + dbex + cafy + dbfy = caex + daey + cbfx + dbfy = (ae + bf)cx + (ae + bf)dy = (ae + bf)(cx + dy) \in R(cx + dy)$, so that

$M \subseteq R(cx+dy)$ and as $R(cx+dy) \subseteq M$, we get $M = R(cx+dy) = Rz$, where $z = cx + dy \in M$. Hence, M is cyclic and as M is faithful, by Remark 2.10, we get $M \cong R$.

Proposition 2.14. If L is a gw −prime submodule of M and $x, y \in M$ such that $(L:x)$ and $(L:y)$ are semiprime ideals of R and $(L:x) \neq (L:y)$, then $L = (L+Rx)\cap(L+Ry)$.
Proof. As $(L:x) \neq (L:y)$, we have $(L:x) \nsubseteq (L:y)$ or $(L:y) \nsubseteq (L:x)$. Let $(L:x) \nsubseteq (L:y)$, then there exists $a \in (L:x)$ and $a \notin (L:y)$, that means $ax \in L$ and $ay \notin L$. Then we get $y \notin L$ (since, if $y \in L$, then $ay \in L$). As L is gw −prime and $(L:y)$ is semiprime, by Corollary 2.4, $(L:y)$ is prime. Let $r \in (L:y)$, then $ry \in L$, so that $ray = ary \in aL \subseteq L$ and thus $r \in (L:ay)$. Hence, $(L:y) \subseteq (L:ay)$. Next, let $r \in (L:ay)$, then $ray \in L$, that is $ra < y > \subseteq L$ and as L is a gw −prime submodule, we get $r^2 < y > \subseteq L$ or $a^2 < y > \subseteq L$, that gives $r^2 \in (L:< y >)$ or $a^2 \in (L:< y >)$ and by Remark 2.3, we have $(L:y) = (L:< y >)$, so that $r^2 \in (L:y)$ or $a^2 \in (L:y)$ and as $(L:y)$ is a prime ideal, we get $r \in (L:y)$ or $a \in (L:y)$ and since $a \notin (L:y)$, so we have $r \in (L:y)$, so that $(L:ay) \subseteq (L:y)$. Hence, we get $(L:y) = (L:ay)$. Now, we have $L \subseteq (L+Rx)\cap(L+Ry)$. It remains to show that $(L+Rx)\cap(L+Ry) \subseteq L$. Let $z \in (L+Rx)\cap(L+Ry)$, so that $z = l_1 + r_1x = l_2 + r_2y$, where $l_1, l_2 \in L, r_1, r_2 \in R$. Next, $az = al_1 + r_1ax = al_2 + r_2ay$, then $r_2ay = al_1 + r_1ax - al_2 \in L$. Hence, we get $r_2 \in (L:ay) = (L:y)$ and then we get $r_2y \in L$, so that $z = l_2 + r_2y \in L$, this gives that $(L+Rx)\cap(L+Ry) \subseteq L$. Hence, we get $(L+Rx)\cap(L+Ry) = L$. If $(L:y) \nsubseteq (L:x)$, then similarly, $(L+Rx)\cap(L+Ry) = L$.

Proposition 2.15. If L is a gw −prime submodule of M and $x, y \in M$, $a \in R$ with $ax \in L$ and $(L:x)$, $(L:y)$ are semiprime, then $L = (L+Rx)\cap(L+Ray)$.
Proof. We have, either $ay \in L$ or $ay \notin L$. First, suppose that $ay \in L$, then $L + Ray \subseteq L + RL \subseteq L + L = L$, so that we get then we $L \subseteq (L+ax)\cap(L+Ray) \subseteq (L+Rx)\cap L = L$, so that we get $L = (L+Rx)\cap(L+Ray)$. Next, suppose that $ay \notin L$, then $a \notin (L:y)$. If $a \in (L:ay)$, then we get $a^2y \in L$, so that $a^2 \in (L:y)$ and as $(L:y)$ is semiprime, we get $a \in (L:y)$, which is a contradiction, so that we get $a \notin (L:ay)$ and as $ax \in L$, we get $a \in (L:x)$. Hence, we get $(L:x) \neq (L:ay)$. To show that $(L:ay)$ is semiprime. Let, $b^2 \in (L:ay)$, where $b \in R$, then $b^2a \in (L:y)$, from this we get $(ba)^2 = b^2a^2 = ab^2a \in a(L:y) \subseteq (L:y)$ and as $(L:y)$ is semiprime, we get $ba \in (L:y)$, so that $b \in (L:ay)$. Hence, $(L:ay)$ is semiprime. Since $(L:x)$ is also semiprime, so by Proposition 2.14, we get $L = (L+Rx)\cap(L+Ray)$.

Proposition 2.16. If L is gw −prime and irreducible submodule of M such that $(L:x)$ is semiprime for all $x \in M$, then L is prime.
Proof. Let $ax \in L$, where $a \in R$. If $z \in M$ is any element, then $az \in M$ and it is given that $(L:x)$ and $(L:az)$ are semiprime ideals, so by Proposition 2.15, we get $L = (L+Rx)\cap(L+Raz)$. As L is irreducible, we get $L = L + Rx$ or $L = L + Raz$. If $L = L + Rx$, then $Rx \subseteq L$ and then $x = 1.x \in Rx \subseteq L$ and if $L = L + Raz$, then $Raz \subseteq L$ and then $az = 1.az \in Raz \subseteq L$, so that $aM \subseteq L$. Hence, L is prime.

Definition 2.17. For a submodule L of M we define $L^* = L\backslash\{0\}$. If N, K are nonzero submodules of M. We say that N is independently related to K if $rx + sy = 0$ for $r, s \in R$ and $x \in N^*, y \in K^*$, then $r = 0 = s$. Otherwise, we say that N is dependently related to K, that is if for $x \in N^*, y \in K^*$ there exist $r, s \in R$ with $r \neq 0$ or $s \neq 0$ such that $rx + sy = 0$ and M is independently related if the distinct submodules of M are pairwise independently related. In particular, if $N = K$, then we say that N is independently related to itself, (or simply N is an independent submodule), if $rx + sy = 0$ for $r, s \in R$ and $x, y \in N^*$, then $r = 0 = s$.

Examples 2.18. **(1)** Consider $\mathbb{Z} \times \mathbb{Z}$ as a $\mathbb{Z}$ −module. Let $N = \{(x, 0): x \in \mathbb{Z}\}$ and $K = \{(0, y): y \in \mathbb{Z}\}$ be two submodules of $\mathbb{Z} \times \mathbb{Z}$. We show that N is independently related to K. Let $m, n \in \mathbb{Z}$ and $(x, 0) \in N^* = N\backslash\{0\}, (0, y) \in K^* = K\backslash\{0\}$ such that $m(x, 0) + n(0, y) = (0, 0)$, then we have $x \neq 0$ and $y \neq 0$. Now, we have $(mx, ny) = (0, 0)$, so that $mx = 0 = ny$ and as $\mathbb{Z}$ is an integral domain, $m = 0 = n$. Hence, N is independently related to K.

(2) Consider $\mathbb{Z}_6$ as a $\mathbb{Z}$ −module. We have $N = \{\overline{0}, \overline{3}\}$ and $K = \{\overline{0}, \overline{2}, \overline{4}\}$ are submodules of $\mathbb{Z}_6$. Now, $\overline{3} \in N^* = N\backslash\{\overline{0}\}$ and $\overline{2} \in K^* = K\backslash\{0\}$ and $2, 3 \in \mathbb{Z}$ such that $2.\overline{3} + 3.\overline{2} = \overline{0}$ but $2 \neq 0$ and $3 \neq 0$, so that N is dependently related to K.

Now we provide some conditions which make a proper submodule of M as gw −prime.

Proposition 2.19. Let R be an integral domain and M is multiplication and independently related. If P is a proper submodule of M with $KL \subseteq P$ and $(P: x) \cap (P: y) = \{0\}$ for all $x \neq y \in M$, then P is gw −prime.

Proof. Since M is multiplication, $K = IM$ and $L = JM$ for some ideals I, J of R. Then, we have $IJM = KL \subseteq P$. If possible suppose that $K \nsubseteq P$ and $L \nsubseteq P$, then there exists $a \in K = IM$ and $b \in L = JM$ such that $a = rx$ and $b = sy$, where, $x, y \in M$, $a, b \notin P$ and $r \in I, s \in J$, then we get $x \notin P$ (since if $x \in P$, then $a = rx \in P$, which is a contradiction). If $x = y$, then we have $sa = srx = rsx = rsy = rb$, that gives $sa - rb = 0$. As $a, b \notin P$, we get $0 \neq a \in K\backslash\{0\} = K^*$ and $0 \neq b \in L\backslash\{0\} = L^*$ and since M is independently related R −module, so that K is independently related to L, that gives $s = 0 = r$ and then we get $a = rx = 0x = 0 \in P$ and $b = sy = 0y = 0 \in P$, which is a contradiction. Hence, we must have $x \neq y$ and thus by the given condition, we get $(P: x) \cap (P: y) = \{0\}$. Now, we have $rsx \in IJM \subseteq P$, then $rs \in (P: x)$. Also, we have $rsy \in IJM \subseteq P$, so that $rs \in (P: y)$. Hence, $rs \in (P: x) \cap (P: y) = \{0\}$, so $rs = 0$ and we get $r = 0$ or $s = 0$. Hence, $a = rx = 0x = 0 \in P$ or $b = sy = 0y = 0 \in P$, the both conclusions are contradiction, and thus $K \subseteq P$ or $L \subseteq P$. Hence, P is prime [26], so that P is gw −prime.

Example 2.20. In the $\mathbb{Z}$ −module $\mathbb{Z}$, $8\mathbb{Z}$ is primary but not prime. We have $2.4 = 8 \in 8\mathbb{Z}$ but $4 \notin 8\mathbb{Z}$ and $2\mathbb{Z} \nsubseteq 8\mathbb{Z}$ (Also, $2 \notin 8\mathbb{Z}$ and $4\mathbb{Z} \nsubseteq 8\mathbb{Z}$), so that $8\mathbb{Z}$ is not prime. To show $8\mathbb{Z}$ is primary. Let $ab \in 8\mathbb{Z}$ for $a, b \in \mathbb{Z}$. Then, $ab = 8k$ for some $k \in \mathbb{Z}$, so that $2|ab$, this gives $2|a$ or $2|b$. If $2|a$, then $a = 2m$ for some $m \in \mathbb{Z}$, so that $a^3 = 8m^3 \in 8\mathbb{Z}$, that gives $a^3\mathbb{Z} \subseteq 8\mathbb{Z}$ and if $2|b$, then $b = 2n$ for some $n \in \mathbb{Z}$. Hence, $b^3 = 8n^3 \in 8\mathbb{Z}$, that gives $b^3\mathbb{Z} \subseteq 8\mathbb{Z}$. Therefore $8\mathbb{Z}$ is primary.

In below a condition is given under which a primary submodule is a prime.

Proposition 2.21. If N is a primary submodule of M with $(N: M)$ is semiprime, then N is prime.

Proof. Let $ax \in N$, where $a \in R$ and $x \in M$. If $x \notin N$, then as N is primary, then $a^nM \subseteq N$ for some $n \in \mathbb{Z}_+$. Hence, $a^n \in (N: M)$. As $(N: M)$ is semiprime, $a \in (N: M)$, this gives $aM \subseteq N$. Hence, N is prime.

It is known, every maximal submodule is gw −prime. However, the zero submodule of the $\mathbb{Z}_4$ −module $\mathbb{Z}_4$ is gw −prime but not prime (see example 2.6 (2)) and hence not maximal.

Now, we prove that under certain conditions which are given in the following proposition that a proper submodule of a cyclic gw −prime module is maximal.

Proposition 2.22. Let $M = Rm$ be a cyclic gw −prime R −module, where $m \in M$. If $ann(m) = \{0\}$ and for each $a \in M$, there exists a homomorphism $f: M \to R$ such that $f(a)a = a$ and the zero ideal of R is semiprime, then every proper submodule of M is maximal.
Proof. Let K be a proper submodule of M. If possible, suppose that there exists a submodule L such that $K \subset L \subseteq M$. Let $y \in L$ and $y \notin K$. Then, as $y \in M$ and as $M = Rm$, we get $y = am$ for some $a \in R$. If $a = 0$, then $y = 0m = 0 \in K$, which is a contradiction. Hence, we get $0 \neq a \in R$. Then $am = y \in L$ and $am = y \notin K$. By the given condition, there exists a homomorphism $f: M \to R$ such that $f(y)y = y$, that is, $f(am)am = am$, then $aaf(m)m = am$. Hence, $a^2 f(m)m = am$. Then, $a(af(m) - 1)m = 0 \in \{0\}$, so that $a(af(m) - 1) < m >= 0 \in \{0\}$ and as M is a gw −prime submodule, we have the zero submodule of M is gw −prime, so that $a^2 < m >= 0$ or $(af(m) - 1)^2 < m >= 0$, then $a^2 m \in a^2 < m >= \{0\}$ or $(af(m) - 1)^2 m \in (af(m) - 1)^2 < m >= \{0\}$. Hence, we get $a^2 m = 0$ or $(af(m) - 1)^2 m = 0$, from which we get $a^2 \in Ann(m) = \{0\}$ or $(af(m) - 1)^2 \in Ann(m) = \{0\}$, so that $a^2 = 0 \in \{0\}$ or $(af(m) - 1)^2 = 0 \in \{0\}$ and so we get $a = 0$ or $af(m) - 1 = 0$. But $a \neq 0$, so that we get $af(m) - 1 = 0$, and then $af(m) = 1$, so that a is a unit in R, thus $a^{-1} \in R$ and as $am \in L$, we get $m = 1.m = a^{-1}am \in a^{-1}L \subseteq L$. Hence, $M = Rm \subseteq RL = L$, so that we get $L = M$. Hence, K is a maximal submodule of M.

Lemma 2.23. If N is a primary submodule of M, then the ideal $(N: M)$ is primary.
Proof. If $(N: M) = R$, then as $1 \in R = (N: M)$, we get $M = 1.M \subseteq N$, so that $N = M$, which contradicts the fact that N is primary ($N \neq M$). Hence, we get $(N: M) \neq R$. Let for $a, b \in R$, we have $ab \in (N: M)$ but $b \notin (N: M)$. Then $abM \subseteq N$ and $bM \nsubseteq N$, so $bm \notin N$ for some $m \in M$, so that $abm \in N$ and as M is primary, we get $a^k M \subseteq N$ for some $k \in \mathbb{Z}_+$, that means $a^k \in (N: M)$. Hence, $(N: M)$ is primary.

3. The effect of localization on gw −prime submodules

Now, we study the effect of localization on gw −prime submodules and we determine those conditions which make a given submodule equivalent to its localization at multiplicative systems. In all what follows in this section, K is a proper submodule of M, S is a multiplicative system in R and P is a prime ideal of R.

First, we provide a condition under which the localization of a proper submodule at any multiplicative system is a prime submodule.

Proposition 3.1. If $(K: a) = 0$ for all $a \notin K$, then K_S is a prime submodule of M_S. In particular, K_P is a prime submodule of M_P.
Proof. As $S \neq \emptyset$, take an $s \in S$. If $K_S = M_S$, then as $S \neq \emptyset$, take an $s \in S$. Now, suppose that $m \in M$, then $\frac{m}{s} \in M_S = K_S$, so that $tm \in K$ for some $t \in S$, so that we get $t \in (K: m)$. If $m \notin K$, then $(K: m) = 0$, so that $t = 0$, that means $0 \in S$, that is a contradiction, so that K_S is proper in M_S. Now, let $\frac{r}{s} . \frac{x}{t} \in K_S$ and $\frac{x}{t} \notin K_S$, where $r \in R, x \in M$ and $s, t \in S$. Then, $\frac{rx}{st} \in K_S$ and $\frac{x}{t} \notin K_S$. Hence, $urx \in K$ for some $u \in S$ and $x \notin K$, then we get $u \in (K: rx)$. If $rx \notin K$, then $(K: rx) = 0$, so that $u = 0$, then we get $0 \in S$, which is a contradiction, so that $rx \in K$, and this gives that $r \in (K: x)$ and as $x \notin K$, we get $(K: x) = 0$, so that $r = 0$. Now, we have

$\frac{r}{s}M_S = \frac{0}{s}M_S = 0 \subseteq K_S$. Hence, K_S is a prime submodule of M_S. Taking $S = R\backslash P$, we get K_P is prime in M_p.

Since every prime submodule is semiprime as well as primary, so that we prove the following corollaries.

Corollary 3.2. If $(K{:}\,a) = 0$ for all $a \notin K$, then K_S is both semiprime and primary. As especial case, K_P is both semiprime and primary in M_P.
Proof. The proof follows directly from the fact that a prime submodule is semiprime as well as primary. By putting $S = R\backslash P$, we get K_P is both semiprime and primary in M_P.

Proposition 3.3. If $(K{:}\,a) = 0$ for all $a \notin K$ and K_S is prime in M_S, then K is prime M. As especial case, K_P is prime in M_P, then K is prime in M.
Proof. Let $rm \in K$, but $m \notin K$, where $r \in R, m \in M$, and we have to show that $rM \subseteq K$. As $S \neq \emptyset$, take an $s \in S$. Now, we have $\frac{r}{s}.\frac{m}{s} = \frac{rm}{ss} \in K_S$. As, K_S is primary, then we get $\frac{m}{s} \in K_S$ or $\frac{r}{s}M_S \subseteq K_S$. If $\frac{m}{s} \in K_S$, then $tm \in K$ for some $t \in S$, that gives $t \in (K{:}\,m)$ and as $m \notin K$, we get $(K{:}\,m) = 0$, so that $t = 0 \in S$, which is a contradiction, so that we have $\frac{r}{s}M_S \subseteq K_S$. Now, let $x \in M$ be any element, then $\frac{rx}{ss} = \frac{r}{s}.\frac{x}{s} \in \frac{r}{s}M_S \subseteq K_S$, so that $trx \in K$ for some $t \in S$, then $t \in (K{:}\,rx)$. If $rx \notin K$, then $(K{:}\,rx) = 0$, so that $t = 0 \in S$, which is a contradiction, so that $rx \in K$. Hence, we get $rM \subseteq K$. Hence, K is prime in M. Put $S = R\backslash P$, the proof of the especial case follows directly.

Now, under some conditions we prove that if the localization of a submodule at multiplicative systems is semiprime, then the submodule itself is semiprime.

Theorem 3.4. If $(K{:}\,a) = 0$ for all $a \notin K$ and K_S is a semiprime in M_S, then K is a semiprime in M. As especial case, if K_P is semiprime in M_P, then K is semiprime in M.
Proof. Let for $r \in R$ and $m \in M$ we have $r^2m \in K$. As, $S \neq \emptyset$, take $s \in S$, then we have $(\frac{r}{s})^2\frac{m}{s} = \frac{r^2}{s^2}\frac{m}{s} = \frac{r^2m}{s^3} \in K_S$, where $\frac{r}{s} \in R_S$ and $\frac{m}{s} \in M_S$. Since K_S is semiprime, we get $\frac{r}{s}.\frac{m}{s} \in K_S$, that is, $\frac{rm}{s^2} \in K_S$, from this we get $trm \in K$ for some $t \in S$, so that $t \in (K{:}\,rm)$. If $rm \notin K$, then we get $(K{:}\,rm) = 0$, so that $t = 0$, this gives that $0 \in S$, which is a contradiction, and thus $rm \in K$. Hence, K is semiprime. Put $S = R\backslash P$, the proof of the especial case follows directly.

Corollary 3.5. Let A be a proper ideal with $(A{:}\,x) = 0$ for all $x \notin A$ and A_S is semiprime in R_S, then A is semiprime in R. As especial case, if A_P is semiprime in R_P, then A is a semiprime in R.
Proof. As R is an R −module, the result follows from Theorem 3.4.

Proposition 3.6. If $(K{:}\,a) = 0$ for all $a \notin K$ and K_S is primary in M_S, then K is primary in M. In particular, if K_P is primary in M_P, then K is primary in M.
Proof. Let for $a \in R, m \in M$, we have $am \in K$, but $m \notin K$ and we have to show that $a^nM \subseteq K$ for some positive integer n. As $S \neq \emptyset$, take an $s \in S$. Now, we have $\frac{a}{s}.\frac{m}{s} = \frac{am}{ss} \in K_S$. As, K_S is primary, then we get $\frac{m}{s} \in K_S$ or $(\frac{a}{s})^nM_S \subseteq K_S$ for some $n \in \mathbb{Z}_+$. If $\frac{m}{s} \in K_S$, then $tm \in K$ for some $t \in S$, that gives $t \in (K{:}\,m)$ and as $m \notin K$, we get $(K{:}\,m) = 0$, so that $t = 0 \in S$,

which is a contradiction, so that we have $(\frac{a}{s})^n M_S \subseteq K_S$. Then, $\frac{a^n}{s^n} M_S \subseteq K_S$. Now, let $x \in M$ be any element, then $\frac{a^n}{s^n} . \frac{x}{s} \in \frac{a^n}{s^n} M_S \subseteq K_S$, so that $\frac{a^n x}{s^{n+1}} \in K_S$, so that $ta^n x \in K$ for some $t \in S$ and then $t \in (K: a^n x)$. If $a^n x \notin K$, then $(K: a^n x) = 0$, so that $t = 0 \in S$, which is a contradiction, so that $a^n x \in K$. Hence, we get $a^n M \subseteq K$. Hence, K is primary in M. Put $S = R\backslash P$, the proof of the especial case follows directly.

Example 3.7. Consider $\mathbb{Z}_{12}$ as a $\mathbb{Z}_{12}$ −module. Take the submodules $N = \{\overline{0}, \overline{2}, \overline{4}, \overline{6}, \overline{8}, \overline{10}\}$ and $L = \{\overline{0}, \overline{4}, \overline{8}\}$ of $\mathbb{Z}_{12}$. Now let $S = \{\overline{1}, \overline{2}, \overline{4}, \overline{8}\}$ which is a multiplicative system in $\mathbb{Z}_{12}$. It is easy to check that $N_S = \left\{\frac{\overline{0}}{\overline{1}}, \frac{\overline{1}}{\overline{1}}, \frac{\overline{2}}{\overline{1}}\right\} = L_S$, but clearly $N \neq L$. This because of that, the submodule L is not prime. Now, we give some conditions under which the converse of the mentioned property in above is true also.

Proposition 3.8. Let N, L be prime submodules of M, $N_S = L_S$ and $S \cap (N: M) = \emptyset = S \cap (L: M)$, then $N = L$.
Proof. Let $x \in N$ and fix an $s \in S$. Then, $\frac{x}{s} \in N_S = L_S$, so that $tx \in L$ for some $t \in S$ and as L is a prime submodule, we get $tM \subseteq L$ or $x \in L$. If $tM \subseteq L$, then $t \in (L: M)$, and since, $S \cap (L: M) = \emptyset$, so that we get $t \notin S$, that is a contradiction, so that $x \in L$. Hence, $N \subseteq L$. Next, let $x \in L$. As, $S \cap (N: M) = \emptyset$, by the same technique as in the above, we get $L \subseteq N$. Hence, $N = L$.

4. Conclusions

In the last section of this paper, we list some conclusions that we observed in the whole results.
(1) It is known that gw −prime modules are not primeful, but if M is a gw −prime module in which the zero submodule is primary and $(0: M)$ is a semiprime ideal, then we can make it as a primeful module by a certain homomorphism $f: M \to R$.
(2) A nonzero multiplication module in which every gw −prime submodule is finitely generated is Noetherian.
(3) An irreducible and gw −prime submodule N of an R −module M for which $(N: x)$ is semiprime for all $x \in M$, is prime.
(4) We can make every proper submodule of a gw −prime module as a maximal by defining a homomorphism from M to R with certain conditions.

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