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An Extension of Some Truncated Distributions Family With Application

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Abstract:
This paper suggests a combination of continuous distributions based on Truncated Gompertz-G (TGO -gd[0,1]) and Truncated Gompertz-Weibull (TGO-wd) distributions. Some of the main properties of the suggested combination such as the distribution function (cdf), the probability function (pdf), survival function (Sf), hazard function (Hrf), quintile function (Qf), moment generating function (mgf), skewness S_K,, kurtosis K_u , entropy, and maximum likelihood estimator (MLE) for the proposed combination of TGO-wd[0,1] have been presented. An experimental estimation is also studied to determine the efficiency of the proposed distributions. The MLE method is used to estimate the parameters of the distributions which represents an important part of obtaining standard information that is used to know the efficiency of the proposed distribution after application. According to the proposed distribution through one real data set, the capabilities of the proposed family show greater flexibility compared to other models.

Keywords*:* Gompertz distribution, Weibull distribution, Entropy, Moments, Truncated distribution.

امتداد لبعض عائلة التوزيعات المبتورة مع التطبيق

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الخلاصة :
تمثل هذه الورقة مجموعة مقترحة من التوزيعات المستمرة بناءً على توزيعات Gompertz-G المبتورة ([tgo-gd[0,1) وتوزيعات (tgo-wd (Gompertz-Weibull. بعض الخصائص الرئيسية للتركيبة المقترحة مثل دالة التوزيع (cdf)، دالة الاحتمالية (pdf)، دالة البقاء (Sf)، دالة الخطر (Hrf)، دالة التجزيئية (Qf)، دالة المولدة للعزوم (mgf)، تم تقديم الانحراف S_K، والتفلطح K_u ، والإنتروبيا، و دالة الامكان الاعظم (MLE) للمجموعة المقترحة من [TGO-wd[0,1.

كما تمت دراسة تقدير تجريبي لمعرفة كفاءة التوزيع المقترح تم استخدام طريقة MLE لتقدير معلمات التوزيعات حيث تمثل جزء مهم للحصول على معايير معلوماتية حيث تستخدم لمعرفة كفاءة التوزيع المقترح بعد التطبيق، أظهرت إمكانيات الأسرة المقترحة وفق التوزيع المقترح عن طريق مجموعة بيانات حقيقية واحدة مرونة أكبر مقارنة بالنماذج الاخرى.

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1. Introduction

several researchers established new families using Gompertz distribution. Gompertz distribution (gd) has been studied by Alizadeh et al. in 2017 to suggest a family that has more flexibility in scanning and observing variant data. Also, the Gompertz family has been used by Oguntunde et al. in 2019 to create a new distribution called Gompertz Fréchet which is applied to engineering data. As well, Eghwerido et al. in 2020 introduced the Gompertz family to generate a distribution named Gompertz-Alpha Power Inverted Exponential .

Then after, Abid et al. in 2017 used the [0,1] Truncated Fréchet-Gamma and Inverted Gamma (TFIG) distributions to introduce [0,1] Truncated family distributions . They discussed some statistical properties of the proposed family. Here, a proposal of [0,1] Truncated Gompertz –Weibull distribution called TGO-wd[0,1] is derived then we study some of its properties. Moreover, the outcomes reveal the competitive flexible for the proposed distribution compared to other distributions that applied to real data [1] [2] [3] [4].

2. [0,1] Truncated Gompertz –G Family Distribution[5]:

O.D. Atanda et.al. in 2020 introduced the cdf and pdf for the Gompertz distribution as follows [5] :

$$F(x,\beta,\theta) = 1 - e^{-\frac{\beta}{\theta}(e^{\theta x}-1)} , \tag{1}$$

$$f(x,\beta,\theta) = \beta e^{\theta x} e^{-\frac{\beta}{\theta}(e^{\theta x}-1)} , \tag{2}$$

where $x \geq 0$, $\beta > 0$, and $\theta > 0$, β is the scale parameter and θ is the shape parameter.

The truncated distribution has a pdf which is given by the following formula [6]:

$$f(x \mid a \leq x \leq b) = \frac{g(x)}{G(b)-G(a)} \quad \text{such that} \quad (x \geq 0), G(b)-G(a) \neq 0. \tag{*}$$

Then the cdf and pdf for tgd[0,1] are given respectively:

$$F_T(x,\beta,\theta) = \frac{1-e^{-\frac{\beta}{\theta}\left(e^{\theta x}-1\right)}}{1-e^{-\frac{\beta}{\theta}\left(e^{\theta}-1\right)}}, \tag{3}$$

$$f_T(x;\beta,\theta) = \frac{\beta e^{\theta x} e^{-\frac{\beta}{\theta}\left(e^{\theta x}-1\right)}}{1-e^{-\frac{\beta}{\theta}\left(e^{\theta}-1\right)}} \quad \text{where} \quad x \geq 0, \text{and } \beta,\theta > 0 . \tag{4}$$

So the cdf for the proposed distribution can be obtained by the compensation of Equation (3) with G(x).

Let $x = t \Rightarrow dx = dt$, then we have:

$$F(x;\beta,\theta) = \int_0^{G(x)} f(t;\beta,\theta)dt \quad \text{such that} \quad t \geq 0 \text{ and } \beta,\theta > 0,$$

$$F_T(x;\beta,\theta) = \frac{1-e^{-\frac{\beta}{\theta}(e^{\theta G(x)}-1)}}{1-e^{-\frac{\beta}{\theta}(e^{\theta}-1)}}, \tag{5}$$

where $x \geq 0$, and $\beta,\theta,a,b > 0$. By deriving Equation (5), the pdf for TGO-wd[0,1] can be found as follows:

$$f_T(x,\beta,\theta) = \frac{\beta e^{\theta G(x)}\, g(x)\, e^{-\frac{\beta}{\theta}\left(e^{\theta G(x)}-1\right)}}{1-e^{-\frac{\beta}{\theta}\left(e^{\theta}-1\right)}} \quad \text{such that } x \geq 0, \text{and } \beta,\theta > 0 , \tag{6}$$

3- The TG-wd[0,1] family:

Assuming that $G(x)$ is the Weibull distribution (WD). Its pdf and cdf are respectively given by:

$$g(x;a,b) = abx^{b-1}e^{-ax^b} \quad \text{such that } x \geq 0,\ a,\ b > 0 \tag{7}$$

$$G(x;a,b) = 1 - e^{-ax^b}. \tag{8}$$

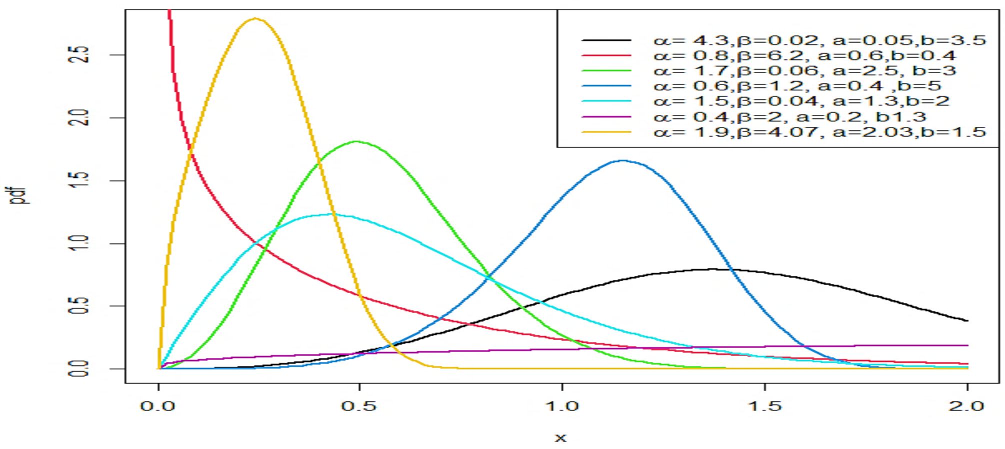
Now by substituting Equation (8) in Equation(5), the cdf can be written as follows:

$$F_T(x;\beta,\theta,a,b) = \frac{1-e^{-\frac{\beta}{\theta}\left(e^{\theta\left(1-e^{-ax^b}\right)}-1\right)}}{1-e^{-\frac{\beta}{\theta}\left(e^{\theta}-1\right)}}, \qquad x \geq 0, \beta, \theta, a, b > 0 \tag{9}$$

and by substituting Equations (7-8) in Equation (6), the pdf for the proposed distribution TGO-wd [0,1] can be written as follows:

$$f_T(x;\beta,\theta,a,b) = \frac{\beta e^{\theta\left(1-e^{-ax^b}\right)}\, abx^{b-1}e^{-ax^b}\, e^{-\frac{\beta}{\theta}\left(e^{\theta\left(1-e^{-ax^b}\right)}-1\right)}}{1-e^{-\frac{\beta}{\theta}\left(e^{\theta}-1\right)}}, \ x \geq 0, \beta, \theta, a, b > 0$$

Figure 1a reveals the plots of the pdf and Figure 1b is the cdf for TGO-wd[0,1] with different parameters values on software R.



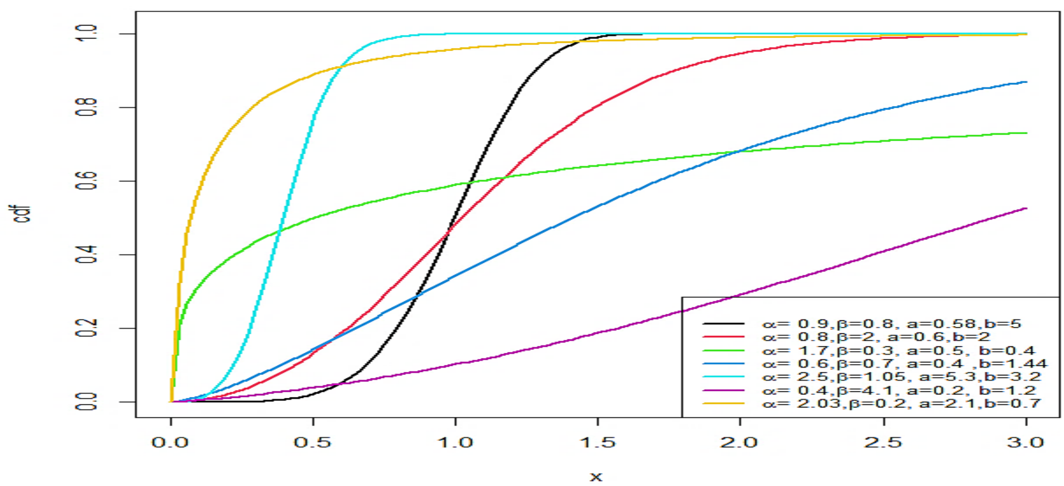


Figure 1 a: The plot of the pdf or TGO-wd[0,1] using variant values for the parameter.
b: The plot of the cdf for TGO-wd[0,1] using different values for the parameter.

4 . Expansion function: Consider the following pdf

$$f(x,\beta,\theta,a,b) = \frac{\beta e^{\theta G(x;a,b)}\, g(x;a,b)\, e^{-\frac{\beta}{\theta}\left(e^{\theta G(x,a,b)}-1\right)}}{1-e^{-\frac{\beta}{\theta}\left(e^{\beta}-1\right)}} . \tag{10}$$

By the expansion of $e^{-\frac{\beta}{\theta}\left(e^{\theta G(x,a,b)}-1\right)}$, the following can be obtained:

$$e^{-\frac{\beta}{\theta}(e^{\theta G(x,a,b)}-1)} = \sum_{k=0}^{\infty} \frac{(-1)^k (\beta/\theta)^k}{k!} (e^{\theta G(x;a,b)} - 1)^k \quad . \tag{11}$$

And using binomial exponential for $(e^{\theta G(x\,;a\,,b)} - 1\,)^k$, we get:

$$(e^{\theta G(x;a,b)} - 1)^{\,k} = \sum_{i=0}^{\infty} (-1)^i \binom{k}{i} e^{\theta G(x;a,b) i} \tag{12}$$

By substituting Equation (12) in Equation (11), we get:

$$e^{-\frac{\beta}{\theta}(e^{\theta G\,(x,a,b)}-1)} = \sum_{k,i=0}^{\infty} \frac{(-1)^{\,k+i} (\beta/\theta)^{\,k}}{k!} \binom{k}{i} e^{\theta G(x;a,b) i} \,. \tag{13}$$

In addition, substituting Equation (13) in Equation (10) gives:

$$= \sum_{k=i=0}^{\infty} \frac{(-1)^{k+i} \binom{k}{i} \beta^{k+1} e^{\theta G(\,x\,;a,b)}\; g(x;a,b)\;\; e^{\theta G(x;a,b) i}}{k!\theta^k \left(1-e^{-\frac{\beta}{\theta}\left(e^{\beta}-1\right)}\right)} \,. \tag{14}$$

Now we put Equations (7-8) in Equation (14) to get:

$$= \sum_{k=i=0}^{\infty} \frac{(-1)^{k+i} \binom{k}{i} \beta^{k+1} e^{\theta\left(1-e^{-ax^b}\right)}\; abx^{b-1} e^{-ax^b}\;\; e^{\theta\left(1-e^{-ax^b}\right) i}}{k!\theta^k \left(1-e^{-\frac{\beta}{\theta}\left(e^{\beta}-1\right)}\right)} \,.$$

$$= \sum_{k=i=0}^{\infty} \frac{(-1)^{k+i} \binom{k}{i} \beta^{k+1} e^{\theta(1+i)\left(1-e^{-ax^b}\right)} abx^{b-1} e^{-ax^b}}{k!\theta^k \left(1-e^{-\frac{\beta}{\theta}\left(e^{\beta}-1\right)}\right)} \quad . \tag{15}$$

By the formula of exponential $e^{\theta(1+i)\left(1-e^{-ax^b}\right)}$, it will give:

$$e^{-\theta(1+i)\left(1-e^{-ax^b}\right)} = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} (\theta(1+i))^j \left(1-e^{-ax^b}\right)^j$$

$$\left(1-e^{-ax^b}\right)^j = \sum_{l=0}^{\infty} (-1)^l \binom{j}{l} e^{-alx^b}.$$

$$e^{-\theta(1+i)\left(1-e^{-ax^b}\right)} = \sum_{j=l=0}^{\infty} \frac{(-1)^{j+l} \binom{j}{l} (\theta(1+i))^j e^{-alx^b}}{j!} \,. \tag{16}$$

We put Equation (16) in Equation (15) to get:

$$= \sum_{k=i=j=l=0}^{\infty} \frac{(-1)^{k+i+j+l} \binom{j}{l}\binom{k}{i} \beta^{k+1} (\theta(1+i))^j e^{-alx^b} abx^{b-1} e^{-ax^b}}{k!\,\theta^k j! \left(1-e^{-\frac{\beta}{\theta}\left(e^{\beta}-1\right)}\right)}$$

$$= \sum_{k=i=j=l=0}^{\infty} \frac{(-1)^{k+i+j+l} \binom{j}{l}\binom{k}{i} \beta^{k+1} (\theta(1+i))^j e^{-ax^b(l+1)} abx^{b-1}}{k!\theta^k j! \left(1-e^{-\frac{\beta}{\theta}\left(e^{\beta}-1\right)}\right)} \,. \tag{17}$$

Where $V_{k,i,j} = \sum_{k=i=j=l=0}^{\infty} \frac{(-1)^{k+i+j+l} \binom{j}{l}\binom{k}{i} \beta^{k+1} (\theta(1+i))^j ab}{k!\theta^k j! \left(1-e^{-\frac{\beta}{\theta}\left(e^{\beta}-1\right)}\right)}$.

Then the pdf of TGO-wd[0,1] is given as follows:

$$f_T(x;\beta,\theta,a,b) = \sum_{k,i,j=0}^{\infty} V_{k,i,j}\, x^{b-1} e^{-a(1+j)x^b} (x \geq 0, \beta, \theta, a, b > 0). \tag{18}$$

5. Some properties of TGO-wd) [0,1]:

Some statistical properties of the proposed TGO-wd[0,1] will be discussed in this part.

5-1. Survival function (Sf)[2]:

The formula of the Sf can be written as follows:

$$S(x,\beta,\theta,a,b) = 1 - F(x,\beta,\theta,a,b). \tag{19}$$

We put Equation (9) in Equation (19) the result gives:

$$S(x,\beta,\theta,a,b) = \frac{e^{-\frac{\beta}{\theta}\left(e^{\theta\left(1-e^{-ax^b}\right)}-1\right)} - e^{-\frac{\beta}{\theta}\left(e^{\theta}-1\right)}}{1-e^{-\frac{\beta}{\theta}\left(e^{\theta}-1\right)}}, \qquad x \geq 0, \beta, \theta, a, b > 0. \tag{20}$$

5-2. The hazard rate function (Hrf) [2]:

We can calculate that the Hrf for TGO-wd[0,1] is given as follows:

$$h(x,\beta,\theta,a,b) = \frac{f_T(x;\beta,\theta,a,b)}{S(x;\beta,\theta,a,b)} \quad . \tag{21}$$

By substituting Equations (10) and (20) in Equation (21), we obtain:

$$h(x,\beta,\theta,a,b) = \frac{\beta e^{\theta(1-e^{-ax^b})}abx^{b-1}e^{-\frac{\beta}{\theta}\left(e^{\theta(1-e^{-ax^b})}-1\right)}}{e^{-\frac{\beta}{\theta}\left(e^{\frac{3}{3}(1-e^{-ax^b})}-1\right)}-e^{-\frac{\beta}{\theta}(e^{\theta}-1)}} .$$

Figure 2 shows the estimated Hrf on R. The function is decreasing and increasing.

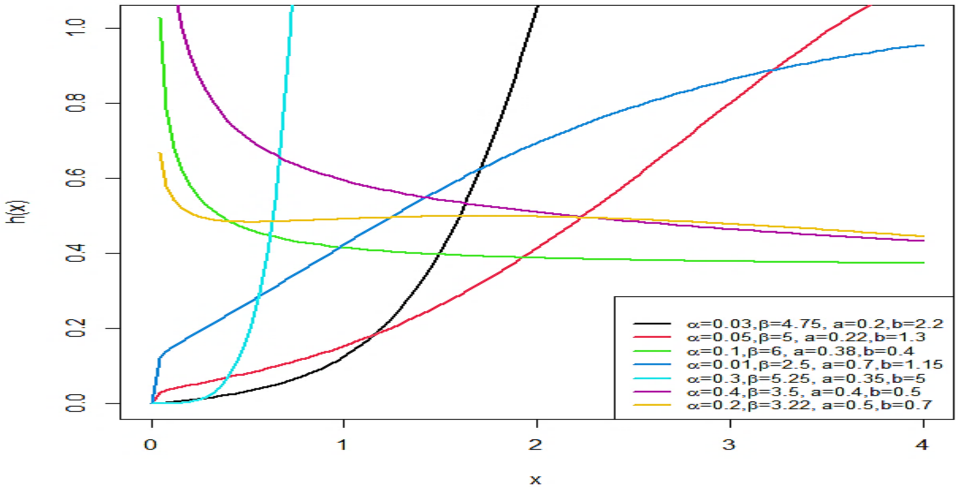


Figure 2:u The plot of the Hrf for TGO-wd[0,1].

5-3. The quintile function [2]:

The quintile function QF is given by the following formula:

$X_q = Q(U) = F^{-1}[F(X_q, \emptyset)]$ such that $\emptyset = (\beta,\theta,a,b)$ then

$$Xq = \sqrt[b]{-\frac{1}{a}\ln\left\{1+\frac{1}{\theta}\ln\left[-\frac{\theta}{\beta}[\ln\left(1-U(1-e^{-\frac{\beta}{\theta}(e^{\theta}-1)})\right)]\right]\right\}} \quad . \tag{22}$$

The median can be calculated by quintile function. Here we will assume that U=0.5 in Equation (18) so the result will be as follows:

$$X_{0.5} = \sqrt[b]{-\frac{1}{a}\ln\left\{1-\frac{1}{\theta}\ln\left[-\frac{\theta}{\beta}[\ln(1-0.5A)+1]\right]\right\}} \quad .$$

5-4. The moment (μ_r) [2]:

The r^{th} moment ($\acute{\mu}_r$) of the TGO-wd[0,1] is calculated by the following formula:

$$\acute{\mu}_r = \int_0^{\infty} x^r f_T(x;\beta,\theta,a,b)dx \tag{23}$$

Now putting Equation (18) in Equation (23) gives:

$$\acute{\mu}_r = \int_0^{\infty} x^r \sum_{k,i,j=0}^{\infty} V_{k,i,j}\, x^{b-1}e^{-a(1+j)x^b}dx , \tag{24}$$

where $V_{k,i,j} = \frac{\beta^{k+1}(-1)^{k+i+j}\binom{k}{i}e^{\theta(1+i)}(\theta(1+i))^j ab}{\theta^k k!\, j!\left(1-e^{-\frac{\beta}{\theta}(e^{\theta}-1)}\right)}$.

Let $y = a(1+j)x^b, x = \frac{y^{1/b}}{(a(1+j))^{1/b}}$, $then\ dx = \frac{y^{\frac{1}{b}-1}}{b(a(1+j))^{\frac{1}{b}}}dy$

Substituting $a(1+j)x^b,\ x\ and\ dx$ in Equation (24) then we obtain:

$\acute{\mu}_r = \sum_{k,i,j=0}^{\infty} V_{k,i,j} \frac{1}{b(a(1+j))^{\frac{r+b}{b}}} \int_0^{\infty} y^{\frac{r+b}{b}-1} e^{-y} dy.$

By Gamma integral $\Gamma(b) = \int_0^{\infty} y^{b-1} e^{-y} dy$ the statement will be as follows:

$$\acute{\mu}_r = E(X^r) = \sum_{k,i,j=0}^{\infty} V_{k,i,j} \frac{1}{b(a(1+j))^{\frac{r+b}{b}}} \Gamma(\frac{r+b}{b}) . \tag{25}$$

Now using Equation (25) when $r = 1, 2, 3$ and 4, then we can calculate that the moments of $E(X), E(x^2), E(X^3)$ and $E(X^4)$ are given as follows:

$E(X) = \sum_{k,i,j=0}^{\infty} V_{k,i,j} \frac{1}{b(a(1+j))^{\frac{1+b}{b}}} \Gamma(\frac{1+b}{b})$,

$E(X^2) = \sum_{k,i,j=0}^{\infty} V_{k,i,j} \frac{1}{b(a(1+j))^{\frac{2+b}{b}}} \Gamma(\frac{2+b}{b})$,

$E(X^3) = \sum_{k,i,j=0}^{\infty} V_{k,i,j} \frac{1}{b(a(1+j))^{\frac{1+b}{b}}} \Gamma(\frac{3+b}{b})$,

$E(X^4) = \sum_{k,i,j=0}^{\infty} V_{k,i,j} \frac{1}{b(a(1+j))^{\frac{4+b}{b}}} \Gamma(\frac{4+b}{b})$.

5-5. Moment Generating Function(mgf):

The mgf is given by the following formula:

$M_x(t) = E(e^{tx})$ with using exponential series of $E(e^{tx})$ then:

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} (E(x^r)). \tag{26}$$

We can calculate that the mgf of the proposed family of distribution by substituting Equation (25) in Equation (26) becomes as follows:.

$M_{(x,\beta,\theta,a,b)}(t) = \sum_{k,i,j=0}^{\infty} \sum_{r=0}^{\infty} \frac{t^r}{r!} V_{k,i,j} \frac{1}{b(a(1+j))^{\frac{r+b}{b}}} \Gamma(\frac{r+b}{b})$,

$$M_{(x;\beta,\theta,a,b)}(t) = \sum_{k,i,j,r=0}^{\infty} W_{k,i,j,r} \frac{1}{b(a(1+j))^{\frac{r+b}{b}}} \Gamma(\frac{r+b}{b}) , \tag{27}$$

where $W_{k,i,j,r} = \frac{\beta^{k+1}(-1)^{k+i+j}\binom{k}{i} e^{\theta(1+i)}(\theta(1+i))^j abt^r}{\theta^k k! j! r! \left(1-e^{-\frac{\beta}{\theta}\left(e^{\theta}-1\right)}\right)}$.

The mean (μ) can be calculated by:

$\mu = E(X) = \sum_{k,i,j=0}^{\infty} V_{k,i,j} \frac{1}{b(a(1+j))^{\frac{1+b}{b}}} \Gamma(\frac{1+b}{b})$.

The variance (σ^2) can be calculated by :

$\sigma^2 = \sum_{k,i,j=0}^{\infty} V_{k,i,j} \frac{1}{b(a(1+j))^{\frac{2+b}{b}}} \Gamma\left(\frac{2+b}{b}\right) - (\sum_{k=i=j=0}^{\infty} V_{k,i,j} \frac{1}{b(a(1+j))^{\frac{1+b}{b}}} \Gamma(\frac{1+b}{b}))^2.$

The skewness of TGO-wd[0,1] is given by the following formula:

$$S_k = \frac{E(X-\mu)^3}{(\sigma^2)^{(3/2)}} = \frac{E\left(X^3\right) - 3E(X)E\left(X^2\right) + 2(E(X))^3}{(\sigma^2)^{(3/2)}}. \tag{28}$$

Substituting each $E(X), E(X^2), E(X^3)$, and σ^2 in Equation (28) to get the following:

$$S_k = \frac{\sum_{k,i,j=0}^{\infty} V_{k,i,j} \frac{1}{b(a(1+j))^{\frac{3+b}{b}}} \Gamma\left(\frac{3+b}{b}\right) - 3\left(\sum_{k,i,j=0}^{\infty} V_{k,i,j} \frac{1}{b(a(1+j))^{\frac{1+b}{b}}} \Gamma\left(\frac{1+b}{b}\right) \sum_{k=i=j=0}^{\infty} V_{k,i,j} \frac{1}{b(a(1+j))^{\frac{2+b}{b}}} \Gamma\left(\frac{2+b}{b}\right)\right) + 2\left(\sum_{k,i,j=0}^{\infty} V_{k,i,j} \frac{1}{b(a(1+j))^{\frac{1+b}{b}}} \Gamma\left(\frac{1+b}{b}\right)\right)^3}{\left(\sum_{k,i,j=0}^{\infty} V_{k,i,j} \frac{1}{b(a(1+j))^{\frac{2+b}{b}}} \Gamma\left(\frac{2+b}{b}\right) - \left(\sum_{k,i,j=0}^{\infty} V_{k,i,j} \frac{1}{b(a(1+j))^{\frac{1+b}{b}}} \Gamma\left(\frac{1+b}{b}\right)\right)^2\right)^{\left(\frac{3}{2}\right)}}$$

The kurtosis (K_u) for TGO-wd[0,1] can be calculated as follows:

$$K_u = \frac{E(X-\mu)^4}{(\sigma^2)^2} = \frac{E(X^4) - 4E(X)E(X^3) + 6(E(X))^2 E(X^2) - 3(E(X))^4}{(\sigma^2)^2} - 3 \ . \qquad (29)$$

Substituting each $E(X), E(X^2), E(X^3)$, and $E(X^4)$ in Equation(29) , it gives: $K_u =$

$$\frac{\sum_{k,i,j=0}^{\infty} V_{k,i,j} \frac{1}{b(a(1+j))^{\frac{4+b}{b}}} \Gamma\left(\frac{4+b}{b}\right) - 4\left[\sum_{k,i,j=0}^{\infty} V_{k,i,j} \frac{1}{b(a(1+j))^{\frac{1+b}{b}}} \Gamma\left(\frac{1+b}{b}\right) \sum_{k,i,j=0}^{\infty} V_{k,i,j} \frac{1}{b(a(1+j))^{\frac{3+b}{b}}} \Gamma\left(\frac{3+b}{b}\right)\right] + 6\left(\sum_{k,i,j=0}^{\infty} V_{k,i,j} \frac{1}{b(a(1+j))^{\frac{1+b}{b}}} \Gamma\left(\frac{1+b}{b}\right)\right)^2 \sum_{k,i,j=0}^{\infty} V_{k,i,j} \frac{1}{b(a(1+j))^{\frac{2+b}{b}}} \Gamma\left(\frac{2+b}{b}\right) - 3\left(\sum_{k,i,j=0}^{\infty} V_{k,i,j} \frac{1}{b(a(1+j))^{\frac{1+b}{b}}} \Gamma\left(\frac{1+b}{b}\right)\right)^4}{\left(\sum_{k,i,j=0}^{\infty} V_{k,i,j} \frac{1}{b(a(1+j))^{\frac{2+b}{b}}} \Gamma\left(\frac{2+b}{b}\right) - \left(\sum_{k,i,j=0}^{\infty} V_{k,i,j} \frac{1}{b(a(1+j))^{\frac{1+b}{b}}} \Gamma\left(\frac{1+b}{b}\right)\right)^2\right)^2} - 3.$$

5-6 The Rényi Entropy [7]:

We can calculate that the Rényi entropy of TGO-wd[0,1] is given as follows:

$$I_s(x) = \frac{1}{1-s} \log \left[\int_{-\infty}^{\infty} f(x,\beta,\theta,a,b)^s \, dx\right] \text{ such that } s > 0 \ and \ s \neq 1. \qquad (30)$$

Then by substituting Equation(18) in Equation (30), we obtain.

$$I_s(x) = \frac{1}{1-s} \log\left[\int_0^{\infty} \sum_{k,i,j=0}^{\infty} V^s{}_{k,i,j} \left[x^{b-1} e^{-a(1+j)x^b}\right]^s \, dx\right].$$

$$I_s(x) = \frac{1}{1-s} \log\left[\int_0^{\infty} \sum_{k,i,j=0}^{\infty} V^s{}_{k,i,j} \, x^{s(b-1)} e^{-s(a(1+j))x^b} \, dx\right]. \qquad (31)$$

Let $y = s(a(1+j))x^b$, $x = \frac{y^{\frac{1}{b}}}{(s(a(1+j)))^{\frac{1}{b}}}$, $dx = \frac{y^{\frac{1}{b}-1}}{b(s(a(1+j)))^{\frac{1}{b}}}$.

Substituting $s(a(1+j))x^b$, $x \ and \ dx$ in Equation (31)

$$= \frac{1}{1-s} \log \sum_{k,i,j=0}^{\infty} V^s{}_{k,i,j} \frac{1}{b(s(a(1+j)))^{s\left(1-\frac{1}{b}\right)+\frac{1}{b}}} \int_0^{\infty} y^{s\left(1-\frac{1}{b}\right)+\frac{1}{b}-1} e^{-y} dy$$

By using the integral $\Gamma(b) = \int_0^{\infty} y^{b-1} e^{-y} dy$, we obtain:

$$= \frac{1}{1-s} \log \sum_{k,i,j=0}^{\infty} V^s{}_{k,i,j} \frac{1}{b(s(a(1+j)))^{s\left(1-\frac{1}{b}\right)+\frac{1}{b}}} \Gamma\left(s\left(1-\frac{1}{b}\right)+\frac{1}{b}\right) , \qquad (32)$$

where $V_{k,i,j} = \frac{\beta^{k+1}(-1)^{k+i+j}\binom{k}{i} e^{\theta(1+i)} (\theta(1+i))^j ab}{\theta^k k! j! \left(1 - e^{-\frac{\beta}{\theta}\left(e^{\theta}-1\right)}\right)}$.

5-7 MLE estimator [13],[14]:

The MLE estimator for the parameters of the TGO-wd[0,1] is applied to the complete samples. Assume $x_1, x_2, \dots, x_n$ are observation values of a proposed distribution which has parameters (β, θ , a, b), the MLE function can be written as follows:

$$L(x; \beta, \theta, a, b) = \Pi_{i=1}^{n} f_T(x; \beta, \theta, a, b). \qquad (36)$$

Then using Equation (10) and Equation (36) to get:

$$L = \Pi_{i=1}^{n}\left\{\frac{\beta e^{\theta\left(1-e^{-ax^{b}}\right)} abx^{b-1}e^{-ax^{b}} e^{-\frac{\beta}{\theta}\left(e^{\theta\left(1-e^{-ax^{b}}\right)}-1\right)}}{1-e^{-\frac{\beta}{\theta}\left(e^{\theta}-1\right)}}\right\} .$$

$$L = \beta^{n} e^{\theta\sum_{i=1}^{n}\left(1-e^{-a(x_i)^{b}}\right)} a^{n} b^{n} \sum_{i=1}^{n}(x_i)^{(b-1)}\, e^{-a\sum_{i=1}^{n}(x_i)^{b}})$$

$$(e^{-\frac{\beta}{\theta}\left(e^{\theta\sum_{i=1}^{n}\left(1-e^{-a(x_i)^{b}}\right)}-1\right)})(1-e^{-\frac{\beta}{\theta}(e^{\theta}-1)})^{-n} \quad (37)$$

Let $LnL = Ln(\Pi_{i=1}^{n} f_T(x;\beta,\theta,a,b))$ therefore.

$$LnL = nLn\beta + \theta\sum_{i=1}^{n}\left(1-e^{-a(x_i)^{b}}\right) + nLna + nLnb + \sum_{i=1}^{n}(b-1)\ln x_i - a\sum_{i=1}^{n}(x_i)^{b} - \frac{\beta}{\theta}\left(e^{\theta\sum_{i=1}^{n}\left(1-e^{-a(x_i)^{b}}\right)}-1\right) - nLn(1-e^{-\frac{\beta}{\theta}\left(e^{\theta}-1\right)}) \quad (38)$$

1- If θ, a and b are known:

$$\frac{\partial LnL}{\partial\beta} = \frac{n}{\beta} - \frac{1}{\theta}\left(e^{\theta\sum_{i=1}^{n}\left(1-e^{-a(x_i)^{b}}\right)}-1\right) - \left(\frac{ne^{-\frac{\beta}{\theta}\left(e^{\theta}-1\right)}}{\theta(1-e^{-\frac{\beta}{\theta}\left(e^{\theta}-1\right)})}\right)$$

2- If $\beta,\ a$ and b are known:

$$\frac{\partial LnL}{\partial\theta} = \sum_{i=1}^{n}\left(1-e^{-a(x_i)^{b}}\right) - \beta\left[\frac{\theta\left(\sum_{i=1}^{n}\left(1-e^{-a(x_i)^{b}}\right)\right)\left(e^{\theta\sum_{i=1}^{n}\left(1-e^{-a(x_i)^{b}}\right)}\right) - \left(e^{\theta\sum_{i=1}^{n}\left(1-e^{-a(x_i)^{b}}\right)}-1\right)}{\theta^{2}}\right] - \frac{n\beta e^{-\frac{\beta}{\theta}\left(e^{\theta}-1\right)}(e^{\theta}(\theta-1)+1}{\theta^{2}(1-e^{-\frac{\beta}{\theta}\left(e^{\theta}-1\right)})}$$

3- If β,θ and b are known:

$$\frac{\partial LnL}{\partial a} = \theta\sum_{i=1}^{n}(x_i)^{b}\, e^{-a\sum_{i=1}^{n}(x_i)^{b}} + \frac{n}{a} - \sum_{i=1}^{n}(x_i)^{b} - \beta[\sum_{i=1}^{n}(x_i)^{b}\, e^{-a(x_i)^{b}} e^{\theta\sum_{i=1}^{n}\left(1-e^{-a(x_i)^{b}}\right)}]$$

4- If β,θ and a are known:

$$\frac{\partial LnL}{\partial b} = \theta a\sum_{i=1}^{n}(x_i)^{b}\ln\left(\sum_{i=1}^{n}(x_i)\right)e^{-a\sum_{i=1}^{n}(x_i)^{b}} + \frac{n}{b} + \frac{1}{\Pi_{i=1}^{n}(x_i)^{(b-1)}}\Pi_{i=1}^{n}(x_i)^{(b-1)}(\mathrm{Ln}(\Pi_{i=1}^{n}(x_i))) - a\sum_{i=1}^{n}(x_i)^{b}\,\mathrm{Ln}(\sum_{i=1}^{n}(x_i)) + \beta a\sum_{i=1}^{n}(x_i)^{b}\,(Ln(\sum_{i=1}^{n}(x_i)))e^{\theta\left(1-e^{-a\sum_{i=1}^{n}(x_i)^{b}}\right)}e^{-a\sum_{i=1}^{n}(x_i)^{b}}.$$

The solutions to the equations $\frac{\partial LnL}{\partial\vartheta}=0, \frac{\partial LnL}{\partial\theta}=0, \frac{\partial LnL}{\partial a}=0$ and $\frac{\partial LnL}{\partial b}=0$ will be done numerically until we get the values of the parameters $\beta,\ \theta,\ a,$ and b.

7. Applications:

The R application software has been done by one real dataset for the TGO-wd[0,1]. A comparison of the values using statistical criteria such as -LL, AIC, CAIC, BIC and HQIC) . They are the negative of the logarithm of the maximum likelihood and the Akiaki, corrected Akiaki, Bayesian, and Hannan-Quinen information criteria, respectively with some distributions like Beta-Weibull [8], Lomax Weibull [9], Exponential Generalized Weibull

[10], Weibull- Weibull [11], Gompertz Weibull, and Weibull . In Table 1, s_k value is less than zero so the dataset curve has left skewed from mean. The value of k_S value is greater than zero so the probability distribution curve is on its peakedness. In Table 2, TGO-wd[0,1] shows more matching with histogram for the real dataset and the statistical criteria provide law values when it is compared with the values of the statistical criteria for other distributions. Therefore, the TGO-wd[0,1] has better performance and more flexibility with real data. Table 3 is showing Kolmogorov-Smirnov test (KS), Statistic and p-value test TGO-wd[0,1] and various distributions for real dataset. Given data set of 63 observations of the strength 1.5 cm (glass fibers) [12]:

0.55, 0.74, 0.77, 0.81, 0.84, 0.93, 1.04, 1.11, 1.13, 1.24, 1.25, 1.27, 1.28, 1.29, 1.30, 1.36,1.39, 1.42, 1.48, 1.48, 1.49,1.49, 1.50, 1.50, 1.51, 1.52, 1.53, 1.54, 1.55, 1.55, 1.58, 1.59,1.60, 1.61, 1.61, 1.61, 1.61,1.62, 1.62, 1.63, 1.64, 1.66, 1.66, 1.66, 1.67, 1.68, 1.68, 1.69,1.70, 1.70, 1.73, 1.76, 1.76,1.77, 1.78, 1.81, 1.82, 1.84, 1.84, 1.89, 2.00, 2.01, 2.24.

Table 1: The statistical summary for the real dataset.

n	*mean*	*sd*	*median*	*min*	*max*	S_k	K_u
63	1.51	0.32	1.59	0.55	2.24	-0.88	0.8

Table 2: The parameters and criteria (-LL,AIC,CAIC BIC,HQIC) values.

Model	*Est. parameters*		*-LL*	*AIC*	*CAIC*	*BIC*	*HQIC*
[0,1]TGO-WD	$\hat{\beta}$ $\hat{\theta}$ $\hat{a}$ $\hat{b}$	0.0857690 4.44503545 0.12010790 3.96616969	12.9324	32.6878	33.3775	41.2603	36.0594
BeW	$\hat{\beta}$ $\hat{\theta}$ $\hat{a}$ $\hat{b}$	0.8366791 0.1490802 0.4111292 5.5399691	14.9757	37.9515	38.6411	46.5240	41.3231
LoW	$\hat{\beta}$ $\hat{\theta}$ $\hat{a}$ $\hat{b}$	1.23989415 12.1140457 0.00356677 6.21407408	15.8074	39.6148	40.3044	48.1873	42.9864
EGW	$\hat{\beta}$ $\hat{\theta}$ $\hat{a}$ $\hat{b}$	0.1613691 1.4823542 1.0609900 4.2575485	16.8245	41.6490	42.3387	50.2216	45.0207
WeW	$\hat{\beta}$ $\hat{\theta}$ $\hat{a}$ $\hat{b}$	3.3410877 0.3324063 0.1430366 1.7300482	15.2068	38.4136	39.1033	46.9862	41.7853
GoW	$\hat{\beta}$ $\hat{\theta}$ $\hat{a}$ $\hat{b}$	0.7743762 0.0324643 0.0813584 5.6134744	15.1884	38.3769	39.0665	46.9494	41.7485
W	$\hat{\beta}$ $\hat{\theta}$ $\hat{a}$ $\hat{b}$	0.75989193 0.07626089 0.05989193 5.77626089	15.2068	34.6994	34.4994	38.6999	36.0995

Table 3: Test statistic of TGO-wd[0,1] and the other models for the real data.

Model	KS Statistic	KS p-value
TGO-wd[0,1]	0.1104885	0.4252945
BeW	0.1532979	0.1035166
LoW	0.1575187	0.0877498
EGW	0.1797335	0.0341437
WeW	0.1522563	0.1077502
Gow	0.1520554	0.1085831
W	0.1523223	0.1074778

Figure 3 shows a histogram of pdf and cdf estimations for TGO-wd[0,1] with other models . The TGO-wd[0,1] is more convergent with data then the proposed distribution is better according to the values of statistical criteria . So, the TGO-wd[0,1] provides the best performance than the other competing models.

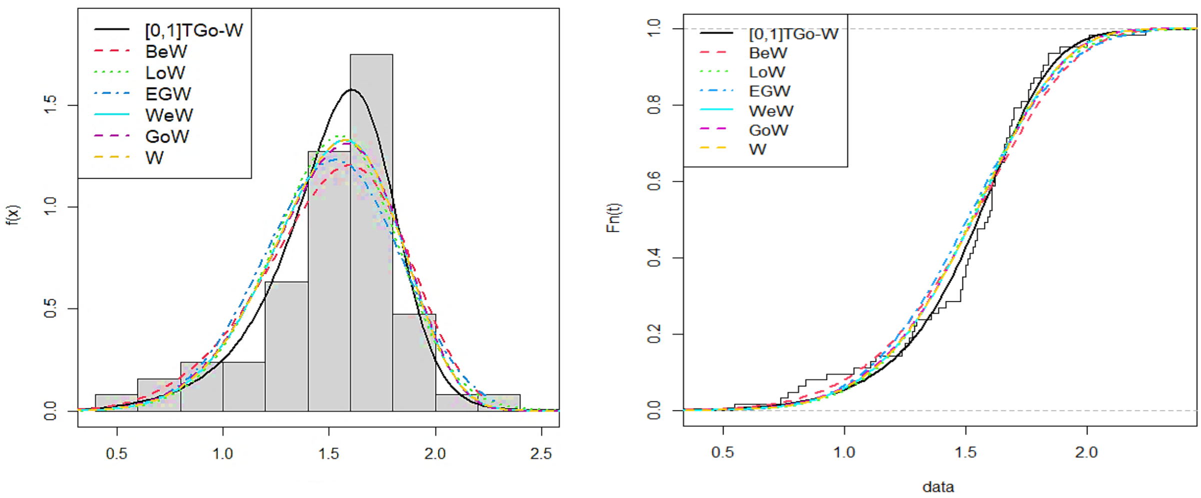


Figure 3: The plot pdf and cdf of the TGO-wd[0,1] and other distribution for real data set.

The plot of (TTT) in Figure 4 shows a data set which is related to an increasing failure rate of real data.

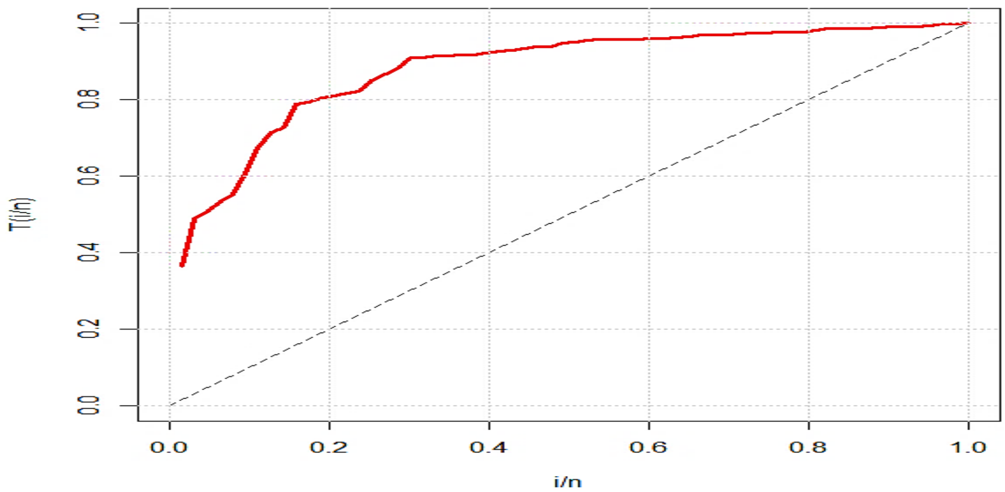


Figure 4: The plot of (TTT) for the third real data in R.

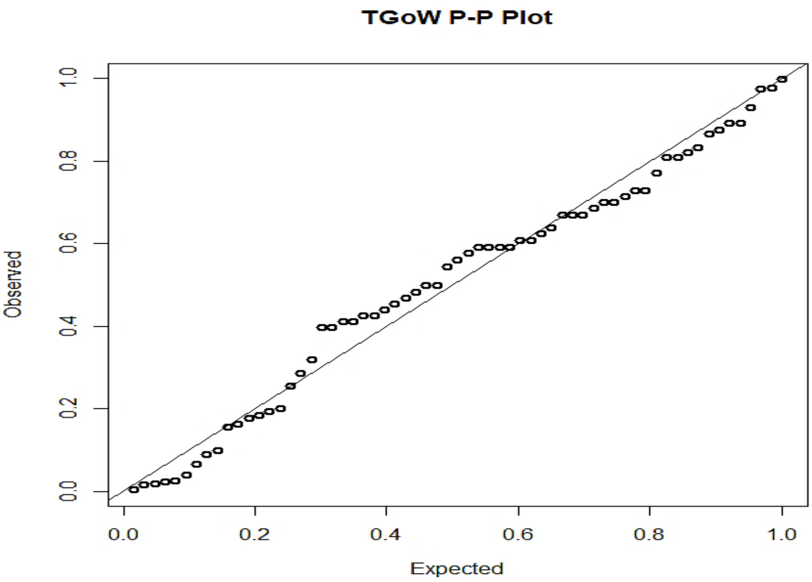


Figure5: The p-p plot for TGO-wd[0,1] distribution.

The TGO-wd[0,1] represents by points as it is noticed in Figure 5 which is closer to the unit line in compared to the other distributions.

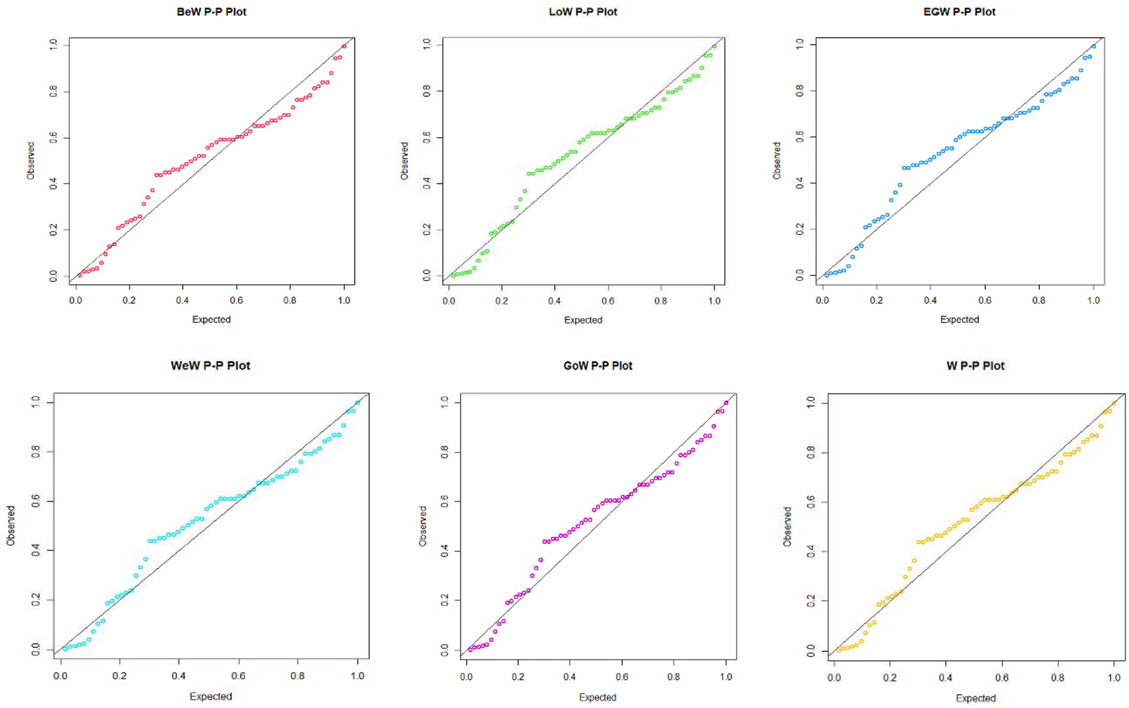


Figure 6: The P-P Plots of the real data for comparative models.

8- Conclusion:

The current study introduces a suggested formula for TGO-wd[0,1]. The MLE method is used to estimate the four parameters $(\beta,\ \theta, a, b)$ of the proposed distribution. some statistical properties for the TGO-wd[0,1] are introduced. The proposed combination distribution shows the best performance with the value of all the statistical criterion LL, AIC, CAIC, BIC and HQIC. The proposed distribution has the best flexibility with the real dataset than the other comparative distributions like Beta-Weibull, Lomax Weibull, Exponential Generalized

Weibull, Weibull- Weibull, Gompertz Weibull, and Weibull . Due to the flexibility of TGO-wd[0,1], the suggested formula can be used in various fields of statistics.

Based on the foregoing of the importance of creating new distributions to cover all statistical fields, and in order to enrich statistical research with competing topics, it was necessary to note and direct attention and look at other methods and ways to create new families of distributions. It is worth to note that we review some of these considerations:

1- Establishing families of distributions using on Left Truncated Gompertz – Exponential distribution.
2- Establishing a new families of distributions using Right Truncated Gompertz – Weibull distribution.
3- Studying the other properties for the proposed family TGO-wd[0,1].

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