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Sum Ideal Graphs Associated to a Given Ideal of a Commutative Ring

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Abstract
The aim of this paper is to introduce and study a new kind of graphs associated to an ideal of a commutative ring. Let $\mathcal{R}$ be a commutative ring with identity, and $I(\mathcal{R})$ be the set of all non-trivial ideals of $\mathcal{R}$ with $S\in I(\mathcal{R})$. The sum ideal graph associated to S, denoted by $\Psi(\mathcal{R}, S)$, is the undirected graph with vertex set $\{A\in I(\mathcal{R}): S\subset A+B$, for some $B\in I(\mathcal{R})\}$ where two ideal vertices A and B are adjacent if and only if $A\neq B$ and $S\subset A+B$. In this paper we establish some of characterizations and results of this kind of graph with providing some examples.

Keywords: Sum ideal graphs, Maximal ideals, Connected graphs.

بيانات جمع المثاليات المقارنة بمثالية معلومة في الحلقات الابدالية

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الخلاصة
الهدف في هذا البحث هو تعريف ودراسة نوع جديد من البيانات المرتبطة بمثالية معلومة للحلقة الابدالية. لتكن $\mathcal{R}$ حلقة ابدالية مع العنصر المحايد ، وان $I(\mathcal{R})$ مجموعة مثاليات غير تافهة للحلقة $\mathcal{R}$. يعرف بيان جمع المثاليات المقارنة بمثالية غير تافهة S على انه البيان الذي مجموعة رؤوسه هي $\{A\in I(\mathcal{R}): S\subset A+B,$ for some $B\in I(\mathcal{R})\}$، وان أي رأسين مثاليين مختلفين A وB متجاورين اذا وفقط اذا $S\subset A+B$. في هذاالبحث سوف نعطي بعض النتائج والمميزات لهذا النوع من البيان مع اعطاء بعض الأمثلة.

1. Introduction

A graph consists of two sets , vertex set and edge set , such that each edge assigned as unordered pair of two distinct vertices. Recently, some kinds of graphs were introduced and studied whose vertex set are elements or ideals of a given ring, and the binary operations of the ring makes the adjacency of the graph. The zero divisor graph was first introduced by Beck I. in [3]. The annihilating-ideal graph of a commutative ring $\mathcal{R}$ was introduced by Behboodi M. in [4]. This kind of graph has been studied, see [1, 2, 7, 8].

In this paper, we introduce and study the notion of sum ideal graph associated to an ideal of a commutative ring with identity in which the set of maximal ideal has a main role to obtain most of its results and characterizations.

Throughout this paper all rings will be finite and commutative with identity, and some basic definitions in [5, 6] will be used. Also we use $\mathcal{R}$, S, $M(\mathcal{R})$, $J(\mathcal{R})$, $V(\Psi)$ and $E(\Psi)$ to denote a

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commutative ring, a non-trivial ideal of $\mathcal{R}$, the set of maximal ideals, the Jacobson radical of $\mathcal{R}$, the vertex set and the edge set of $\Psi(\mathcal{R}, S)$ respectively.

2. The sum ideal graph associated to a given ideal of $\mathcal{R}$

In this section, we introduce the notion of sum ideal graph associated to a given ideal, we give some of its basic properties.

Definition2.1: Let $\mathcal{R}$ be a commutative ring with identity, and $I(\mathcal{R})$ be the set of all non-trivial ideals of $\mathcal{R}$ with $S \in I(\mathcal{R})$. The sum ideal graph associated to S, denoted by $\Psi(\mathcal{R}, S)$, is an undirected graph with vertex set $\{A \in I(\mathcal{R}): S \subset A+B$, for some $B \in I(\mathcal{R})\}$ where distinct ideal vertices A and B are adjacent if and only if S is a proper subset of A+B.

Example1: Let $\mathcal{R}=\mathbb{Z}_{48}$ and S= (6). The graph $\Psi(\mathbb{Z}_{48}, (6))$ is:

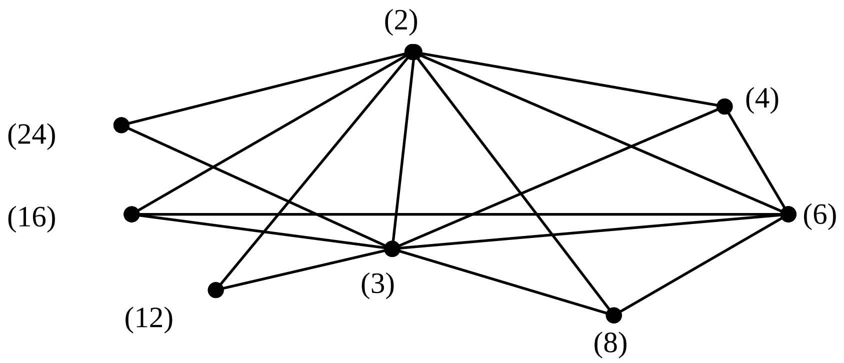


Figure 1- The graph $\Psi(\mathbb{Z}_{48}, (6))$

Before starting our main results, we give the following lemma.

Lemma 2.2: Let {I, J} be an edge in $\Psi(\mathcal{R}, S)$. If $K \in I(\mathcal{R})$ such that $I \subset K$, then K is adjacent to J in $\Psi(\mathcal{R}, S)$.

Proof: Suppose that {I, J} is an edge in $\Psi(\mathcal{R}, S)$. Then $S \subset I+J$. Since $I \subset K$, we have $S \subset K+J$. This means that, K is adjacent to J in $\Psi(\mathcal{R}, S)$.

We start this section with the following main result.

Proposition2.3:

1. If $E(\Psi) \neq \emptyset$, then $M(\mathcal{R}) \cap V(\Psi) \neq \emptyset$. Furthermore, for every I, $J \in M(\mathcal{R})$ with $I \neq J$, I and J are adjacent in $\Psi(\mathcal{R},S)$.
2. For every I, $J \in M(\mathcal{R})$ with $I \neq J$, $\Psi(\mathcal{R}, I)$ and $\Psi(\mathcal{R}, J)$ are identical.

Proof:

1. Assume that $E(\Psi) \neq \emptyset$. Let $I \in V(\Psi)$, then there exists a vertex J of $\Psi(\mathcal{R},S)$ such that $S \subset I+J$. If either $I \in M(\mathcal{R})$ or $J \in M(\mathcal{R})$, then the prove terminates. Now, assume that $I \notin M(\mathcal{R})$. Then there exists $M \in M(\mathcal{R})$ such that $I \subset M$. If J=M, then the prove completed. Suppose that $J \neq M$. Then by Lemma 2.2, M is adjacent to J in $\Psi(\mathcal{R}, S)$. Thus M is a maximal ideal vertex in $M(\mathcal{R})$. Assume that I, $J \in M(\mathcal{R})$. Then $S \subset I+J=\mathcal{R}$. Thus {I, J} is an edge in $\Psi(\mathcal{R}, S)$.
2. Let I, $J \in M(\mathcal{R})$ with $I \neq J$. If {A, B} is an edge in $\Psi(\mathcal{R}, I)$, then $I \subset A+B$. Then the maximally of I gives that $A+B=\mathcal{R}$. Obviously, $J \subset A+B$. Thus {A, B} is an edge in $\Psi(\mathcal{R}, J)$. Similarly, we can show that every edge of $\Psi(\mathcal{R}, J)$ is an edge of $\Psi(\mathcal{R}, I)$. Hence $\Psi(\mathcal{R}, I)$ and $\Psi(\mathcal{R}, J)$ are identical.

The next result shows that $\Psi(\mathcal{R}, S)$ is a null graph under certain conditions.

Proposition2.4: Let $S \in M(\mathcal{R})$. Then $\Psi(\mathcal{R}, S)$ is a null graph if and only if $\mathcal{R}$ is a local ring .

Proof: Suppose that $\mathcal{R}$ is a local ring. Then $M(\mathcal{R}) = \{S\}$. Since every non-trivial ideal contained in S, we have $S \not\subset I+J$, for every I, $J \in I(\mathcal{R})$. Thus $\Psi(\mathcal{R}, S)$ is a null graph.

Conversely, suppose that $\Psi(\mathcal{R}, S)$ is a null graph. Then by Proposition2.3, $\mathcal{R}$ has exactly one maximal ideal. This means that $\mathcal{R}$ is a local ring.

Remark2.5: Let $K \in V(\Psi)-\{S\}$. Then {K, S} is an edge in $\Psi(\mathcal{R},S)$ if and only if $S \subset S+K$, this means that $S+K \neq S$. Equivalently, $K \not\subset S$.

The next result shows the adjacency between S and all ideal vertices in $\Psi(\mathcal{R},S)$.

Proposition2.6:

1. If $E(\Psi) \neq \emptyset$, then the ideal vertex S is adjacent to all $I \in M(\mathcal{R})$.

2. If K∈V(Ψ) is adjacent to S, then K is adjacent to at least one maximal ideal in Ψ(ℛ,S).
3. If S is a minimal ideal vertex of ℛ, then S is adjacent to all ideal vertex K in Ψ(ℛ, S).

Proof:

1. Let I∈M(ℛ)-{S}. It is clear that S⊆S+I and I⊄S. Thus S⊂S+I. Hence S adjacent to I in Ψ(ℛ, S) for every ideal vertices I≠S.
2. Since K and S are adjacent, S⊂S+K. If S ∈M(ℛ), then the prove terminates. If K ∈M(ℛ), then by Proposition 2.3, K is adjacent to all elements of M(ℛ). Assume that S, K∉ M(ℛ). Then S contained properly in a maximal ideal say M. It follows from S⊂S+K that S⊂M+K. Thus K is adjacent to a maximal ideal M.

The proof of the third part follows from Remark 2.5.

Next, we turn to the following result.

Proposition2.7: Let I, J∈M(ℛ) with I≠J. If K is an ideal vertex of Ψ(ℛ, S) which is not contain in J, then K and J are adjacent ideal vertex in Ψ(ℛ, S).

Proof: By Proposition 2.3 S⊂I+J. Since K⊄J, we have J⊂K+J. Then the maximally of J gives K+J=ℛ. Thus S ⊂K+J. Hence {K, J} is an edge in Ψ(ℛ, S).

In the next result we demonstrate the partite of Ψ(ℛ, I).

Theorem2.8: If ℛ has exactly two maximal ideals I and J, then:

1. K⊈I+J, for all K∈V(Ψ)-{I, J}.
2. Ψ(ℛ, I) and Ψ(ℛ,J) are complete bipartite graphs.

Proof:

1. Since I, J ∈ M(ℛ), we have {I, J} is an edge of Ψ(ℛ,I). Let K be any non-maximal ideal vertex in Ψ(ℛ,I). Then there exists an ideal vertex L in Ψ(ℛ,I) such that I⊂K+L and L≠K. Now we have the following cases for L:

Case1: If L=I, then I⊂K+L follows that I⊂K+I. Since I∈ M(ℛ), I+K=ℛ≠I. Therefore K⊄ I. Thus K⊂J. Similarly, we can verify that K⊂ I, but K⊄ J when L=J.

Case2: If L≠I,J, then there exists I∈ M(ℛ) such that L⊂I. Since I⊂K+L, we have I⊂K+I. It follows that K+I=ℛ. Thus K⊄I and K⊂J. Similarly, if L contained properly in J we get K⊄J and K⊂I.

2. From Proposition 2.3, Ψ(ℛ,I) and Ψ(ℛ,J) are identical. It is enough to show that Ψ(ℛ,I) is a bi-partite graph. If K⊂I and K⊄J, for all K∈V(Ψ)-{I,J}, then by Proposition2.7, J is adjacent to every ideal vertex K in Ψ(ℛ,I) and we take V_1={I}∪{K∈V(Ψ); K⊄ J} , V_2={J}. Similarly, if K⊂J and K⊄I for all K∈V(Ψ)-{I,J}, we can choose V_1={I}, V_2={J}∪{K∈V(Ψ); K⊄ I}. In both cases, the graph is star. Assume that some of ideal vertices T, M∈V(Ψ)-{I, J} contained properly in I and J, respectively. Now, we can take V_1={I}∪{T∈V(Ψ); T⊄ J} and V_2={J}∪{M∈V(Ψ); M⊄ I}. Since T⊄ J and M⊄ I, I⊂T+M it means that {T, M} is an edge in Ψ(ℛ,I), in this case the graph is a complete bipartite graph. Similarly, we can prove that Ψ(ℛ,J) is a complete bipartite graph.

Example2: Consider the ring of integers modulo 30, $\mathbb{Z}_{36}$.

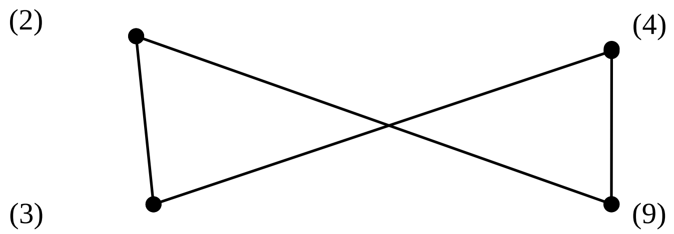


Figure 2-The graph Ψ($\mathbb{Z}_{36}$, (3))

Obviously, Ψ($\mathbb{Z}_{36}$, (3)) is a complete bipartite graph.

Corollary 2.9: If M(ℛ)={S, K}with S≠K, then the girth of Ψ(ℛ,S)) is either equal to 4 or ∞.

Proof: The prove follows from Proposition 2.3 and Theorem2.8.

Next, we shall give the converse of Theorem 2.8.

Proposition2.10: If Ψ(ℛ, S) is a bipartite graph, then |M(ℛ)|≤2

Proof: Let Ψ(𝓡, S) be a bipartite graph with partite sets V_1 and V_2. Since every two distinct maximal ideal vertex are adjacent, then each of V_1 and V_2 contains at most one maximal ideal vertex. Thus │M(𝓡)│≤2.

The next main result shows the adjacency between maximal and non-maximal ideals of 𝓡 in Ψ(𝓡, S).

Theorem2.11: Every non- maximal ideal vertex in Ψ(𝓡,S) is adjacent to at least one maximal ideal vertex.

Proof: Let K∉M(𝓡) be an ideal vertex in Ψ(𝓡, S). We have the following cases:

Case1: Let M(𝓡)={I}. If S=I, then by Proposition2.4, Ψ(𝓡, S) is null graph. Suppose that S≠I. Clearly, S⊂I. Thus S⊂I+S. Let K≠S be any ideal vertex of Ψ(𝓡, S). If S⊂K, then by Lemma 2.2, K is adjacent to I in Ψ(𝓡, S). Assume that S⊄K. Since S ⊆S+K and S contains properly in I, then I adjacent to K in Ψ(𝓡, S).

Case2: Let │M(𝓡)│≥ 2. From Proposition2.6, S is adjacent to all maximal ideal vertices. If S ∉M(𝓡), there exists H∈M(𝓡) such that S⊂H. It follows from S ⊆S+K that S⊂H+K. Thus K is adjacent to H. Now assume that S∈M(𝓡). Then we have two subcases for K and S:

Subcase1: Let K⊄S. Then S≠ S+ K. Thus S⊂K+S. Hence K is adjacent to S.

Subcase2: Let K⊂S. Since S ⊆S+K and S is a maximal ideal then S+K=S. Thus we have S⊄ S+K. That means, S and K are not adjacent ideal vertices in the graph Ψ(𝓡, S). Since K is an ideal vertex in Ψ(𝓡, S), then there exists L∈ V(Ψ) such that S⊂ K+L. Now, if L∈M(𝓡), then the proof is completed. Otherwise, there exists W∈M(𝓡)contains properly L. It follows that S⊂K+L⊂K+W. Thus K is adjacent to W in Ψ(𝓡, S).

Example3: The following graph shows that every non-maximal ideal vertex is adjacent to a maximal ideal vertex.

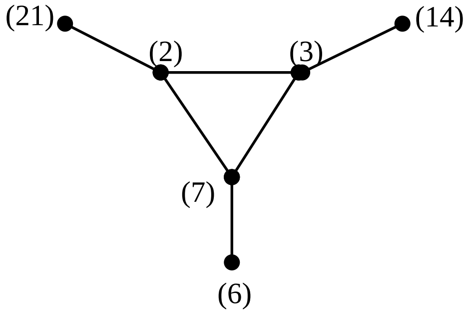


Figure 3-The graph Ψ($\mathbb{Z}_{42}$, (7))

The next result shows that Ψ(𝓡, S) contains a star with the same vertex set of Ψ(𝓡, S).

Proposition2.12: If S∈V(Ψ)- M(𝓡), then M(𝓡) contains an element that adjacent to all ideal vertices of Ψ(𝓡, S). Moreover, Ψ(𝓡, S) and I(𝓡) has the same cardinality.

Proof: Since S∉M(𝓡), there exists M∈M(𝓡) such that S⊂M Thus S⊂I+M, for any I∈I(𝓡)-{S,M}. This means that, M is adjacent to all ideal vertices of Ψ(𝓡, S). Consequently, the order of the graph Ψ(𝓡, S) is equal to the number of all non-trivial ideals of 𝓡.

In the next result we give the necessary and sufficient condition for an ideal of 𝓡 to be ideal vertex of Ψ(𝓡, S).

Theorem2.13: Let S∈ M(𝓡). Then K∈V(Ψ) if and only if K⊈J(𝓡).

Proof: Let K∈V(Ψ) assume that K⊆J(𝓡). Then K+M=M, for every M∈ M(𝓡). Therefore, S⊄K+M for every M∈M(𝓡). This means that K is not adjacent to every I∈ M(𝓡). This contradicts Theorem2.11. Therefore, K⊈J(𝓡).

Conversely, assume that K⊈J(𝓡), then there exists I∈ M(𝓡) such that K⊄I. Since I⊂K+I, then S⊂𝓡=K+I. This means that K∈V(Ψ).

Example 4: Consider the ring $\mathbb{Z}_{54}$.

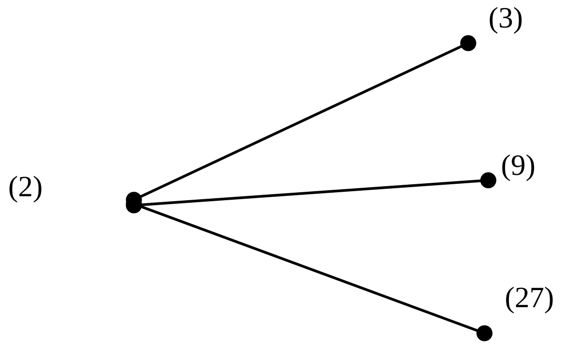


Figure 4-The graph $\Psi(\mathbb{Z}_{54}, (3))$

Clearly, the ideals (6) and (18) are not vertices of $\Psi(\mathbb{Z}_{54}, (3))$, since (6), (18) ⊆J($\mathcal{R}$)=(6)
In the next result we find the upper bound of the girth of Ψ($\mathcal{R}$,S).

Theorem2.14: If Ψ($\mathcal{R}$, S) contains a cycle, then the girth of Ψ($\mathcal{R}$,S)) is less than or equal to four.

Proof: If $\mathcal{R}$ is a local ring and Ψ($\mathcal{R}$,S) is a null graph, then S∉M($\mathcal{R}$) by Proposition2.4. Let I∈M($\mathcal{R}$) and K∈V(Ψ) such that K⊄S. Then by Remark 2.5, S is adjacent to K in Ψ($\mathcal{R}$, S). Furthermore, we have I adjacent to both S and K. Thus C_3:I, S, K, I is a cycle in Ψ($\mathcal{R}$, S). If │M($\mathcal{R}$)│>2, then by Proposition2.3 we can easily find a cycle of length three. Suppose that M($\mathcal{R}$)={I, J} with I≠ J. Obviously, I and J are adjacent ideal vertices because I+J= $\mathcal{R}$. If either S=I or S=J, then by Corollary2.9, the girth of Ψ($\mathcal{R}$,S)) is equal to 4. Assume that neither S=I nor S=J. This yields that S∉M($\mathcal{R}$). From Proposition2.6, S is adjacent to both I and J. Thus C_3: I, S, J, I is a cycle in Ψ($\mathcal{R}$, S).

In the following result, we find the value of girth of Ψ($\mathcal{R}$,S).

Proposition2.15: If Ψ($\mathcal{R}$, S) contains an edge {I,J}such that I, J∉M($\mathcal{R}$) and neither I⊆J nor I⊆J. Then the girth of Ψ($\mathcal{R}$, S) is equal to three.

Proof: Suppose that {I,J}is an edge in Ψ($\mathcal{R}$, S) such that I, J∉M($\mathcal{R}$) and neither I⊆J nor I⊆J. Then we have I+J≠I and I+J≠J. Thus I, J⊂I+J. Since I, J∉M($\mathcal{R}$), then I+J≠$\mathcal{R}$. By Lemma 2.2, I+J is adjacent to both I and J. Thus C_3: I, (I+J), J, I is a cycle in Ψ($\mathcal{R}$, S). Hence the girth of Ψ($\mathcal{R}$, S) is equal to three.

3. Connectivity of Ψ($\mathcal{R}$, S)

In this section we investigate the connectivity of Ψ($\mathcal{R}$,S) and some basic concepts related to connectivity.

We start this section with the following main result.

Theorem 3.1: The graph Ψ($\mathcal{R}$, S) is connected with diam(Ψ($\mathcal{R}$, S))≤3.

Proof: Let I, J∈V(Ψ) with I≠J. If I+J=$\mathcal{R}$, then by Proposition2.3 I is adjacent to J in Ψ($\mathcal{R}$, S). Assume that I+J≠$\mathcal{R}$. We have the following cases for I and J:

Case1: If I∈M($\mathcal{R}$) and J∉M($\mathcal{R}$), then by Theorem2.11, there exists M∈M($\mathcal{R}$) adjacent to J in Ψ($\mathcal{R}$, S). If M=I, then P_1: I, J is a path in Ψ($\mathcal{R}$, S). Suppose that M≠I. From Proposition2.3, M is also adjacent to I. Thus P_2: J, M, I is a path in Ψ($\mathcal{R}$, S). Similarly, we can find a path between I and J of length at most two, when J∈M($\mathcal{R}$) and I∉M($\mathcal{R}$).

Case 2: If I, J∉M($\mathcal{R}$), then by Theorem2.11, there exist H, L ∈M($\mathcal{R}$) such that I and J are adjacent to H and L respectively. If H=L, then we have a path P_2: I, H, J in Ψ($\mathcal{R}$, S). Suppose that H≠L. By Proposition2.3, H and L are adjacent ideal vertices in Ψ($\mathcal{R}$, S). Thus P_3:I, H, L, J is a path in Ψ($\mathcal{R}$, S). From each case, we have shown that the graph Ψ($\mathcal{R}$, S) is connected and diam(Ψ($\mathcal{R}$,S))≤3.

In the next result we show that the central vertex set of Ψ($\mathcal{R}$,S) contains a maximal ideal of $\mathcal{R}$.

Theorem3.2: There exists at least one maximal ideal of $\mathcal{R}$ which is a central vertex of Ψ($\mathcal{R}$, S).

Proof: If $\mathcal{R}$ is a local ring and Ψ($\mathcal{R}$, S)≠∅, then by Theorem2.12, Ψ($\mathcal{R}$, S) contains a maximal ideal which is a central vertex of Ψ($\mathcal{R}$, S). Now, suppose that $\mathcal{R}$ is not a local ring and S∉M($\mathcal{R}$). Again by Theorem2.12, there exists I∈ M($\mathcal{R}$) such that I is adjacent to all ideal vertex of Ψ($\mathcal{R}$, S). Thus rad(Ψ($\mathcal{R}$, S))= e(I)=1. Thus I is a central vertex of Ψ($\mathcal{R}$, S). Suppose that S ∈ M($\mathcal{R}$). From Proposition2.4, │M($\mathcal{R}$)│>1. We have the following cases:

Case1: If M($\mathcal{R}$)={S, I}with S≠I, then by Theorem2.8, the graph Ψ($\mathcal{R}$, S) is a bi-partite graph with partite sets V_1 and V_2. If Ψ($\mathcal{R}$, S) is a star, then either S or I is a central vertex. Assume that Ψ($\mathcal{R}$, S) is not a star. Then by the same theorem the partition V_1 and V_2 are $V_1=\{I\}\cup\{K\in V(\Psi); K\not\subset S\}$and $V_2=\{S\}\cup\{K\in V(\Psi); K\not\subset I\}$. Thus rad(Ψ(𝓡,S))=2=e(I). This means that I is a central of Ψ(𝓡, S).

Case2: Suppose that |M(𝓡)|>2. If V(Ψ)=M(𝓡), then the prove is done. Assume that Ψ(𝓡, S) has a non-maximal ideal vertex K. By Theorem 2.13, K⊈J(𝓡), then there exists I∈M(𝓡) which does not contain K. Hence S⊂K+I=𝓡. Clearly, if there exists M∈M(𝓡) such that K⊂ M, then S⊄K+M=M. It follows that Ψ(𝓡,S) is not complete. From Theorem 2.11 and Proposition 2.3, rad(Ψ(𝓡,S))=e(P), for some P∈M(𝓡) adjacent to K in Ψ(𝓡,S).

Example5: In the following graph, e((2))= rad(Ψ($\mathbb{Z}_{56}$, (4)) and the maximal ideal (2) is a central vertex of Ψ($\mathbb{Z}_{56}$, (4)).

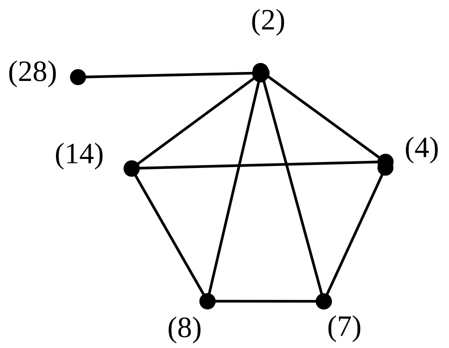


Figure 5- The graph Ψ($\mathbb{Z}_{56}$, (4))

In the next result we demonstrate that M(𝓡) includes all cut vertices of Ψ(𝓡, S).

Theorem3.3: If I is a cut vertex of Ψ(𝓡, S), then I is a maximal ideal of 𝓡.

Proof: Suppose that I is a cut vertex of Ψ(𝓡, S). Then the graph Ψ(𝓡, S)-I is disconnected. Assume that I∉M(𝓡). Let V_1 and V_2 be any two components of Ψ(𝓡, S)-I with N∈ V_1 and M∈ V_2. If M, N∉M(𝓡), then by Theorem2.11 there exist K, L∈ M(𝓡) such that {M, K} and {N, L} are edges in V_1 and V_2 respectively. By Proposition 2.3, K and L are adjacent in Ψ(𝓡, S). Suppose that M∈M(𝓡) and N∉M(𝓡). Then there exist H∈M(𝓡) ∩V_2 such that N is adjacent to H in V_2. If M, N∈M(𝓡) we get the same result. In each case we conclude that there exists two adjacent vertices in different component. This is impossible. Therefore I∈M(𝓡).

4. Completeness of Ψ(𝓡, S)

In this section we explain the minimally of S and the completeness of Ψ(𝓡, S).

We start this section with the following results.

Theorem 4.1: If the graph Ψ(𝓡, S) is complete, then S is a minimal ideal of 𝓡.

Proof: Suppose that Ψ(𝓡, S) is complete graph and S is not a minimal ideal. Then there is a non-trivial ideal K of 𝓡 such that K⊂S. It follows that S⊄S+K. This means that S and K are not adjacent ideal vertices. This contradicts that Ψ(𝓡, S) is a complete. Hence S is a minimal ideal of 𝓡.

The converse of Theorem4.1 may not be true, as the following example shows.

Example6: Consider the ring of integers modulo 54.

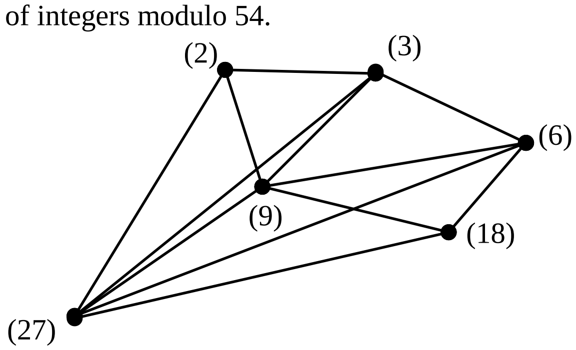


Figure 6- The graph Ψ($\mathbb{Z}_{54}$, (27))

Clearly, S=(27) is a minimal ideal, but Ψ($\mathbb{Z}_{54}$,(27)) is not a complete graph.

The converse of Theorem4.1 will be true, if we determine the number of ideals of 𝓡.

Proposition4.2: Suppose that $\mathcal{R}$ has four-non-trivial ideals I, J, K and S with M($\mathcal{R}$)={I, J} and S=I.J≠(0). Then the graph Ψ($\mathcal{R}$, S) is complete if and only if S is a minimal ideal of $\mathcal{R}$.
Proof: It is obvious from Theorem4.1 that S is a minimal ideal of $\mathcal{R}$ when Ψ($\mathcal{R}$, S) is complete.
Conversely, let S be a minimal ideal of $\mathcal{R}$. Since I and J are maximal ideals, then S is adjacent to I, J and K by Theorem2.6. Since K∉M($\mathcal{R}$), K contained in at least one maximal ideal of $\mathcal{R}$, let be I. Then S⊂I+K and S⊂J+K=$\mathcal{R}$. This means that, both I and J are adjacent to K. Hence any two distinct ideal vertices of I, J, K and S are adjacent in Ψ($\mathcal{R}$, S).
Example7: The graph Ψ($\mathbb{Z}_{18}$,(6)) is a complete graph.

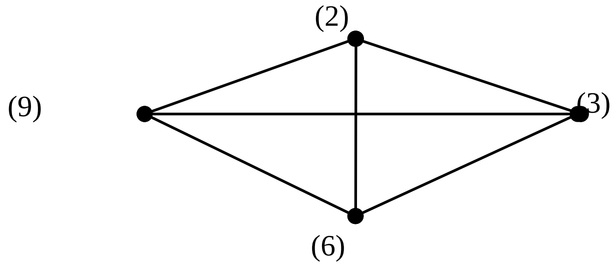
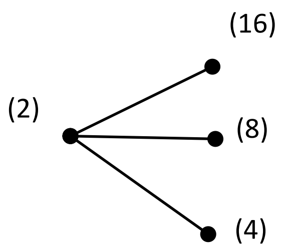


Figure 7- The graph Ψ($\mathbb{Z}_{18}$, (6))

The next result investigate the completeness of Ψ($\mathcal{R}$, S).
Theorem 4.3: If I($\mathcal{R}$) consists of the chain S= I_1⊂ I_2⊂...⊂I_n =I, then Ψ($\mathcal{R}$, S) is complete graph.
Proof: Obviously, I is adjacent to all J∈I($\mathcal{R}$) in Ψ($\mathcal{R}$, S).
Since S= I_1⊂ I_2⊂...⊂I_n=I, we have S⊂I_i+I_j, for every i, j=1,2,...,n with i≠j. Thus S is adjacent to I_i for i=2,...,n and every two distinct ideals of $\mathcal{R}$ are adjacent in Ψ($\mathcal{R}$, S). Hence Ψ($\mathcal{R}$, S) is a complete graph.
In the next result we find the chromatic number of Ψ($\mathcal{R}$, S).
Theorem 4.4: If the ideals of $\mathcal{R}$ consists of the chain I_1⊂ I_2⊂... I_{n-1} ⊂I_n with n≥ 3, then the chromatic number of Ψ($\mathcal{R}$, I_m) is χ(Ψ($\mathcal{R}$, I_m)) = n − (m − 1), for every m=1, 2, …, n-1.
Proof: If m=1, then the graph Ψ($\mathcal{R}$, I_m) is complete and the formula is satisfied. Let m=2. Then I_2⊂ I_i+I_j, for all i, j=2,...,n with i≠j. Thus there is a complete subgraph K_{n-1} of Ψ($\mathcal{R}$, I_2) whose vertices are I_2, I_3, …, I_n. So, we have n-1 different colours of K_{n-1}. On the other hand I_1⊂ I_2 and I_2⊄ I_1+ I_2. This means that I_2 is not adjacent to I_1. Thus I_1 and I_2 have the same colour. Hence χ(Ψ($\mathcal{R}$, I_2))=n-1=n-(m-1), when m=2. In general, if 1≤m≤n-1, then Ψ($\mathcal{R}$, I_m) contains a complete subgraph whose vertices are I_m, I_{m+1},…, I_{n-1}. Since I_1⊂ I_2⊂…⊂ I_m , every two of vertices I_1, I_2,…, I_m are not adjacent. So, the vertices I_m, I_{m+1},…, I_{n-1} have n-m different colours but I_1, I_2,…, I_m have the same colour. Thus χ(Ψ($\mathcal{R}$, I_m)) =n-m+1=n − (m − 1).
Example8: Consider the following graphs:



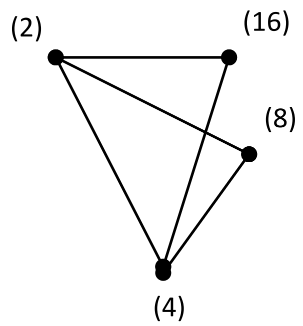


Figure 8- The graph Ψ($\mathbb{Z}_{32}$, (4)) **Figure 9-** The graph Ψ($\mathbb{Z}_{32}$, (8))

It is clear from Figure- that $\Psi(\mathbb{Z}_{32},(4))$ is $K_{1,3}$, so we can choose two distinct colours for the sets {(2)} and {(16), (8), (4)} respectively. Hence $\chi(\Psi(\mathbb{Z}_{32},(4)))=2$.

From Figure-9, we can choose three distinct colours for the sets {(2)} and {(16), (8)} and {(4)} respectively. thus $\chi(\Psi(\mathbb{Z}_{32},(4)))=3$

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