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Reliability Function of the Special (3+1) Cascade Model

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Abstract
In this paper, a reliability function is found for the special case of cascade model where the model consists of (3 + 1) units and trace random variables that represent the strength and stress Rayleigh distribution, respectively. The mathematical formula of the reliability function was estimated by estimating the parameters of Rayleigh distribution in the five different estimation methods, Maximum likelihood method, Percentile method, Least Squares method, Weighted Least Squares method and Regression method. Using the MATLAB software, A Monte Carlo simulation is performed to compare the results of different estimation methods using the mean square error criterion. The simulation results showed that the best estimation methods for estimating the reliability model are ML estimator and Pr estimator.

Keywords: Rayleigh Distribution, Monte Carlo, Mean Square Error, Unit, Percentile Method.

دالة المعولية لنموذج كاسكاد(3+1) الخاص

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الخلاصة
في هذه البحث، تم ايجاد دالة المعولية للحالة الخاصة لنموذج كاسكاد حيث يتكون النموذج من (3 + 1) وحدة وتتبع المتغيرات العشوائية التي تمثل المتانة والاجهاد توزيع رالي. تم تقدير الصيغة الرياضية لدالة المعولية من خلال تقدير معلمات توزيع رالي بخمس طرق تقدير مختلفة هي طريقة الإمكان الاعظم، طريقة النسبة المئوية ، طريقة المربعات الصغرى ، طريقة المربعات الصغرى الموزونة ، طريقة الانحدار. باستخدام برنامج ماتلاب ، تم إجراء محاكاة مونت كارلو لمقارنة نتائج طرق التقدير المختلفة باستخدام معيار متوسط مربع الخطأ. أظهرت نتائج المحاكاة أن أفضل طرق التقدير لتقدير نموذج المعولية هي مقدر الإمكان الاعظم ومقدر النسبة المئوية.

1. Introduction

The rapid development in all fields, especially the industrial field, has led to emergence of many giant industrial machines and complex devices. These devices can malfunction and which lead to work stoppage and waste of time and effort, therefore, knowing the reliability

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of any unit enables us to avoid the loss of time and efforts. Therefore, interest in reliability and knowledge of its amount has increased.

The amount of reliability can be determined by the function $\mathcal{R} = pr(X < Y)$ [1], where the random variable x represents the strength and the random variable y represents the stress where the unit resists with its strength x, the stress y is exposed and continued to work until the stress overcome the strength and unit stops working [2]. Cascade models are a special type of standby redundancy system, and it is hierarchical standby redundancy. Work stoppage can be avoided when the model unit fails in the cascade model, where the failed unit is replaced by a new one of the spare units, and the new stress changes k twice as much as the previous one [3].

There are many previous works in the literatures such as: Khaleel and Karam founded an estimate of the reliability function for a special cascade model consisting of two basic components and one standby component ([3] and [4]). Ahmed and batah estimated the reliability of the stress- strength model with eight different estimation methods and also made a simulation to compare the different methods [5]. Cetinkaya and Genc studied the distribution for the reliability setup of a system exposed to certain stresses assuming that the standard two-sided power distribution is the primary distribution [6]. Bai et al. estimated the reliability of the multicomponent stress-strength model assuming that the stress variables follow the Weibull and the strength variables follow exponential based on Gumbel copula within the Type-I gradually hybrid control system [7]. Sarhan and Tolba estimated R when X and Y are two independent random variables following the two-parameter Weibull distributions; and the power variable X is subject to SSPALT [8].

The (3+1) cascade model contains four units, three units are basic and necessary for the work of the model, while the fourth unit will be a redundancy in an active-standby state. Suppose the cascade model (3+1) which contains the four units, namely, $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ and $\mathcal{C}_4$ such that $\mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{C}_3$ are the basic units is the model that needs to work where the unit $\mathcal{C}_4$ is a redundancy unit that is in active standby mode. Consider the random variables X_1, X_2, X_3 and X_4 are the unit strengths $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ and $\mathcal{C}_4$, respectively, also the random variables Y_1, Y_2, Y_3 and Y_4 represent the stresses placed on the units . Here, in the event that the unit $\mathcal{C}_1$ is stop working, it is replaced by the unit $\mathcal{C}_4$, where $X_4 = \mathcal{m}X_1$ and $Y_4 = kY_1$, if the unit $\mathcal{C}_2$ stops, it is replaced by the unit $\mathcal{C}_4$, where $X_4 = \mathcal{m}X_2$ and $\mathrm{Y}_4 = k\mathrm{Y}_2$, also if the unit $\mathcal{C}_3$ is stop working, it is replaced by the unit $\mathcal{C}_4$, where $X_4 = \mathcal{m}X_3$ and $\mathrm{Y}_4 = k\mathrm{Y}_3$ where (k) is the stress attenuation factor and $(\mathcal{m})$ is the strength attenuation factor, such that $0 < \mathcal{m} < 1$ and $k > 1$ [4].

The aim of this paper is to find the mathematical formula for the reliability in a special (3+1) cascade model by assuming that random variables follow the Rayleigh distribution and estimate them using the methods of estimation , namely, ML, Pr, LS, WLS and Rg as well as comparing the results using the mean square error to find the best estimator.

2. The mathematical formula

Suppose that the (3+1) model which contains three base units are necessary for the functioning of the model and the fourth unit is redundant standby, where the random variables of the strength are $\imath$ =1,2,3,4 and the random variables of the stress are $X_\imath$; and $Y_\jmath$; $\jmath$ =1,2,3,4, all variables are independently and identically where $X \sim R(2, \lambda)$ and $Y \sim R(2, \vartheta)$ [5].

The CDF of $\mathrm{R}(2, \lambda)$ is given as follows:

$$\mathrm{F(x)} = 1 - \mathrm{e}^{-\lambda \mathrm{x}^2} \quad \mathrm{x} > 0; \lambda_\ell > 0 \ ; \ \ell = 1,2,3,4 . \tag{1}$$

and the CDF of $R(2,\vartheta)$ is given by :

$$G(y) = 1 - e^{-\vartheta y^2} \quad y > 0;\ \vartheta_j > 0\ ; j = 1,2,3,4 \ . \tag{2}$$

The reliability model can be written as follows:

$$\mathcal{R} = pr[x_1 \geq y_1, x_2 \geq y_2, x_3 \geq y_3] + pr[x_1 < y_1, x_2 \geq y_2, x_3 \geq y_3, x_4 \geq y_4]$$
$$+pr[x_1 \geq y_1, x_2 < y_2, x_3 \geq y_3, x_4 \geq y_4] + pr[x_1 \geq y_1, x_2 \geq y_2, x_3 < y_3, x_4 \geq y_4]$$

$$\mathcal{R} = \mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3 + \mathcal{P}_4 \tag{3}$$

$$\mathcal{P}_1 = pr[x_1 \geq y_1, x_2 \geq y_2, x \geq y_3]$$
$$= pr[x_1 \geq y_1]\, pr[x_2 \geq y_2]\, pr[x_3 \geq y_3]$$
$$= \left[\int_0^\infty \left(\overline{F}_{x_1}(y_1)\right) g(y_1)dy_1\right]\left[\int_0^\infty \left(\overline{F}_{x_2}(y_2)\right) g(y_2)dy_2\right]\left[\int_0^\infty \left(\overline{F}_{x_3}(y_3)\right) g(y_3)dy_3\right]$$

Where $\overline{F}(x) = 1 - F(x)$

$$\mathcal{P}_1 =$$
$$\left[\int_0^\infty \left(e^{-\lambda_1 y_1^2}\right) 2\vartheta_1 y_1 e^{-\vartheta_1 y_1^2} dy_1\right]\left[\int_0^\infty \left(e^{-\lambda_2 y_2^2}\right) 2\vartheta_2 y_2 e^{-\vartheta_2 y_2^2} dy_2\right]\left[\int_0^\infty \left(e^{-\lambda_3 y_3^2}\right) 2\vartheta_3 y_3 e^{-\vartheta_3 y_3^2} dy_3\right]$$
$$= \left[\int_0^\infty 2\vartheta_1 y_1 e^{-(\lambda_1+\vartheta_1)y_1^2} dy_1\right]\left[\int_0^\infty 2\vartheta_2 y_2 e^{-(\lambda_2+\vartheta_2)y_2^2} dy_2\right]\left[\int_0^\infty 2\vartheta_3 y_3 e^{-(\lambda_3+\vartheta_3)y_3^2} dy_3\right]$$

Now, will get $\mathcal{P}_1$:

$$\mathcal{P}_1 = \left[\frac{\vartheta_1}{\lambda_1+\vartheta_1}\right]\left[\frac{\vartheta_2}{\lambda_2+\vartheta_2}\right]\left[\frac{\vartheta_3}{\lambda_3+\vartheta_3}\right] \tag{4}$$

To derive $\mathcal{P}_2$ will start

$$\mathcal{P}_2 = pr[x_1 < y_1, x_2 \geq y_2, x_3 \geq y_3, x_4 \geq y_4]$$
$$= pr[x_1 < y_1, x_2 \geq y_2, x_3 \geq y_3, \mathcal{M}x_1 \geq \mathcal{K}y_1]$$
$$= pr[x_1 < y_1, \mathcal{M}x_1 \geq \mathcal{K}y_1] p[x_2 \geq y_2]\, p[x_3 \geq y_3]$$
$$= \left[\int_0^\infty \left(F_{x_1}(y_1)\right)\left(\overline{F}_{x_1}\left(\frac{k}{m}y_1\right)\right) g(y_1)dy_1\right]\left[\int_0^\infty \left(\overline{F}_{x_2}(y_2)\right) g(y_2)dy_2\right]$$
$$.\left[\int_0^\infty \left(\overline{F}_{x_3}(y_3)\right) g(y_3)dy_3\right]$$
$$= \left[\int_0^\infty \left(1 - e^{-\lambda_1 y_1^2}\right)\left(e^{-\lambda_1\left(\frac{k}{m}\right)^2 y_1^2}\right) 2\vartheta_1 y_1 e^{-\vartheta_1 y_1^2} dy_1\right]\left[\int_0^\infty \left(e^{-\lambda_2 y_2^2}\right) 2\vartheta_2 y_2 e^{-\vartheta_2 y_2^2} dy_2\right]$$
$$.\left[\int_0^\infty \left(e^{-\lambda_3 y_3^2}\right) 2\vartheta_3 y_3 e^{-\vartheta_3 y_3^2} dy_3\right]$$
$$\mathcal{P}_2 = \left[\int_0^\infty \left(e^{-\lambda_1\left(\frac{k}{m}\right)^2 y_1^2}\right) 2\vartheta_1 y_1 e^{-b_1 y_1^2} dy_1 - \int_0^\infty \left(e^{-\lambda_1 y_1^2}\right)\left(e^{-\lambda_1\left(\frac{k}{m}\right)^2 y_1^2}\right) 2\vartheta_1 y_1 e^{-\vartheta_1 y_1^2} dy_1\right]$$
$$.\left[\int_0^\infty \left(e^{-\lambda_2 y_2^2}\right) 2\vartheta_2 y_2 e^{-\vartheta_2 y_2^2} dy_2\right]\left[\int_0^\infty \left(e^{-\lambda_3 y_3^2}\right) 2\vartheta_3 y_3 e^{-\vartheta_3 y_3^2} dy_3\right]$$
$$\mathcal{P}_2 = \left[\int_0^\infty 2\vartheta_1 y_1 e^{-\left(\lambda_1\left(\frac{k}{m}\right)^2+\vartheta_1\right)y_1^2} dy_1 - \int_0^\infty 2\vartheta_1 y_3 e^{-\left(\lambda_1+\lambda_1\left(\frac{k}{m}\right)^2+\vartheta_1\right)y_1^2} dy_1\right]$$
$$.\left[\int_0^\infty 2\vartheta_2 y_2 e^{-(\lambda_2+\vartheta_2)y_2^2} dy_2\right].\left[\int_0^\infty 2\vartheta_3 y_3 e^{-(\lambda+\vartheta_3)y_3^2} dy_3\right].$$

Then

$$\mathcal{P}_2 = \left[\frac{\lambda_1\vartheta_1}{\left(\lambda_1\left(\frac{k}{m}\right)^2+\vartheta_1\right)\left(\lambda_1\left(1+\left(\frac{k}{m}\right)^2\right)+\vartheta_1\right)}\right]\left[\frac{\vartheta_2}{\lambda_2+\vartheta_2}\right]\left[\frac{\vartheta_3}{\lambda_3+\vartheta_3}\right] \tag{5}$$

For $\mathcal{P}_3$ we can write :

$$\mathcal{P}_3 = pr[x_1 \geq y_1, x_2 < y_2, x_3 \geq y_3, x_4 \geq y_4]$$
$$= pr[x_1 \geq y_1, x_2 < y_2, x_3 \geq y_3, \mathcal{M}x_2 \geq \mathcal{K}y_2]$$
$$= pr[x_1 \geq y_1] pr[x_2 < y_2, \mathcal{M}x_2 \geq \mathcal{K}y_2]\, pr[x_3 \geq y_3]$$
$$= \left[\frac{\vartheta_1}{\lambda_1+\vartheta_1}\right]\left[\frac{\lambda_2\vartheta_2}{\left(\lambda_2\left(\frac{k}{m}\right)^2+\vartheta_2\right)\left(\lambda_2\left(1+\left(\frac{k}{m}\right)^2\right)+\vartheta_2\right)}\right]\left[\frac{\vartheta_3}{\lambda_3+\vartheta_3}\right] \ . \tag{6}$$

Also, for $\mathcal{P}_4$

$$\mathcal{P}_4 = pr[x_1 \geq y_1, x_2 \geq y_2, x_3 < y_3, x_4 \geq y_4]$$
$$= pr[x_1 \geq y_1, x_2 \geq y_2, x_3 < y_3, \mathcal{M}x_2 \geq \mathcal{K}y_2]$$

$$= pr[x_1 \geq y_1]pr[x_2 \geq y_2]pr[x_3 < y_3, \mathcal{M}x_3 \geq \mathcal{K}y_3]$$

$$\mathcal{P}_4 = \left[\frac{\vartheta_1}{\lambda_1+\vartheta_1}\right]\left[\frac{\vartheta_2}{\lambda_2+\vartheta_2}\right]\left[\frac{\lambda_3\vartheta_3}{\left(\lambda_3\left(\frac{k}{m}\right)^2+\vartheta_3\right)\left(\lambda_3\left(1+\left(\frac{k}{m}\right)^2\right)+\vartheta_3\right)}\right]. \quad (7)$$

Finally, we can write the mathematical formula for the reliability of the (3+1) model by substituting equations (4-7) in equation (3), we get :

$$\mathcal{R} = \left[\frac{\vartheta_1}{\lambda_1+\vartheta_1}\right]\left[\frac{\vartheta_2}{\lambda_2+\vartheta_2}\right]\left[\frac{\vartheta_3}{\lambda_3+\vartheta_3}\right] + \left[\frac{\lambda_1\vartheta_1}{\left(\lambda_1\left(\frac{k}{m}\right)^2+\vartheta_1\right)\left(\lambda_1\left(1+\left(\frac{k}{m}\right)^2\right)+\vartheta_1\right)}\right]\left[\frac{\vartheta_2}{\lambda_2+\vartheta_2}\right]\left[\frac{\vartheta_3}{\lambda_3+\vartheta_3}\right] + \left[\frac{\vartheta_1}{\lambda_1+\vartheta_1}\right]$$
$$\left[\frac{\lambda_2\vartheta_2}{\left(\lambda_2\left(\frac{k}{m}\right)^2+\vartheta_2\right)\left(\lambda_2\left(1+\left(\frac{k}{m}\right)^2\right)+\vartheta_2\right)}\right]\left[\frac{\vartheta_3}{\lambda_3+\vartheta_3}\right] + \left[\frac{\vartheta_1}{\lambda_1+\vartheta_1}\right]\left[\frac{\vartheta_2}{\lambda_2+\vartheta_2}\right]\left[\frac{\lambda_3\vartheta_3}{\left(\lambda_3\left(\frac{k}{m}\right)^2+\vartheta_3\right)\left(\lambda_3\left(1+\left(\frac{k}{m}\right)^2\right)+\vartheta_3\right)}\right]. \quad (8)$$

3.Estimation

3-1Maximum likelihood Method

Assume that $x_1, x_2 \dots, x_n$, we can use the general form function L as follows [9]:

$$\mathrm{L}(x_1, x_2, x_3 \dots, x_n, 2, \lambda) = \mathrm{f}(x_1; 2, \lambda)\mathrm{f}(x_2; 2, \lambda)\mathrm{f}(x_3; 2, \lambda) \dots \mathrm{f}(x_n; 2, \lambda)$$
$$= 2^n\lambda^n \prod_{i=1}^{n} x_i \mathrm{e}^{-\sum_{i=1}^{n}\lambda x_i^2} \quad . \quad (9)$$

Taking the natural logarithm of equation (9) :

$$\ln \mathrm{L} = n\mathrm{Ln}2 + n\mathrm{Ln}\lambda + \sum_{i=1}^{n} \ln x_i - \lambda \sum_{i=1}^{n} x_i^2. \quad (10)$$

Derive equation (10):

$$\frac{\partial \ln \mathrm{L}}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i^2 . \quad (11)$$

Equating the equation (11) to zero :

$$\frac{n}{\hat{\lambda}} - \sum_{i=1}^{n} x_i^2 = 0 . \quad (12)$$

From equation (12) the estimator of λ is given as follows

$$\hat{\lambda}_{ML} = \frac{n}{\sum_{i=1}^{n} x_i^2} . \quad (13)$$

Suppose that $X_{1_{i_1}}; i_1 = 1, 2, \dots, n_1$ $X_{2_{i_2}}; i_2 = 1, 2, \dots, \mathrm{n}_2$ $X_{3_{i_3}}; i_3 = 1, 2, \dots, \mathrm{n}_3$ and $X_{4_{i_4}}; i_4 = 1, 2, \dots, \mathrm{n}_4$ are strength random samples from $\mathrm{R}(2, \lambda_1)$, $\mathrm{R}(2, \lambda_2)$, $\mathrm{R}(2, \lambda_3)$ and $\mathrm{R}(2, \lambda_4)$, with samples size n_1 , $\mathrm{n}_2, \mathrm{n}_3$ and n_4 , respectively :

$$\hat{\lambda}_{\zeta ML} = \frac{n_\zeta}{\sum_{i_\zeta=1}^{n_\zeta} x_{\zeta_{i_\zeta}}} , \zeta = 1,2,3,4 . \quad (14)$$

The ML estimator for ϑ_1, ϑ_2 , ϑ_3 and ϑ_4 is as follows:

$$\hat{\vartheta}_{\zeta ML} = \frac{m_\zeta}{\sum_{j_\zeta=1}^{m_\zeta} y_{\zeta_{j_\zeta}}} , \zeta = 1,2,3,4. \quad (15)$$

Substituting equations(14-15) in (8), we get :

$$\hat{\mathcal{R}}_{ML} = \left[\frac{\hat{\vartheta}_{1ML}}{\hat{\lambda}_{1ML}+\hat{\vartheta}_{1ML}}\right]\left[\frac{\hat{\vartheta}_{2ML}}{\hat{\lambda}_{2ML}+\hat{\vartheta}_{2ML}}\right]\left[\frac{\hat{\vartheta}_{3ML}}{\hat{\lambda}_{3ML}+\hat{\vartheta}_{3ML}}\right] + \left[\frac{\hat{\lambda}_{1ML}\hat{\vartheta}_{1ML}}{\left(\hat{\lambda}_{1ML}\left(\frac{k}{m}\right)^2+\hat{\vartheta}_{1ML}\right)\left(\hat{\lambda}_{1ML}\left(1+\left(\frac{k}{m}\right)^2\right)+\hat{\vartheta}_{1ML}\right)}\right]$$
$$\left[\frac{\hat{\vartheta}_{2ML}}{\hat{\lambda}_{2ML}+\hat{\vartheta}_{2ML}}\right]\left[\frac{\hat{\vartheta}_{3ML}}{\hat{\lambda}_{3ML}+\hat{\vartheta}_{3ML}}\right] + \left[\frac{\hat{\vartheta}_{1ML}}{\hat{\lambda}_{1ML}+\hat{\vartheta}_{1ML}}\right]\left[\frac{\hat{\lambda}_{2ML}\hat{\vartheta}_{2ML}}{\left(\hat{\lambda}_{2ML}\left(\frac{k}{m}\right)^2+\hat{\vartheta}_{2ML}\right)\left(\hat{\lambda}_{2ML}\left(1+\left(\frac{k}{m}\right)^2\right)+\hat{\vartheta}_{2ML}\right)}\right]$$
$$\left[\frac{\hat{\vartheta}_{3ML}}{\hat{\lambda}_{3ML}+\hat{\vartheta}_{3ML}}\right] + \left[\frac{\hat{\vartheta}_{1ML}}{\hat{\lambda}_{1ML}+\hat{\vartheta}_{1ML}}\right]\left[\frac{\hat{\vartheta}_{2ML}}{\hat{\lambda}_{2ML}+\hat{\vartheta}_{2ML}}\right]\left[\frac{\hat{\lambda}_{3ML}\hat{\vartheta}_{3ML}}{\left(\hat{\lambda}_{3ML}\left(\frac{k}{m}\right)^2+\hat{\vartheta}_{3ML}\right)\left(\hat{\lambda}_{3ML}\left(1+\left(\frac{k}{m}\right)^2\right)+\hat{\vartheta}_{3ML}\right)}\right]. \quad (16)$$

3-2 Percentile Method:

In this method, we start with equation (1):

$$\mathrm{F}\left(\mathrm{x}_{(\mathrm{i})}\right) = 1 - \mathrm{e}^{-\lambda x_{(\mathrm{i})}^2}$$

$$ln\left(1 - F(x_i)\right) = -\lambda x_{(\mathrm{i})}^2$$

$$x_{(i)} = \left(\frac{-\ln\left(1-F\left(x_{(i)}\right)\right)}{\lambda}\right)^{\frac{1}{2}} . \quad (17)$$

The value P_i ; $i = 1,2,\dots,n$ is used where :

$$x_{(i)} = \left(\frac{-\ln(1-P_i)}{\lambda}\right)^{\frac{1}{2}} . \quad (18)$$

The minimizing of the following equation is

$$\sum_{i=1}^{n}[x_i - F(x_i)]^2 . \quad (19)$$

Substitution equation (17) in (18), will get:

$$\sum_{i=1}^{n}\left[x_{(i)} - \left(\frac{-\ln(1-P_i)}{\lambda}\right)^{\frac{1}{2}}\right]^2 . \quad (20)$$

Derive equation (20) for λ, we get:

$$\sum_{i=1}^{n} 2\left[\left(x_{(i)}\right) - \lambda^{-\frac{1}{2}}(-\ln(1-P_i))^{\frac{1}{2}}\right]\left(\frac{1}{2}\lambda^{-\left(\frac{1}{2}+1\right)}\right)(-\ln(1-P_i))^{\frac{1}{2}} = 0 .$$

The Pr estimator of λ is:

$$\hat{\lambda}_{Pr} = \left[\frac{\sum_{i=1}^{n}(-\ln(1-P_i))}{\sum_{i=1}^{n}\left(x_{(i)}\right)(-\ln(1-P_i))^{\frac{1}{2}}}\right]^2 . \quad (21)$$

Percentile estimators of the $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ and $(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4)$ are :

$$\hat{\lambda}_{\zeta Pr} = \left[\frac{\sum_{i_\zeta=1}^{n_\zeta}\left(-\ln\left(1-P_{i_\zeta}\right)\right)}{\sum_{i_\zeta=1}^{n_\zeta}\left(x_{\zeta_{(i_\zeta)}}\right)\left(-\ln\left(1-P_{i_\zeta}\right)\right)^{\frac{1}{2}}}\right]^2 ;\ \zeta = 1,2,3,4 . \quad (22)$$

and

$$\hat{\vartheta}_{\zeta Pr} = \left[\frac{\sum_{i_\zeta=1}^{n_\zeta}\left(-\ln\left(1-P_{i_\zeta}\right)\right)}{\sum_{i_\zeta=1}^{n_\zeta}\left(y_{\zeta_{(i_\zeta)}}\right)\left(-\ln\left(1-P_{i_\zeta}\right)\right)^{\frac{1}{2}}}\right]^2 ;\ \zeta = 1,2,3,4 . \quad (23)$$

Now, replacement equations (22) and (23) in equation (8) :

$$\hat{\mathcal{R}}_{Pr} = \left[\frac{\hat{\vartheta}_{1Pr}}{\hat{\lambda}_{1Pr}+\hat{\vartheta}_{1Pr}}\right]\left[\frac{\hat{\vartheta}_{2Pr}}{\hat{\lambda}_{2Pr}+\hat{\vartheta}_{2Pr}}\right]\left[\frac{\hat{\vartheta}_{3Pr}}{\hat{\lambda}_{3Pr}+\hat{\vartheta}_{3Pr}}\right] + \left[\frac{\hat{\lambda}_{1Pr}\hat{\vartheta}_{1Pr}}{\left(\hat{\lambda}_{1Pr}\left(\frac{k}{m}\right)^2+\hat{\vartheta}_{1Pr}\right)\left(\hat{\lambda}_{1Pr}\left(1+\left(\frac{k}{m}\right)^2\right)+\hat{\vartheta}_{1Pr}\right)}\right]$$
$$\left[\frac{\hat{\vartheta}_{2Pr}}{\hat{\lambda}_{2Pr}+\hat{\vartheta}_{2Pr}}\right]\left[\frac{\hat{\vartheta}_{3Pr}}{\hat{\lambda}_{3Pr}+\hat{\vartheta}_{3Pr}}\right] + \left[\frac{\hat{\vartheta}_{1Pr}}{\hat{\lambda}_{1Pr}+\hat{\vartheta}_{1Pr}}\right]\left[\frac{\hat{\lambda}_{2Pr}\hat{\vartheta}_{2Pr}}{\left(\hat{\lambda}_{2Pr}\left(\frac{k}{m}\right)^2+\hat{\vartheta}_{2Pr}\right)\left(\hat{\lambda}_{2Pr}\left(1+\left(\frac{k}{m}\right)^2\right)+\hat{\vartheta}_{2Pr}\right)}\right]$$
$$\left[\frac{\hat{\vartheta}_{3Pr}}{\hat{\lambda}_{3Pr}+\hat{\vartheta}_{3Pr}}\right] + \left[\frac{\hat{\vartheta}_{1Pr}}{\hat{\lambda}_{1Pr}+\hat{\vartheta}_{1Pr}}\right]\left[\frac{\hat{\vartheta}_{2Pr}}{\hat{\lambda}_{2Pr}+\hat{\vartheta}_{2Pr}}\right]\left[\frac{\hat{\lambda}_{3Pr}\hat{\vartheta}_{3Pr}}{\left(\hat{\lambda}_{3Pr}\left(\frac{k}{m}\right)^2+\hat{\vartheta}_{3Pr}\right)\left(\hat{\lambda}_{3Pr}\left(1+\left(\frac{k}{m}\right)^2\right)+\hat{\vartheta}_{3Pr}\right)}\right]. \quad (24)$$

3-3 Least Squares Method (LS):

This method is used to minimize the non-parametric $(\hat{F})$ and parametric (F) functions [10]:

$$S(2,\lambda) = \sum_{i=1}^{r}\left(\hat{F}\left(x_{(i)}\right) - F\left(x_{(i)}\right)\right)^2$$
$$= \sum_{i=1}^{n}\left(\hat{F}\left(x_{(i)}\right) - \left(1 - e^{-\lambda x_{(i)}^2}\right)\right)^2 . \quad (25)$$

The linear form of the CDF is as follows:

$$F\left(x_{(i)}\right) = 1 - e^{-\lambda x_{(i)}^2}$$
$$-\ln\left(1 - F\left(x_{(i)}\right)\right) = \lambda x_{(i)}^2 . \quad (26)$$

Replace $\hat{F}(x_{(i)})$ by the value :

$\hat{F}(x_{(i)}) = \frac{i}{n+1}\ ; i = 1,2,\ldots,n$.

The equation $S(2,\lambda)$ becomes :

$$S(2,\lambda) = \sum_{i=1}^{n}\left(q_i - \lambda x_{(i)}^2\right)^2 \quad . \qquad (27)$$

Where $q_{(i)} = -\ln\left(1 - \hat{F}(x_{(i)})\right) = -\ln(1 - P_i)$

We derive equation (27) for λ to get:

$$\frac{\partial S(2,\lambda)}{\lambda} = \sum_{i=1}^{n} 2\left(q_i - \lambda x_{(i)}^2\right)\left(-x_{(i)}^2\right) = 0$$

$$-\sum_{i=1}^{n} q_i x_{(i)}^2 + \lambda \sum_{i=1}^{n} x_{(i)}^4 = 0$$

Then we get $\hat{\lambda}_{LS}$:

$$\hat{\lambda}_{LS} = \frac{\sum_{i=1}^{n} q_i\, x_{(i)}^2}{\sum_{i=1}^{n} x_{(i)}^4} \quad . \qquad (28)$$

The LS estimator of the $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ and $(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4)$ are given as follows :

$$\hat{\lambda}_{\zeta LS} = \frac{\sum_{i_\zeta=1}^{n_\zeta} q_{i_\zeta}\, x_{\zeta_{(i_\zeta)}}^2}{\sum_{i_\zeta=1}^{n_\zeta} x_{\zeta_{(i_\zeta)}}^4}\ , \zeta = 1,2,3,4 \ . \qquad (29)$$

and

$$\hat{\vartheta}_{\zeta LS} = \frac{\sum_{j_\zeta=1}^{m_\zeta} q_{j_\zeta}\, y_{\zeta_{(j_\zeta)}}^2}{\sum_{j_\zeta=1}^{m_\zeta} y_{\zeta_{(j_\zeta)}}^4}\ , \zeta = 1,2,3,4 \ . \qquad (30)$$

Substituting equations(29) and (30) in (8), we get :

$$\hat{\mathcal{R}}_{LS} = \left[\frac{\hat{\vartheta}_{1LS}}{\hat{\lambda}_{1LS}+\hat{\vartheta}_{1LS}}\right]\left[\frac{\hat{\vartheta}_{2LS}}{\hat{\lambda}_{2LS}+\hat{\vartheta}_{2LS}}\right]\left[\frac{\hat{\vartheta}_{3LS}}{\hat{\lambda}_{3LS}+\hat{\vartheta}_{3LS}}\right] + \left[\frac{\hat{\lambda}_{1LS}\hat{\vartheta}_{1LS}}{\left(\hat{\lambda}_{1LS}\left(\frac{k}{m}\right)^2+\hat{\vartheta}_{1LS}\right)\left(\hat{\lambda}_{1LS}\left(1+\left(\frac{k}{m}\right)^2\right)+\hat{\vartheta}_{1LS}\right)}\right]$$
$$\left[\frac{\hat{\vartheta}_{2LS}}{\hat{\lambda}_{2LS}+\hat{\vartheta}_{2LS}}\right]\left[\frac{\hat{\vartheta}_{3LS}}{\hat{\lambda}_{3LS}+\hat{\vartheta}_{3LS}}\right] + \left[\frac{\hat{\vartheta}_{1LS}}{\hat{\lambda}_{1LS}+\hat{\vartheta}_{1LS}}\right]\left[\frac{\hat{\lambda}_{2LS}\hat{\vartheta}_{2LS}}{\left(\hat{\lambda}_{2LS}\left(\frac{k}{m}\right)^2+\hat{\vartheta}_{2LS}\right)\left(\hat{\lambda}_{2LS}\left(1+\left(\frac{k}{m}\right)^2\right)+\hat{\vartheta}_{2LS}\right)}\right]$$
$$\left[\frac{\hat{\vartheta}_{3LS}}{\hat{\lambda}_{3LS}+\hat{\vartheta}_{3LS}}\right] + \left[\frac{\hat{\vartheta}_{1LS}}{\hat{\lambda}_{1LS}+\hat{\vartheta}_{1LS}}\right]\left[\frac{\hat{\vartheta}_{2LS}}{\hat{\lambda}_{2LS}+\hat{\vartheta}_{2LS}}\right]\left[\frac{\hat{\lambda}_{3LS}\hat{\vartheta}_{3LS}}{\left(\hat{\lambda}_{3LS}\left(\frac{k}{m}\right)^2+\hat{\vartheta}_{3LS}\right)\left(\hat{\lambda}_{3LS}\left(1+\left(\frac{k}{m}\right)^2\right)+\hat{\vartheta}_{3LS}\right)}\right]. \qquad (31)$$

3-4 Weighted Least Squares method:

In this method, we start with the following equation:

$$Q = \sum_{i=1}^{n} w_i\left(\hat{F}(x_{(i)}) - F(x_{(i)})\right)^2 \ . \qquad (32)$$

Where $w_i = \frac{1}{Var[F(x_{(i)})]} = \frac{(n+1)^2(n+2)}{i(n-i+1)}, i = 1,2,\ldots,n$.

Let x_1, x_2, x_4 ..., and x_n be a random sample for $R(2,\lambda)$.The minimization of the equation becomes [9]:

$$Q(2,\lambda) = \sum_{i=1}^{n} w_i\left(\hat{F}(x_{(i)}) - \left(1 - e^{-\lambda x_{(i)}^2}\right)\right)^2 \ . \qquad (33)$$

We follow the same steps in equations (26) and (27) , then we get:

$$Q(2,\lambda) = \sum_{i=1}^{n} w_i\left(q_i - \lambda x_{(i)}^2\right)^2 \quad . \qquad (34)$$

Derive equation (34) for λ:

$$\frac{\partial Q(2,\lambda)}{\partial\lambda} = \sum_{i=1}^{n} 2w_i\left(q_i - \lambda x_{(i)}^2\right)\left(-x_{(i)}^2\right)$$

$$-\sum_{i=1}^{n} w_i q_i\, x_{(i)}^2 + \hat{\lambda}\sum_{i=1}^{n} w_i x_{(i)}^4 = 0$$

The Weighted Least Square estimator of λ , $\hat{\lambda}_{(WLS)}$, is given as follows:

$$\hat{\lambda}_{WLS} = \frac{\sum_{i=1}^{n} w_i q_i x_{(i)}^2}{\sum_{i=1}^{n} w_i x_{(i)}^4} \quad . \tag{35}$$

Now, the WLS estimators of the $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ and $(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4)$ are respectively givenas follows:

$$\hat{\lambda}_{\zeta WLS} = \frac{\sum_{i_\zeta=1}^{n_\zeta} w_{i_\zeta} q_{i_\zeta} x_{\zeta_{(i_\zeta)}}^2}{\sum_{i_\zeta=1}^{n_\zeta} w_{i_\zeta} x_{\zeta_{(i_\zeta)}}^4}, \zeta = 1,2,3,4 \quad . \tag{36}$$

and

$$\hat{\vartheta}_{\zeta WLS} = \frac{\sum_{j_\zeta=1}^{n_\zeta} w_{j_\zeta} q_{j_\zeta} y_{\zeta_{(j_\zeta)}}^2}{\sum_{j_\zeta=1}^{n_\zeta} w_{j_\zeta} y_{\zeta_{(j_\zeta)}}^4}, \zeta = 1,2,3,4 \quad . \tag{37}$$

Where $w_j = \frac{1}{Var[G(y_{(j)})]} = \frac{(m+1)^2(m+2)}{j(m-j+1)}, j = 1,2, \dots, m.$

Now, replacement equations (36) and (37) in equation (8) to get:

$$\hat{\mathcal{R}}_{WLS} = \left[\frac{\hat{\vartheta}_{1WLS}}{\hat{\lambda}_{1WLS}+\hat{\vartheta}_{1WLS}}\right]\left[\frac{\hat{\vartheta}_{2LWS}}{\hat{\lambda}_{2WLS}+\hat{\vartheta}_{2LS}}\right]\left[\frac{\hat{\vartheta}_{3WLS}}{\hat{\lambda}_{3WLS}+\hat{\vartheta}_{3LS}}\right] + \left[\frac{\hat{\lambda}_{1WLS}\hat{\vartheta}_{1WLS}}{\left(\hat{\lambda}_{1WLS}\left(\frac{k}{m}\right)^2+\hat{\vartheta}_{1WLS}\right)\left(\hat{\lambda}_{1WLS}\left(1+\left(\frac{k}{m}\right)^2\right)+\hat{\vartheta}_{1WLS}\right)}\right]$$
$$\left[\frac{\hat{\vartheta}_{2WLS}}{\hat{\lambda}_{2WLS}+\hat{\vartheta}_{2WLS}}\right]\left[\frac{\hat{\vartheta}_{3WLS}}{\hat{\lambda}_{3WLS}+\hat{\vartheta}_{3WLS}}\right] +$$
$$\left[\frac{\hat{\vartheta}_{1WLS}}{\hat{\lambda}_{1WLS}+\hat{\vartheta}_{1WLS}}\right]\left[\frac{\hat{\lambda}_{2WLS}\hat{\vartheta}_{2WLS}}{\left(\hat{\lambda}_{2WLS}\left(\frac{k}{m}\right)^2+\hat{\vartheta}_{2WLS}\right)\left(\hat{\lambda}_{2WLS}\left(1+\left(\frac{k}{m}\right)^2\right)+\hat{\vartheta}_{2WLS}\right)}\right]$$
$$\left[\frac{\hat{\vartheta}_{3WLS}}{\hat{\lambda}_{3WLS}+\hat{\vartheta}_{3WLS}}\right] +$$
$$\left[\frac{\hat{\vartheta}_{1WLS}}{\hat{\lambda}_{1WLS}+\hat{\vartheta}_{1WLS}}\right]\left[\frac{\hat{\vartheta}_{2WLS}}{\hat{\lambda}_{2WLS}+\hat{\vartheta}_{2WLS}}\right]\left[\frac{\hat{\lambda}_{3WLS}\hat{\vartheta}_{3WLS}}{\left(\hat{\lambda}_{3WLS}\left(\frac{k}{m}\right)^2+\hat{\vartheta}_{3WLS}\right)\left(\hat{\lambda}_{3WLS}\left(1+\left(\frac{k}{m}\right)^2\right)+\hat{\vartheta}_{3WLS}\right)}\right] . \tag{38}$$

3-5 Regression Method:

In this method, we can start by the standard regression equation [9]:

$$z_i = a + bu_i + e_i \quad . \tag{39}$$

Where z_i is the dependent variable, u_i is the independent variable and e_i is error random variable independent.

Let $x_1, x_2, \dots,$ and x_n be a random sample of $R(2, \lambda)$.

Take the logarithm for equation (1):

$$F(x_{(i)}) = 1 - e^{-\lambda x_{(i)}^2}$$
$$\left(1 - F(x_{(i)})\right)^{-1} = e^{\lambda x_{(i)}^2}$$
$$\mathrm{Ln}\left[\left(1 - F(x_{(i)})\right)^{-1}\right] = \lambda x_{(i)}^2$$

Changing $F(x_{(i)})$ by the plotting position P_i

$$\mathrm{Ln}[(1 - P_i)^{-1}] = \lambda x_{(i)}^2 \quad . \tag{40}$$

By comparison between equations (40) and (39):

$$z_i = \mathrm{Ln}[(1 - P_i)^{-1}], a = 0, b = \lambda, u_i = x_{(i)}^2 \quad ; i = 1,2, \dots, n \quad . \tag{41}$$

Where b can be estimated by the minimizing summation of the squared error with respect to b, then we get:

$$\hat{b} = \frac{n\sum_{i=1}^{n} z_i u_i - \sum_{i=1}^{n} z_i \sum_{i=1}^{n} u_i}{n\sum_{i=1}^{n}(u_i)^2 - \left(\sum_{i=1}^{n} u_i\right)^2} \tag{42}$$

By substation (41) in (42), the estimator for λ, $\hat{\lambda}_{(Rg)}$, is given as follows:

$$\hat{\lambda}_{Rg} = \frac{\mathrm{n}\sum_{\mathrm{i}=1}^{\mathrm{n}} x_{(\mathrm{i})}^{2} \,\mathrm{Ln}\left[(1-\mathrm{P_i})^{-1}\right] - \sum_{\mathrm{i}=1}^{\mathrm{n}} x_{(\mathrm{i})}^{2} \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{Ln}\left[(1-\mathrm{P_i})^{-1}\right]}{\mathrm{n}\sum_{\mathrm{i}=1}^{\mathrm{n}} x_{(\mathrm{i})}^{4} - \left[\sum_{\mathrm{i}=1}^{\mathrm{n}} x_{(\mathrm{i})}^{2}\right]^{2}} \;. \quad (43)$$

The Rg estimators of the $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ and $(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4)$ are :

$$\lambda_{\zeta Rg} = \frac{\mathrm{n}_\zeta \sum_{\mathrm{i}_\zeta=1}^{\mathrm{n}_\zeta} x_{\zeta_{(\mathrm{i}_\zeta)}}^{2} \ln\left[\left(1-\mathrm{P}_{\mathrm{i}_\zeta}\right)^{-1}\right] - \sum_{\mathrm{i}_\zeta=1}^{\mathrm{n}_\zeta} x_{\zeta_{(\mathrm{i}_\zeta)}}^{2} \sum_{\mathrm{i}_\zeta=1}^{\mathrm{n}_\zeta} \ln\left[\left(1-\mathrm{P}_{\mathrm{i}_\xi}\right)^{-1}\right]}{\mathrm{n}_\zeta \sum_{\mathrm{i}_\zeta=1}^{\mathrm{n}_\zeta} x_{\zeta_{(\mathrm{i}_\zeta)}}^{4} - \left[\sum_{\mathrm{i}_\zeta=1}^{\mathrm{n}_\zeta} x_{\zeta_{(\mathrm{i}_\zeta)}}^{2}\right]^{2}} \;, \zeta = 1,2,3,4 \;. \quad (44)$$

and

$$\hat{\vartheta}_{\zeta Rg} = \frac{\mathrm{m}_\zeta \sum_{\mathrm{j}_\zeta=1}^{\mathrm{m}_\zeta} y_{\zeta_{(\mathrm{j}_\zeta)}}^{2} \ln\left[\left(1-\mathrm{P}_{\mathrm{j}_\zeta}\right)^{-1}\right] - \sum_{\mathrm{j}_\zeta=1}^{\mathrm{m}_\zeta} y_{\zeta_{(\mathrm{j}_\zeta)}}^{2} \sum_{\mathrm{j}_\zeta=1}^{\mathrm{m}_\zeta} \ln\left[\left(1-\mathrm{P}_{\mathrm{j}_\zeta}\right)^{-1}\right]}{\mathrm{m}_\zeta \sum_{\mathrm{j}_\zeta=1}^{\mathrm{m}_\zeta} y_{\zeta_{(\mathrm{j}_\zeta)}}^{4} - \left[\sum_{\mathrm{j}_\zeta=1}^{\mathrm{m}_\zeta} y_{\zeta_{(\mathrm{j}_\zeta)}}^{2}\right]^{2}} \;, \zeta = 1,2,3,4 \;. \quad (45)$$

From equation (45) we have $\mathrm{z_j} = \ln\left[\left(1-\mathrm{P_j}\right)^{-1}\right], \mathrm{a} = 0, \mathrm{b} = \theta, \mathrm{u_j} = y_{(\mathrm{j})}^{2}$;$\mathrm{j} = 1,2,\dots,\mathrm{m}$

Now, we put equations (44) and (45) in equation (8) to get:

$$\begin{aligned}\hat{\mathcal{R}}_{Rg} = &\left[\frac{\hat{\vartheta}_{1Rg}}{\hat{\lambda}_{1Rg}+\hat{\vartheta}_{1Rg}}\right]\left[\frac{\hat{\vartheta}_{2Rg}}{\hat{\lambda}_{2Rg}+\hat{\vartheta}_{2Rg}}\right]\left[\frac{\hat{\vartheta}_{3Rg}}{\hat{\lambda}_{3Rg}+\hat{\vartheta}_{3Rg}}\right] + \left[\frac{\hat{\lambda}_{1Rg}\hat{\vartheta}_{1Rg}}{\left(\hat{\lambda}_{1Rg}\left(\frac{k}{m}\right)^{2}+\hat{\vartheta}_{1Rg}\right)\left(\hat{\lambda}_{1Rg}\left(1+\left(\frac{k}{m}\right)^{2}\right)+\hat{\vartheta}_{1Rg}\right)}\right]\\ &\left[\frac{\hat{\vartheta}_{2Rg}}{\hat{\lambda}_{2Rg}+\hat{\vartheta}_{2Rg}}\right]\left[\frac{\hat{\vartheta}_{3Rg}}{\hat{\lambda}_{3Rg}+\hat{\vartheta}_{3Rg}}\right] + \left[\frac{\hat{\vartheta}_{1Rg}}{\hat{\lambda}_{1Rg}+\hat{\vartheta}_{1Rg}}\right]\left[\frac{\hat{\lambda}_{2Rg}\hat{\vartheta}_{2Rg}}{\left(\hat{\lambda}_{2Rg}\left(\frac{k}{m}\right)^{2}+\hat{\vartheta}_{2Rg}\right)\left(\hat{\lambda}_{2Rg}\left(1+\left(\frac{k}{m}\right)^{2}\right)+\hat{\vartheta}_{2Rg}\right)}\right]\\ &\left[\frac{\hat{\vartheta}_{3Rg}}{\hat{\lambda}_{3Rg}+\hat{\vartheta}_{3Rg}}\right] + \left[\frac{\hat{\vartheta}_{1Rg}}{\hat{\lambda}_{1Rg}+\hat{\vartheta}_{1Rg}}\right]\left[\frac{\hat{\vartheta}_{2Rg}}{\hat{\lambda}_{2Rg}+\hat{\vartheta}_{2Rg}}\right]\left[\frac{\hat{\lambda}_{3Rg}\hat{\vartheta}_{3Rg}}{\left(\hat{\lambda}_{3Rg}\left(\frac{k}{m}\right)^{2}+\hat{\vartheta}_{3Rg}\right)\left(\hat{\lambda}_{3Rg}\left(1+\left(\frac{k}{m}\right)^{2}\right)+\hat{\vartheta}_{3Rg}\right)}\right]. \quad (46)\end{aligned}$$

4. Simulations

We conduct simulation of the results of the five estimation methods where the comparison is made using MSE. It is replicated 10000 times. Different size of samples small, medium and large were taken independently [10].

4-1Algorithm:

The MATLAB program is used to write simulation algorithms for the purpose of estimating the reliability of the model, according to the following steps :

1-Let$(x_{11}, x_{12}, \dots, x_{1n_1})$, $(x_{21}, x_{22}, \dots, x_{2n_2})$, $(x_{31}, x_{32}, \dots, x_{3n_3})$, $(x_{41}, x_{42}, \dots, x_{4n_4})$ and $(y_{11}, y_{12}, \dots, y_{1m_1})$, $(y_{21}, y_{22}, \dots, y_{2m_2})$, $(y_{31}, y_{32}, \dots, y_{3m_3})$, $(y_{41}, y_{42}, \dots, y_{4m_4})$ are random samples of sizes $(n_1, n_2, n_3, n_4, m_1, m_2, m_3, m_4) =$ (30,30,30,30,30,30,30,30), (60, 60, 60, 60, 60, 60,60, 60), and (90,90,90,90,90,90, 90,90) are generated from the Rayleigh distribution.

2- In Table 1, parameters values are selected for six experiments $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \vartheta_2, \vartheta_2, \vartheta_3, \vartheta_4)$.

Table 1: Values of parameters and Reliability.

Experiment	$\mathcal{K}$	$\mathcal{M}$	λ_1	λ_2	λ_3	ϑ_1	ϑ_2	ϑ_3	$\mathcal{R}$
1	1.1	0.95	0.5	0.5	0.5	0.5	0.5	0.5	0.2209
2	1.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.1318
3	1.2	0.8	1	1	1	2	2	2	0.4158
4	1.2	0.8	0.7	0.7	0.7	3	3	3	0.7047
5	1.2	0.8	2	2	2	1.8	1.8	1.8	0.1526
6	1.2	0.8	2	2	2	1.5	1.5	1.5	0.1132

3- Parameters $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \vartheta_2, \vartheta_2, \vartheta_3, \vartheta_4)$ are estimated (ML, Pr, LS, WLS and Rg) in equations: (14),(15), (22),(23),(29) ,(30), (36),(37),(44) and (45), respectively.
4- $\mathcal{R}$ was estimated in equations: (16),(24),(31),(38) and (46).
5- The mean is calculated according to formula: Mean $= \frac{\sum_{i=1}^{L} \hat{\mathcal{R}}_i}{L}$.
6- Finally, the mean square error is used to compare the results according to the formula: $\text{MSE}(\hat{\mathcal{R}}) = \frac{1}{L}\sum_{i=1}^{L}(\hat{\mathcal{R}}_i - \mathcal{R})^2$.

4-2 Simulation Results

After performing the above steps for different sample size $(n_1, n_2, n_3, n_4, m_1, m_2, m_3, m_4) =$ $(30,30,30,30,30,30,30,30)$, $(60,60,60,60,60,60,60,60)$, and $(90,90,90,90,90,90, 90,90)$,we get the following:

Table 2: The results of simulation trial (1).

Sizes of simples	*Criterion*	*ML*	*LS*	*WLS*	*Pr*	*Rg*	*Best*
(30,30,30,30,30,30,30,30)	Mean	0.2202	0.2202	0.2199	0.2201	0.2279	ML & Pr
	MSE	0.0021	0.0026	0.0036	0.0021	0.0067	
(60,60,60,60,60,60,60,60)	Mean	0.2207	0.2207	0.2203	0.2206	0.2285	
	MSE	0.0010	0.0013	0.0024	0.0010	0.0037	
(90,90,90,90,90,90, 90,90)	Mean	0.2204	0.2204	0.2201	0.2204	0.2282	
	MSE	0.0007	0.0009	0.0019	0.0007	0.0026	

Table 3: The results of simulation trial (2).

Sizes of simples	*Criterion*	*ML*	*LS*	*WLS*	*Pr*	*Rg*	*Best*
(30,30,30,30,30,30,30,30)	Mean	0.1316	0.1313	0.1313	0.1317	0.1313	ML & Pr
	MSE	0.0009	0.0011	0.0015	0.0009	0.0028	
(60,60,60,60,60,60,60,60)	Mean	0.1318	0.1319	0.1320	0.1318	0.1323	
	MSE	0.0004	0.0006	0.0010	0.0004	0.0015	
(90,90,90,90,90,90, 90,90)	Mean	0.1316	0.1319	0.1324	0.1316	0.1327	
	MSE	0.0003	0.0004	0.0008	0.0003	0.0010	

Table 4: The results of simulation trial (3).

Sizes of simples	*Criterion*	*ML*	*LS*	*WLS*	*Pr*	*Rg*	*Best*
(30,30,30,30,30,30,30,30)	Mean	0.4099	0.4084	0.4055	0.4099	0.4131	ML & Pr
	MSE	0.0034	0.0042	0.0061	0.0034	0.0110	
(60,60,60,60,60,60,60,60)	Mean	0.4126	0.4117	0.4084	0.4125	0.4199	
	MSE	0.0017	0.0021	0.0038	0.0017	0.0059	
(90,90,90,90,90,90, 90,90)	Mean	0.4140	0.4136	0.4112	0.4140	0.4238	
	MSE	0.0011	0.0015	0.0031	0.0011	0.0043	

Table 5: The results of simulation trial (4).

Sizes of simples	*Criterion*	*ML*	*LS*	*WLS*	*Pr*	*Rg*	*Best*
(30,30,30,30,30,30,30,30)	Mean	0.6959	0.6934	0.6881	0.6959	0.7119	ML & Pr
	MSE	0.0026	0.0033	0.0049	0.0026	0.0097	
(60,60,60,60,60,60,60,60)	Mean	0.6999	0.6981	0.6923	0.7000	0.7227	
	MSE	0.0013	0.0016	0.0030	0.0013	0.0053	
(90,90,90,90,90,90, 90,90)	Mean	0.7015	0.7003	0.6952	0.7015	0.7280	
	MSE	0.0008	0.0011	0.0023	0.0008	0.0040	

Table 6: The results of simulation trial (5)

Sizes of simples	*Criterion*	*ML*	*LS*	*WLS*	*Pr*	*Rg*	*Best*
(30,30,30,30,30,30,30,30)	Mean	0.1525	0.1522	0.1521	0.1526	0.1552	ML & Pr
	MSE	0.0013	0.0016	0.0022	0.0013	0.0040	
(60,60,60,60,60,60,60,60)	Mean	0.1526	0.1525	0.1523	0.1527	0.1554	
	MSE	0.0006	0.0008	0.0014	0.0006	0.0021	
(90,90,90,90,90,90, 90,90)	Mean	0.1528	0.1530	0.1534	0.1528	0.1563	
	MSE	0.0004	0.0005	0.0011	0.0004	0.0015	

Table 7: The results of simulation trial (6).

Sizes of simples	*Criterion*	*ML*	*LS*	*WLS*	*Pr*	*Rg*	*Best*
(30,30,30,30,30,30,30,30)	Mean	0.1135	0.1138	0.1143	0.1136	0.1173	ML & Pr
	MSE	0.0008	0.0010	0.0015	0.0008	0.0027	
(60,60,60,60,60,60,60,60)	Mean	0.1134	0.1136	0.1141	0.1134	0.1164	
	MSE	0.0004	0.0005	0.0010	0.0004	0.0015	
(90,90,90,90,90,90, 90,90)	Mean	0.1134	0.1134	0.1138	0.1134	0.1158	
	MSE	0.0002	0.0003	0.0005	0.0002	0.0007	

6- Finally, the mean square error is used to compare the results according to the formula: $\text{MSE}(\hat{\mathcal{R}}) = \frac{1}{\text{L}}\sum_{i=1}^{\text{L}}(\hat{\mathcal{R}}_i - \mathcal{R})^2$.

4-2 Simulation Results

After performing the above steps for different sample size $(n_1, n_2, n_3, n_4, m_1, m_2, m_3, m_4) =$ $(30,30,30,30,30,30,30,30), (60,60,60,60,60,60,60,60),$ and $(90,90,90,90,90,90, 90,90)$,we get the following:

4-3 Discussion of results

When looking at Table 1, which contains the values of the parameters and reliability of the six experiments, it turns out the following:

If experiments 1 and 2 are compared, it is clear that when the value of the attenuation factor K increases and the value of the attenuation factor m decreases, the reliability value decreases and vice versa. If experiments 4 and 3 are compared with experiments 5 and 6, it is clear that when the parameter values $\vartheta_2, \vartheta_2, \vartheta_3$ and ϑ_4 increase and the parameter values $\lambda_1, \lambda_2, \lambda_3$ and λ_4 decrease, the reliability value of the model increases and vice versa.

With regard to Tables (2-6) of the Monte Carlo simulation results, it turns out that the performance of ML and Pr estimators are the best for estimating the reliability of the model, and this is for all six experiments and various sample sizes.

5. Conclusions

We came to these conclusions after conducting the simulation:

1.The conclusion is made from Table 1.

I-As the numerical value of the parameters $\lambda_1, \lambda_2, \lambda_3$ $, and\ \lambda_4$ increases, the reliability value of the model decreases.

II- As the numerical value of the parameters $\vartheta_2, \vartheta_2, \vartheta_3$ and ϑ_4 increases, the reliability value of the model increases.

III- With the increase in the $\frac{k}{m}$ value, the reliability value decreases.

2.The conclusion is made from Tables (2-7) that the best estimator for $\mathcal{R}$ is the ML estimator and the Pr estimator for 6 experiments with different sample sizes.

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