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Modeling the Influence of Time Delay in Controlling Infectious Diseases with A Stage-Structure

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Abstract

This research aims to construct a stage-structured mathematical model to evaluate the impact of time delay on infectious disease control. The solution's characteristics are described. Every possible equilibrium point has been determined. The local stability of all equilibrium points for time delay values is investigated. It has been shown that the Hopf bifurcation takes place in the vicinity of the endemic equilibrium point. The periodic dynamics' stability and direction are investigated. Numerical simulations were provided to confirm the theoretical findings and understand the effects of varying parameter values.

Keyword: Infection diseases, Stability, Time delay, Hopf bifurcation.

نمذجة تأثير التأخير الزمني في السيطرة على الأمراض المعدية ذات مراحل عمريه

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الخلاصة

يهدف هذا البحث إلى إنشاء نموذج رياضي ذات مراحل عمريه لتقييم تأثير التأخير الزمني في السيطره على الأمراض المعدية. تم وصف خصائص الحل. تم تحديد وجود جميع النقاط التوازن. تمت دراسة الاستقرار المحلي لجميع نقاط التوازن لجميع قيم التأخير الزمني. لقد ثبت أن تشعب هوبف يحدث بالقرب من نقطة التوازن الوبائيه. تمت دراسة استقرار واتجاه الديناميكيات الدورية. تم تقديم عمليات المحاكاة العددية لتأكيد النتائج النظرية وفهم تأثيرات قيم المعلمات المختلفة.

الكلمات المفتاحية: الأمراض المعدية، الاستقرار، التأخير الزمني، تشعب هوبف.

1. Introduction

Infectious diseases remain a major worldwide health worry, necessitating efficient control techniques to limit their effects on populations. So, Mathematical modeling is crucial for comprehension and predicting disease spread. By utilizing mathematical equations and numerical simulations, many researchers have proposed many types of epidemic models for

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understanding disease transmission mechanisms; see [1–7]. The (SIR) model, introduced by Kermack and McKendrick [8], is a mathematical model for studying the spread of contagious illnesses. It divides the total population into three types: susceptible (S), infected (I), and recovered (R). Many researchers have studied the (SIR) epidemic model without delay; see [9- 17].

Time delays are often incorporated into epidemiological models to account for a variety of factors that influence the spread and progression of infectious diseases. These delays represent the time it takes for certain processes to occur; for example, the latent period, also known as the incubation period, indicates the time between infection and the onset of contagiousness or symptoms. It is an essential factor in predicting the spread of infectious diseases because, during the incubation period, individuals may be unaware that they are infected and may spread the disease to others. The latent period of influenza A H1N1 can range from days to years [18]. Several epidemiological models involving time delays in various parameters have been suggested and investigated in many literatures for example Rui et al. [19] suggested an epidemic model type of SIR with time delay describing a constant infectious period. Edoardo et al. [20] studied the SIR epidemic model with distributed time delays. In [21] Haojie et al. formulated a delayed SIR epidemic model with a convex incidence rate. Naji and Hussien [22] have proposed and examined an SEIR epidemic model incorporating a time delay attributed to the incubation period. B. Li et al. [23] conducted an investigation into the stability and local bifurcations of a discrete-time SIR epidemic model.In the absence of a vaccine, Raid and Hassan formed a mathematical model for the dynamics of the SIR epidemic within a stage-structure population [24]. Due to the presence of SIR-type disease, the population can be divided into three compartments: susceptible, infected, and removed. Since the susceptible population has a stage structure, it is separated into two classes: immature susceptible and mature susceptible. They proposed that there are two ways for the disease to spread: through contact and through outside influence, which led to the following system:

$$\begin{aligned}
\frac{dS_1}{dt} &= \Lambda - \alpha S_1 - \gamma_1 S_1 - \beta_1 S_1 I - dS_1, \\
\frac{dS_2}{dt} &= \alpha S_1 - \gamma_2 S_2 - \beta_2 S_2 I - dS_2 \ , \\
\frac{dI}{dt} &= \gamma_1 S_1 + \beta_1 S_1 I + \gamma_2 S_2 + \beta_2 S_2 I - dI - \Psi I \ , \\
\frac{dR}{dt} &= \Psi I \ - dR \ .
\end{aligned} \tag{1}$$

In system (1) , $S_1(t)$ and $S_2(t)$ represent immature susceptible and mature susceptible at time t respectively; $I(t)$ and $R(t)$ represent the infected and recover at time t , respectively; The recruitment rate is represented by $\Lambda > 0$; α is grown up rate ; external incidence rates for $S_1(t)$ and $S_2(t)$ are denoted by $\gamma_1$and γ_2 respectively ; contact rates for $S_1(t)$ and $S_2(t)$ are represented by β_1 and β_2 respectively; d is natural death from $S_1(t)$, $S_2(t)$, $I(t)$ and $R(t)$; Ψ is th recover rate.

Diseases may not be transmitted to susceptible individuals when they come into touch with infected individuals (I). They often require some time for transmission. This time called the delay. It takes for an infection to be transmitted from an infected person to a susceptible person after touch. In the first step in this paper, taking into account the aforementioned, system (1) is modified to involve the delay time in the incidence rat . The resulting system is described as follows:

$$\begin{aligned}
\frac{dS_1}{dt} &= \Lambda - \alpha S_1 - \gamma_1 S_1 - \beta_1 S_1(t-\tau)I(t-\tau) - dS_1,\\
\frac{dS_2}{dt} &= \alpha S_1 - \gamma_2 S_2 - \beta_2 S_2(t-\tau)I(t-\tau) - dS_2 \ ,\\
\frac{dI}{dt} &= \gamma_1 S_1 + \beta_1 S_1(t-\tau)I(t-\tau) + \gamma_2 S_2 + \beta_2 S_2(t-\tau)I(t-\tau) - dI - \Psi I,\\
\frac{dR}{dt} &= \Psi I \ - dR \ .
\end{aligned} \tag{2}$$

Here τ is a time delay representing the incubation period of the disease.The infectious agent first develops in the host during the incubation period, and it is only after this time that the infected become contagious. As a result, the number of actively infected at time t appears through contacts between the actual population of susceptible and infected at the time $(t-\tau)$. Every parameter in system (2) has the same biological meaning as every parameter in system (1) .

From the previous system, we can conclude that $R(t)$ does not effect on $S_1(t)$, $S_2(t)$ and $I(t)$. It is possible to omit this equation without losing generality. So, for the purpose of the study, we take into consideration the following reduced system.

$$\begin{aligned}
\frac{dS_1}{dt} &= \Lambda - \alpha S_1 - \gamma_1 S_1 - \beta_1 S_1(t-\tau)I(t-\tau) - dS_1,\\
\frac{dS_2}{dt} &= \alpha S_1 - \gamma_2 S_2 - \beta_2 S_2(t-\tau)I(t-\tau) - dS_2 \ ,\\
\frac{dI}{dt} &= \gamma_1 S_1 + \beta_1 S_1(t-\tau)I(t-\tau) + \gamma_2 S_2 + \beta_2 S_2(t-\tau)I(t-\tau) - dI - \Psi I.
\end{aligned} \tag{3}$$

Here $S_1(\sigma) > 0, S_2(\sigma) > 0$ and $I(\sigma) \geq 0$ for $\sigma \in [-\tau, 0)$.

The paper is organized as follows: In section 2, we investigate the positivity and bounds of solutions to the modified system . In section 3, we examine the stability and existence of the bifurcation. In section 4 , the stability and direction of the bifurcating periodic solution are established.In section 5, numerical simulations have been used for understanding the dynamics of the system (3). A brief conclusion is presented in section 6.

2. Positive and Boundedness.

The system's (3) equations monitor populations. Therefore, it must be demonstrated to show that all state variables with nonnegative initial conditions will remain positive and bounded for all time t . The following theorem investigates the positivity and boundedness of the system.

Theorem 1. The solutions of the system (3) are positive and bounded for t ≥ 0.

Proof. Since the system's right-hand side interaction functions are continuous and have continuous partial derivatives, they are Lipschtizaine. Then the system (3) has a unique solution $(S_1(t), S_2(t), I(t))$ in $\mathbb{R}_+^3$ with initial condition .

Now, we show that all the solutions are nonnegative. From system (3), we obtain:

$$\begin{aligned}
\frac{dS_1}{dt} &\geq -S_1(\alpha + \gamma_1 + \beta_1 I + d),\\
\frac{dS_2}{dt} &\geq -S_2(\gamma_2 + \beta_2 I + d),\\
\frac{dI}{dt} &\geq -I(d + \Psi).
\end{aligned}$$

Thus, it implies

$$S_1(t) \geq S_1(0) exp - \left\{\int_0^t \ (\alpha + \gamma_1 + \beta_1 I(\zeta) + d) d(\zeta)\right\},$$

$$S_2(t) \geq S_2(0) exp - \left\{\int_0^t \ (\gamma_2 + \beta_2 I(\zeta) + d) d(\zeta)\right\},$$

$$I(t) \geq I(0)exp - \left\{\int_0^t (d+\Psi)d(\zeta)\right\}.$$

As a result, each of the system's (3) solutions is non – negative, because of the positivity of the exponential function and the initial conditions.

Now , the we demonstrate that the system's solutions are bounded for all $t \geq 0$.

Let $M(t) = S_1(t) + S_2(t) + I(t)$ is the total population.

Then, system (3) provides that:

$$\frac{dM}{dt} = \Lambda - dM \ .$$

By using Gronwell's lemma, it follows that:

$$M(t) \leq M(0)e^{-dt} + \frac{\Lambda}{d}(1 - e^{-dt}),$$

Thus, we obtain

$$\lim_{n\to\infty} \quad M(t) \leq \frac{\Lambda}{d} \,,$$

As a result, it follows that the solutions are bounded.

3. The Local stability and Hopf Bifurcation .

The existence and the analysis of stability are performed in this part. Time delay does not affect on the location or number of equilibrium points, as a result, system (3) has two equilibrium points, see [24] and it can be represented as follows:

- Uninfected equilibrium point (UEP) denoted $E^0 = (S_1^0, S_2^0, 0)$, where

$$S_1^0 = \frac{\Lambda}{\alpha + d} \tag{4.a}$$

$$S_2^0 = \frac{\alpha\Lambda}{d(\alpha + d)} \tag{4.b}$$

It is obvious that E^0 exists, if $I = 0$ and the following condition holds :

$$\gamma_1 = \gamma_2 = 0 \tag{5}$$

- Endemic equilibrium point (ENEP) is denoted by $E^* = (S_1^*, S_2^*, I^*)$, where

$$S_1^* = \frac{\Lambda}{\alpha + \gamma_1 + \beta_1 I^* + d}\,, \tag{6.a}$$

$$S_2^* = \frac{\alpha\Lambda}{(\gamma_2 + \beta_2 I^* + d)(\alpha + \gamma_1 + \beta_1 I^* + d)} \ . \tag{6.b}$$

While I^* represents the positive root of the following equation:

$$\kappa_1 (I^*)^3 + \kappa_2 (I^*)^2 + \kappa_3 I^* + \kappa_4 = 0. \tag{7}$$

Here

$$\kappa_1 = -(d + \Psi)\beta_1\beta_2 < 0$$
$$\kappa_2 = \Lambda\beta_1\beta_1 - (d+\Psi)[\beta_1(\gamma_1 + d) + \beta_2(\alpha + \gamma_1 + d)]$$
$$\kappa_3 = [\Lambda\beta_1(\gamma_2 + d) + \Lambda\beta_2(\alpha + \gamma_1)] - (d+\Psi)[(\alpha + \gamma_1 + d)(\gamma_2 + d)]$$
$$\kappa_4 = \Lambda[\gamma_1(\gamma_2 + d) + \alpha\gamma_2] > 0$$

Clearly, E^* exists if $I \neq 0$ and the following condition holds:

$$\kappa_2 < 0 \quad OR \quad \kappa_3 > 0 \tag{8}$$

Now, we first determine the linearized matrix to examine the system's (3) stability at the obtained equilibrium points E^0 and E^* . At any point, say (S_1, S_2, I), the general Jacobian matrix (J) for the system (3) can be expressed as follows:

$$J = [\, q_{ij}]_{3\times 3}, \; i,j = 1,2,3 \tag{9}$$

where

$q_{11} = -[(\alpha + \gamma_1 + d) + \beta_1 I e^{-\lambda\tau}] + ; \; q_{12} = 0 \; ; \; q_{13} = -\beta_1 S_1 e^{-\lambda\tau}$;

$q_{21} = \alpha \; ; \; q_{22} = -[\gamma_2 + d + \beta_2 I e^{-\lambda\tau}] \; ; \; q_{23} = -\beta_2 S_2 e^{-\lambda\tau}$;

$q_{31} = \square_1 + \beta_1 I e^{-\lambda\tau}$; $q_{32} = \square_2 + \beta_2 I e^{-\lambda\tau}$;
$q_{33} = (\beta_1 S_1 + \beta_2 S_2)e^{-\lambda\tau} - (d + \Psi)$

Hence, the characteristic equation of the matrix (9) can be expressed as follows:

$$Q_1(\lambda) + Q_2(\lambda)e^{-\lambda\tau} = 0 \tag{10}$$

Here $Q_1(\lambda)$ and $Q_2(\lambda)$ are polynomails of λ.
The J of the system (3) at the E^0 is

$$J_o = \begin{bmatrix} -(\alpha + d) & 0 & -\beta_1 S_1^0 e^{-\lambda\tau} \\ \alpha & -d & 0 \\ 0 & 0 & (\beta_1 S_1^0 + \beta_2 S_2^0)e^{-\lambda\tau} - (d + \Psi)) \end{bmatrix} \tag{11}$$

Therefore, the eigenvalues of J_0 may be expressed as $\lambda_{S_1}^0 = -(\alpha + d) < 0, \lambda_{S_2}^0 = -d < 0$. Conversely , We utilize the following equation to find the 3rd root.

$$\lambda_I^0 + (d + \Psi) - (\beta_1 S_1^0 + \beta_2 S_2^0)e^{-\lambda\tau} = 0 \tag{12}$$

Thus ;

1. If $\tau = 0$, then equation (12) has an eigenvalue $\lambda_I^0 = -(d + \Psi) + (\beta_1 S_1^0 + \beta_2 S_2^0)$ which is negative under the condition:

$$R_0 < 1 \ , R_0 = \frac{\beta_1 S_1^0 + \beta_2 S_2^0}{d + \Psi} \tag{13}$$

where R_0 represents the basic reproduction number.
Hense , the UEP is locally asymptotically stable, when $R_0 < 1$

2. When $\tau > 0$, we assume that equation (12) has two roots which are $\lambda = \varrho \pm i\omega^0$ where $\varrho = 0, \omega^0 > 0$. Consequently, the direct calculation, by replacing $\lambda = i\omega^0$ in equation (12) and then separating the real and imaginary parts, provides that:

$$\begin{aligned} d + \Psi &= (\beta_1 S_1^0 + \beta_2 S_2^0)\cos\omega^0\tau \\ -\omega^0 &= (\beta_1 S_1^0 + \beta_2 S_2^0)\sin\omega^0\tau \end{aligned} \tag{14}$$

Squaring and adding the two sides of the previous equations gives

$$\omega^0 = \pm\sqrt{((\beta_1 S_1^0 + \beta_2 S_2^0)^2 - (d + \Psi)^2} \tag{15}$$

Note that the condition (13) shows that $\omega^0(\tau)$ cannot be real when $\tau > 0$, it contradicts the assumption. Thus, the characteristic equation's root (12) cannot be purely imaginary. As a result, E^0 is asymptotically stable for all $\tau \geq 0$.

The J of system (3) evaluated at E^* is

$$J_* = \begin{bmatrix} -(e_1 + e_2 e^{-\lambda\tau}) & 0 & -e_3 e^{-\lambda\tau} \\ \alpha & -(e_4 + e_5 e^{-\lambda\tau}) & -e_6 e^{-\lambda\tau} \\ \gamma_1 + e_2 e^{-\lambda\tau} & \gamma_2 + e_5 e^{-\lambda\tau} & (e_3 + e_6)e^{-\lambda\tau} - a_7 \end{bmatrix} \tag{16}$$

Where
$e_1 = \alpha + \square_1 + d$; $e_2 = \beta_1 I^*$; $e_3 = \beta_1 S_1^*$; $e_4 = \square_2 + d$; $e_5 = \beta_2 I^*$;
$e_6 = \beta_2 S_2^*$; $e_7 = d + \Psi$.

The characteristic equation of J_* is

$$\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 + (b_4\lambda^2 + b_5\lambda + b_6)e^{-\lambda\tau} = 0 \quad (17)$$

where

$b_1 = e_1 + e_4 + e_7;$

$b_2 = e_1(e_4 + e_7) + e_4e_7;$

$b_3 = e_4e_7(\alpha + e_4);$

$b_4 = e_2 + e_5 - (e_3 + e_6);$

$b_5 = e_1(e_3 + e_5 + e_6) + e_4(e_2 + e_3 + e_6) + e_7(e_2 + e_5) + \gamma_1e_3 + \gamma_2e_6 - e_2(\beta_2S_1^* + e_{6-}e_5);$

$$b_6 = e_3\big(e_4(e_1 + \gamma_2) + \alpha\,\gamma_2 + e_5(\gamma_1 - e_4)\big) - de_6(\alpha + e_2 + e_4) + e_4e_5e_7 + e_2(e_4 + e_5).$$

Thus,

1. If $\tau = 0$, then equation (17) becomes

$$\lambda^3 + (b_1 + b_4)\lambda^2 + (b_2 + b_5)\lambda + (b_3 + b_6) = 0 \quad (18)$$

Routh-Hurwitz criterion determines that system (3) for $\tau = 0$ is locally asymptotically stable at ENEP, if and only if the following requirements are met.

$$\beta_1S_1^* + \beta_2S_2^* < d + \Psi \quad (19)$$

$$\beta_1S_1^*\ \mathcal{M}_1^* < \mathcal{M}_2^*(\mathcal{M}_3^* + \mathcal{M}_4^*) - \mathcal{M}_5^*\mathcal{M}_6^* \quad (20)$$

Where

$\mathcal{M}_1^* = \alpha(\gamma_2 + e_5) + (e_4 + e_5)(\gamma_1 + e_2);$

$\mathcal{M}_2^* = e_1 + e_2 + e_4 + e_5 + e_7 - (e_3 + e_6);$

$\mathcal{M}_3^* = (e_1 + e_2)(e_4 + e_5);$

$\mathcal{M}_4^* = -(e_1 + e_2)(e_3 + e_6 - a_7) + e_3(\gamma_1 + e_2);$

$\mathcal{M}_5^* = (e_3 + e_6) - (e_4 + e_5 + a_7);$

$\mathcal{M}_6^* = -(e_4 + e_5)(e_3 + e_6 - a_7) + e_6(\gamma_2 + e_5).$

2- when $\tau > 0$, assume that equation (17) has two roots that are entirely imaginary, and they take the forms $\pm i\omega^*(\ \omega^* > 0)$, if in addition to condition (19), the following condition hold

$$b_3 < b_6 \quad (21)$$

The following equations are produced by substituting $\lambda = i\omega^*$ ($\omega^* > 0$) into equation (17) and separating the real and imaginary components

$$\begin{aligned} (b_4\,\omega^{*2} - b_6)\,sin\,\omega^*\tau + b_5\,\omega^*\,cos\,\omega^*\tau &= \omega^{*3} - b_2\,\omega^* \\ b_5\,\omega^*\,sin\,\omega^*\tau + (b_6 - b_4\omega^{*2})\,cos\,\omega^*\tau &= b_1\,\omega^{*2} - b_3 \end{aligned} \quad (22)$$

The following algebraic equation is obtained by squaring, adding, and eliminating the above equations

$$\omega^{*6} + \sigma_1\omega^{*4} + \sigma_2\omega^{*2} + \sigma_3 = 0 \quad (23)$$

where

$\sigma_1 = b_1^2 - b_4^2 - 2b_2;$

$\sigma_2 = b_2^2 - b_5^2 - 2b_1b_3 + 2\,b_4\,b_6\,;$

$\sigma_3 = b_3^2 - b_6^2.$

Substituting $\mathcal{L} = \omega^{*2}$ in equation (23) yields the following 3rd-order equation:

$$f(\mathcal{L}) = \mathcal{L}^3 + \sigma_1\mathcal{L}^2 + \sigma_2\mathcal{L} + \sigma_3 = 0 \tag{24}$$

Clearly, condition (19) and condition (21) implies that $\sigma_3 < 0$. By applying Descartes' rule of sign, the equation (24) has a positive root that is unique, say $\omega_0^* = \sqrt{\mathcal{L}}$. Hense ω_0^* is also the positive root of equation (23). Thus equation (17) has roots that are entirely imaginary and represented by $\pm i\omega_0^*$ from equation (22) after substituting ω_0^* , we obtain:

$$\cos\ \omega_0^*\tau = \frac{\wp_1}{\wp_2}$$

Here

$$\wp_1 = (\ b_5 - b_1\, b_4\ )\ (\ \omega_0^*)^4 + (\ b_1\, b_6 + b_3\, b_4 - b_2\, b_5\ )(\ \omega_0^*)^2 - b_3\, b_6$$
$$\wp_2 = b_4^2(\ \omega_0^*)^4 + (\ b_5^2 - 2b_4\, b_6)(\ \omega_0^*)^2 + b_6^2$$

Then, τ_n corresponding to ω_0^* , we have

$$\tau_n = \frac{1}{\omega_0^*}(\cos^{-1}(\frac{\wp_1}{\wp_2}) + 2\pi n)\ ; n = 0,1,2, \ldots \tag{25}$$

Define $\tau_0 = min_{n\geq 0}\, \tau_n$.

From equation (17), it follows that system (3) has roots, that are represented by $\lambda(\tau) = \epsilon(\tau) + i\omega^*(\tau)$ such that $\epsilon(\tau_0) = 0$ and $\omega^*(\tau_0) = \omega_0^* > 0$. As a result, the following theorem is obtained

Theorem 2 . If requirements (19-20) are satisfied then there is $\tau_0 > 0$ such that ENEP of system (3) is locally asymptotically stable for $\tau < \tau_0$, and the ENEP becomes unstable when $\tau > \tau_0$ under the requirement (19-21). Finally, for $\tau = \tau_0$, the system (3) undergoes a Hopf bifurcation, if the following requirement is met:

$$3(\omega_0^{*2})^2 + 2\sigma_1\omega_0^{*2} + \sigma_2 \neq 0 \tag{26}$$

Proof. For $= 0$, the ENEP is locally asymptotically stable under conditions (19-20) .The Hopf bifurcation's existence will be established for $\tau = \tau_0$, if we can demonstrate that the transcendental characteristic equation (19) has roots which are represented by $\pm i\omega_0^*$ at $\tau = \tau_0$, as a result, if we can prove

$$\left[\frac{d(Re\lambda(\tau))}{d\tau}\right]_{\tau=\tau_0} \neq 0.$$

Differentiating equation (17) with respect to τ provides that :

$$\left[3\lambda^2 + 2\, b_1\lambda + b_2 + (2\, b_4\lambda + b_5)e^{-\lambda\tau} - \tau(\, b_4\lambda^2 + b_5\lambda + b_6)e^{-\lambda\tau}\right]\frac{d\lambda}{d\tau} = \lambda(\, b_4\lambda^2 + b_5\lambda + b_6)e^{-\lambda\tau}$$

(27)

$$\left[\frac{d\lambda}{d\tau}\right]^{-1} = \frac{(3\lambda^2 + 2\, b_1\lambda + b_2)e^{\lambda\tau}}{\lambda(\, b_4\lambda^2 + b_5\lambda + b_6)} + \frac{2\, b_4\lambda + b_5}{\lambda(\, b_4\lambda^2 + b_5\lambda + b_6)} - \frac{\tau}{\lambda} \tag{28}$$

Since for $\tau = \tau_0,\ and\ \lambda = i\omega_0^*$ as a result, we obtain

$$\left[\frac{d\lambda}{d\tau}\right]^{-1} = \left[\frac{\left((\mathrm{b}_2 - 3\,\omega_0^{*2}) + 2i\,\mathrm{b}_1\omega_0^*\right)(\cos\omega_0^*\tau + i\sin\omega_0^*\tau)}{-b_5\omega_0^{*2} + i\omega_0^*(b_6 - b_4\omega_0^{*2})}\right] + \left[\frac{\mathrm{b}_5 + 2i\,\mathrm{b}_4\omega_0^*}{-b_5\omega_0^{*2} + i\omega_0^*(b_6 - b_4\omega_0^{*2})}\right] - \frac{\tau_0}{i\omega_0^*}$$

So ,we can obtain

$$\left[\frac{d(Re\lambda)}{d\tau}\right]_{\lambda=i\omega_0^*} = \left[Re(\frac{d\lambda}{d\tau})^{-1}\right]_{\lambda=i\omega_0^*} . \tag{29}$$

It is obvious that:

$$Re\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\tau=\tau_0} = Re\left[\frac{\left((b_2 - 3\,\omega_0^{*2}) + 2i\,b_1\omega_0^*\right)(cos\,\omega_0^*\tau + i\,sin\,\omega_0^*\tau)}{-b_5\omega_0^{*2} + i(\,\omega_0^*)^2(b_6 - b_4\omega_0^{*2})}\right] + Re\left[\frac{b_5 + 2i\,b_4\omega_0^*}{-b_5\omega_0^{*2} + i\omega_0^*(b_6 - b_4\omega_0^{*2})}\right]$$

Hence, we have

$$Re\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\tau=\tau_0} = \frac{\omega_0^{*2}[3\omega_0^{*4} + (2b_1^2 - 4b_2 - 2b_4^2)\omega_0^{*2} + (b_2^2 - 2b_1b_3 - b_5^2 + 2b_4b_6)]}{(b_5\omega_0^{*2})^2 + (b_6 - b_4\omega_0^{*2})^2} = \frac{f'(\omega_0^{*2})}{(b_5\omega_0^{*2})^2 + (b_6 - b_4\omega_0^{*2})^2}$$

Here $f'(\omega_0^{*2}) = 3(\omega_0^{*2})^2 + 2\sigma_1\omega_0^{*2} + \sigma_2 \neq 0$ due to condition (26). So, we can show that

$$Sign\,\left\{\frac{d}{d\tau}(Re\lambda)|_{\tau=\tau_0}\right\} = Sign\left\{Re\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\tau=\tau_0}\right\} = Sign\{f'(\omega_0^{*2})\}$$

Thus, $Sign\,\left\{\frac{d}{d\tau}(Re\lambda)|_{\tau=\tau_0}\right\} \neq 0$, indicating that a Hopf bifurcation occurs at $\tau = \tau_0$.
If we presume that $\frac{d}{d\tau}(Re\lambda)|_{\tau=\tau_0} < 0$, that leads to the conclusion that the characteristic equation possesses roots with positive real parts at $\tau = \tau_0$. This conflicts with the local stability of the positive equilibrium point. Therefore, we must infer that $\left[\frac{d(Re\lambda)}{d\tau}\right]_{\tau=\tau_0} > 0$ under the condition (26). As a result, the transversality condition holds, and Hopf bifurcation occurs at $\tau = \tau_0,\ and\ \lambda = i\omega_0^*$.

4. The Direction and Stability of the Hopf Bifurcation.

The above investigation established the requirements for system (3) to have a Hopf bifurcation which occurs at ENEP when $\tau = \tau_0$. In this section, we examine the orientation of Hopf bifurcation and the essential requirements for the stability of the emerging periodic solution in the system (3). We achieve this through the application of the center manifold theorem and normal form theory presented by Hassard et al [25].

Theorem 3.

If the following quantities are determined, then the stability and direction of a bifurcating periodic solution can be identified

1. If $M_2 > 0 (M_2 < 0)$, then the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solutions exist for $\tau > \tau_0$ $(\tau < \tau_0)$
2. If $\nu_2 < 0$ $(\nu_2 > 0)$,then the periodic solutions are stable (unstable)
3. If $T_2 > 0$ $(T_2 < 0)$, then the periodic solutions are increasing (decreasing)

Here M_2, ν_2, and T_2 are given

$$\left.\begin{aligned} C_1(0) &= \frac{i}{2w_0\tau_0}\left(g_{11}\ g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2}, \\ M_2 &= -\frac{Re\{C_1(0)\}}{Re\left\{\frac{d\lambda}{d\tau}(\tau_0)\right\}}, \\ \nu_2 &= 2Re\{C_1(0)\}, \\ T_2 &= \frac{-Im\{C_1(0)\} + M_2\ Im\left\{\frac{d\lambda}{d\tau}(\tau_0)\right\}}{w_0\tau_0}. \end{aligned}\right\} \tag{30}$$

Proof. Let $u_1(t) = S_1(t) - S_1^*$, $u_2(t) = S_2(t) - S_2^*$, $u_3(t) = I(t) - I^*$, and $\tau = \tau_0 + \varkappa$, where τ_0 is define by equation (25) and $\varkappa \in \mathbb{R}$. Then system (3) can be transformed into a functional differential equation in $C = C([-1,0], \mathbb{R}^3)$ as follows:

$$u'(t) = L_\varkappa(u_t) + f(\varkappa, u_t), \tag{31}$$

where

$u(t) = (u_1(t), u_2(t), u_3(t))^T \in C = C([-1,0], \mathbb{R}^3)$ and $L_\varkappa: C \to \mathbb{R}^3$, $f: \mathbb{R} \times C \to \mathbb{R}^3$

are given by:

$$L_\varkappa(\varphi) = (\varkappa + \tau_0)[G_1\varphi(0) + G_1\varphi(-1)] \tag{32}$$

and the nonlinear term is

$$f(\varkappa, \varphi) = (\varkappa + \tau_0)\begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}$$

where

$$G_1 = \begin{bmatrix} f_{100}^{(1)} & 0 & 0 \\ f_{1000}^{(2)} & f_{0100}^{(2)} & 0 \\ f_{100000}^{(3)} & f_{010000}^{(3)} & f_{001000}^{(3)} \end{bmatrix} = \begin{bmatrix} -e_1 & 0 & 0 \\ \alpha & -e_4 & 0 \\ \square_1 & \square_2 & -e_7 \end{bmatrix},$$

$$G_2 = \begin{bmatrix} f_{010}^{(1)} & 0 & f_{001}^{(1)} \\ 0 & f_{0010}^{(2)} & f_{0001}^{(2)} \\ f_{000100}^{(3)} & f_{000010}^{(3)} & f_{000001}^{(3)} \end{bmatrix} = \begin{bmatrix} -e_2 & 0 & -e_3 \\ 0 & -e_5 & -e_6 \\ e_2 & e_5 & e_3 + e_6 \end{bmatrix}.$$

with $e_i, i = 1, \ldots\ldots, 7$ are given in the J_* , while.

$$F_1 = \sum_{i+j+k \geq 2} \frac{1}{i!j!k!} f_{ijk}^{(1)} \varphi_1^i(0) \tilde{\varphi}_1^j(-1)\, \tilde{\varphi}_3^k(-1),$$
$$F_2 = \sum_{i+m+l+k \geq 2} \frac{1}{i!m!l!k!} f_{imlk}^{(2)} \varphi_1^i(0) \varphi_2^m(0)\, \tilde{\varphi}_2^l(-1)\, \tilde{\varphi}_3^k(-1),$$
$$F_3 = \sum_{i+m+q+j+l+k \geq 2} \frac{1}{i!m!q!j!l!k!} f_{imqjlk}^{(3)} \varphi_1^i(0) \varphi_2^m(0) \varphi_3^q(0) \tilde{\varphi}_1^j(-1)\, \tilde{\varphi}_2^l(-1) \tilde{\varphi}_3^k(-1),$$

where, $\varphi(\vartheta) = (\varphi_1(\vartheta), \varphi_2(\vartheta), \varphi_3(\vartheta)) \in C, -1 \leq \vartheta \leq 0$, with

$$f^{(1)}_{ijk}\varphi_1^i(0)\tilde{\varphi}_1^j(-1)\,\tilde{\varphi}_3^k(-1)=\frac{\partial^{i+j+k}f^{(1)}}{\partial\varphi_1^i\tilde{\varphi}_1^m\tilde{\varphi}_3^n}\Big|_{(\varphi_1,\tilde{\varphi}_1,\tilde{\varphi}_3)=(0,-1,-1)},$$
$$f^{(2)}_{imlk}\varphi_1^i(0)\varphi_2^m(0)\,\tilde{\varphi}_2^l(-1)\,\tilde{\varphi}_3^k(-1)=\frac{\partial^{i+m+l+k}f^{(2)}}{\partial\varphi_1^i\varphi_2^m\tilde{\varphi}_2^l\,\tilde{\varphi}_3^k}\Big|_{(\varphi_1,\varphi_2,\tilde{\varphi}_2,\tilde{\varphi}_3)=(0,0,-1,-1)},$$
$$f^{(3)}_{\mathrm{imq}jlk}\varphi_1^i(0)\varphi_2^m(0)\varphi_3^q(0)\tilde{\varphi}_1^j(-1)\,\tilde{\varphi}_2^l(-1)\tilde{\varphi}_3^k(-1)=\frac{\partial^{i+m+q+j+l+k}f^{(3)}}{\partial\varphi_1^i\varphi_2^m\varphi_3^q\tilde{\varphi}_1^j\,\tilde{\varphi}_2^l\tilde{\varphi}_3^k}\Big|_{(\varphi_1,\varphi_2,\varphi_2\,,\tilde{\varphi}_1,\tilde{\varphi}_2,\tilde{\varphi}_3)=(0,0,0,-1,-1,-1)}.$$

By the Riesz representation theorem, there exists a matrix and there is a 3×3 matrix function: $\mathfrak{T}(\vartheta,\varkappa),\vartheta\in[-1,0]$, such that

$$L_\varkappa(\varphi)=\int_{-1}^{0}d\mathfrak{T}(\vartheta,\varkappa)\varphi(\vartheta),\ \ \varphi\in C. \tag{33}$$

In fact, it is possible to choose

$$\mathfrak{T}(\vartheta,\varkappa)=(\varkappa+\tau_0)(G_1\delta(\vartheta)-G_2\delta(\vartheta+1)), \tag{34}$$

Here , δ is the Dirac delta function

For $\varphi\in \mathrm{C}([-1,0],\mathbb{R}^3)$, we define

$$\mathcal{A}(\varkappa)\varphi(\vartheta)=\begin{cases}\frac{d\varphi(\vartheta)}{d\vartheta}, & \vartheta\in[-1,0),\\ \\ \int_{-1}^{0}d\mathfrak{T}(m,\varkappa)\varphi(m), & \vartheta=0,\end{cases} \tag{35}$$

and

$$\mathcal{P}(\varkappa)\varphi(\theta\,\vartheta)=\begin{cases}0, & \vartheta\in[-1,0)\\ \\ f(\varkappa,\varphi), & \vartheta=0.\end{cases} \tag{36}$$

Thus, system (3) is equivalent to operator differential equation:

$$u'(t)=\mathcal{A}(\varkappa)\,u_t+\mathcal{P}(\varkappa)\,u_t \tag{37}$$

where, $u_t=u(t+\vartheta),\ \vartheta\in[-1,0]$.
Now, for $\mathfrak{y}\in C^1([-1,0],(\mathbb{R}^3)^*)$, the adjoint operater $\mathcal{A}^*$ of $\mathcal{A}(0)$ is defined as:

$$\mathcal{A}^*\mathfrak{y}(m)=\begin{cases}-\frac{d\mathfrak{y}(m)}{dm}, & m\in(0,1]\\ \\ \int_{-1}^{0}\mathfrak{y}(-t)d\varphi^T(t,0), & m=0\end{cases} \tag{38}$$

and a bilinear form

$$\langle\mathfrak{y}(\mathcal{S}),\varphi(\vartheta)\rangle=\bar{\mathfrak{y}}(0)\ \ \varphi(0)-\int_{\vartheta=-1}^{0}\int_{\in=0}^{\vartheta}\bar{\mathfrak{y}}(\in-\theta)\ \ d\mathfrak{T}(\vartheta)\ \ \varphi(\in)\ \ d\in, \tag{39}$$

where, $\mathfrak{T}(\vartheta)=\mathfrak{T}(\vartheta,0)$, $\mathcal{A}(0)$ and $\mathcal{A}^*$ are obviously adjoint operators. For $\varkappa=0$,
by using a straightforward calculation, we can determine that $p(\vartheta)=(1,p_1,p_2)^T\,e^{i\omega_0^*\tau_0\vartheta}$ be

the eigenvector of $\mathcal{A}(0)$ which corresponding to $i\omega_0^*\tau_0$ and $p^*(m) = D(1, p_1^*, p_2^*)^T \ e^{-i\omega_0^*\tau_0 m}$ be the eigenvector of $\mathcal{A}^*$ which corresponding to $-i\omega_0^*\tau_0$

where

$$p_1 = -\frac{f_{1000}^{(2)}+f_{0001}^{(2)} \ e^{-i\tau_0\omega_0^*} \ p_2}{(f_{0100}^{(2)}+f_{0010}^{(2)} \ e^{-i\tau_0\omega_0^*})-i\omega_0^*},\ p_2 = \frac{iw_0-(f_{110}^{(1)}+ \ f_{010}^{(1)}e^{-i\tau_0\omega_0^*})}{f_{10000}^{(2)}+f_{00010}^{(2)} \ e^{-i\tau_0\omega_0^*}},$$
$$p_1^* = -\frac{(f_{010000}^{(3)}+f_{000010}^{(3)} \ e^{-i\tau_0\omega_0^*})p_2^*}{f_{0100}^{(2)}+f_{0010}^{(2)} \ e^{-i\tau_0\omega_0^*}+i\omega_0^*},$$
$$p_2^* = -\frac{\left[f_{110}^{(1)}+f_{010}^{(1)} \ e^{-i\tau_0\omega_0^*}+i\omega_0^*\right]\left[f_{0100}^{(2)}+f_{0010}^{(2)} \ e^{-i\tau_0\omega_0^*}+i\omega_0^*\right]}{f_{1000}^{(2)}\left[f_{010000}^{(3)}+f_{000010}^{(3)} \ e^{-i\tau_0\omega_0^*}\right]+\left[f_{100000}^{(3)}++f_{00100}^{(3)} \ e^{-i\tau_0\omega_0^*}\right]\left[f_{0100}^{(2)}+f_{0010}^{(2)} \ e^{-i\tau_0\omega_0^*}+i\omega_0^*\right]}.$$

From bilinear inner product (39), we get:

$$\begin{aligned}\langle p^*(m), p(\vartheta)\rangle = \bar{\mathcal{D}}(1 + \bar{p}_1^* p_1 + \bar{p}_2^* p_2 + \tau_0 e^{-i\omega_0^*\tau_0}(f_{010}^{(1)} + f_{001}^{(1)} p_2)\\ +\bar{p}_1^*\tau_0 e^{-i\omega_0^*\tau_0}\left(f_{0010}^{(2)} p_1 + f_{0001}^{(2)} p_2\right)\\ +\bar{p}_2^*\tau_0 e^{-i\omega_0^*\tau_0}\left(f_{000100}^{(3)} + f_{000010}^{(3)} p_1 + f_{000001}^{(3)} p_2\right)\end{aligned} \tag{40}$$

$$\text{Let}, \mathcal{D} = \begin{pmatrix} 1 + \bar{p}_1^* p_1 + \bar{p}_2^* p_2 + \tau_0 e^{-i\omega_0^*\tau_0}(f_{010}^{(1)} + f_{001}^{(1)} p_2 \\ +\bar{p}_1^*\tau_0 e^{-i\omega_0^*\tau_0}\left(f_{0010}^{(2)} p_1 + f_{0001}^{(2)} p_2\right) \\ +\bar{p}_2^*\tau_0 e^{-i\omega_0^*\tau_0}\left(f_{000100}^{(3)} + f_{000010}^{(3)} p_1 + f_{000001}^{(3)} p_2\right) \end{pmatrix}^{-1}$$

where, $\mathcal{D}$ is a complex number and $\bar{\mathcal{D}}$ is the conjugate for $\mathcal{D}$, then $\langle p^*, p\rangle = 1$ and $\langle p^*, \bar{p}\rangle = 0$.

In the following, the characteristics of the Hopf bifurcation are determined using similar arguments as in [25]:

$$\left.\begin{aligned} g_{20} &= 2\tau_0\bar{\mathcal{D}}(\mathfrak{B}_1 + \mathfrak{B}_5\bar{p}_1^* + \mathfrak{B}_9\bar{p}_2^*) \\ g_{11} &= \tau_0\bar{\mathcal{D}}(\mathfrak{B}_2 + \mathfrak{B}_6\bar{p}_1^* + \mathfrak{B}_{10}\bar{p}_2^*) \\ g_{02} &= 2\tau_0\bar{\mathcal{D}}(\mathfrak{B}_3 + \mathfrak{B}_7\bar{p}_1^* + \mathfrak{B}_{11}\bar{p}_2^*) \\ g_{21} &= 2\tau_0\bar{\mathcal{D}}(\mathfrak{B}_4 + \mathfrak{B}_8\bar{p}_1^* + \mathfrak{B}_{12}\bar{p}_2^*) \end{aligned}\right\} \tag{41}$$

where

$\mathfrak{B}_1 = f_{011}^{(1)} \ p_2 \ e^{-2i\omega_0^*\tau_0}$,

$\mathfrak{B}_2 = f_{011}^{(1)} \ (p_1 + \bar{p}_2)$,

$\mathfrak{B}_3 = f_{011}^{(1)} \ \bar{p}_2 e^{-2i\omega_0^*\tau_0}$,

$$\mathfrak{B}_4 = f_{011}^{(1)} \left(p_2 w_{11}^{(1)}(-1)e^{-i\omega_0^*\tau_0} + \tfrac{1}{2} \ \bar{p}_2 \ w_{20}^{(1)}(-1)e^{i\omega_0^*\tau_0} + \tfrac{1}{2} \ w_{20}^{(3)}(-1)e^{i\omega_0^*\tau_0} + w_{11}^{(3)}(-1) + w_{11}^{(3)}(-1)e^{-i\omega_0^*\tau_0}\right),$$

$\mathfrak{B}_5 = f_{0011}^{(2)} \ p_1 p_2 e^{i\omega_0^*\tau_0}$,

$\mathfrak{B}_6 = 2f_{0011}^{(2)} \ p_1 p_2$,

$\mathfrak{B}_7 = f_{0011}^{(2)}\bar{p}_1 \ \bar{p}_2 e^{2i\omega_0^*\tau_0}$,

$$\mathfrak{B}_8 = f_{0011}^{(2)}\left(\frac{\bar{p}_2}{2} w_{20}^{(2)}(-1)e^{i\omega_0^*\tau_0} + p_2 w_{11}^{(2)}(-1)e^{-i\omega_0^*\tau_0} + p_1 w_{11}^{(3)}(-1)e^{-i\omega_0^*\tau_0} + \frac{\bar{p}_1}{2} w_{20}^{(3)}(-1)e^{i\omega_0^*\tau_0}\right),$$

$$\mathfrak{B}_9 = f_{000101}^{(3)}\ p_2\ e^{-2i\omega_0^*\tau_0} + f_{000011}^{(3)}\ p_1 p_2\ e^{2i\omega_0^*\tau_0},$$

$$\mathfrak{B}_{10} = f_{000101}^{(3)}(p_2 + \bar{p}_2) + f_{000011}^{(3)}\ (p_1\bar{p}_2 + \bar{p}_1 p_2),$$

$$\mathfrak{B}_{11} = f_{000101}^{(3)}\ \bar{p}_2\ e^{2i\omega_0^*\tau_0} + f_{000011}^{(3)} p_1 p_2 e^{-2i\omega_0^*\tau_0},$$

$$\mathfrak{B}_{12} = f_{000101}^{(3)}\left(p_2 w_{11}^{(1)}(-1)e^{-i\omega_0^*\tau_0} + \frac{1}{2}\bar{p}_2 w_{20}^{(1)}(-1)e^{i\omega_0^*\tau_0} + w_{11}^{(3)}(-1)e^{-i\omega_0^*\tau_0} + \frac{1}{2} w_{20}^{(3)}(-1)e^{i\omega_0^*\tau_0}\right) + f_{000011}^{(3)}\left(\frac{\bar{p}_2}{2} w_{20}^{(2)}(-1)e^{i\omega_0^*\tau_0} + p_2 w_{11}^{(2)}(-1)e^{-i\omega_0^*\tau_0} + p_1 w_{11}^{(3)}(-1)e^{-i\omega_0^*\tau_0} + \frac{\bar{p}_1}{2} w_{20}^{(3)}(-1)e^{i\omega_0^*\tau_0}\right)$$

.

with

$$w_{20}(\vartheta) = \frac{ig_{20}}{\omega_0^*\tau_0}\ p(0)\ e^{i\omega_0^*\tau_0\vartheta} + \frac{i\bar{g}_{02}}{3\omega_0^*\tau_0}\ \bar{p}(0)\ e^{-i\omega_0^*\tau_0\vartheta} + \mathcal{E}_1\ e^{2i\omega_0^*\tau_0\vartheta}. \tag{42}$$

$$w_{11}(\vartheta) = -\frac{ig_{11}}{\omega_0^*\tau_0}\ p(0)\ e^{i\omega_0^*\tau_0\vartheta} + \frac{i\bar{g}_{11}}{\omega_0^*\tau_0}\bar{p}(0)\ e^{-i\omega_0^*\tau_0\vartheta} + \mathcal{E}_2. \tag{43}$$

Notice that, $\mathcal{E}_1 = \left(\mathcal{E}_1^{(1)}, \mathcal{E}_1^{(2)}, \mathcal{E}_1^{(3)}\right)^T$ and $\mathcal{E}_2 = \left(\mathcal{E}_2^{(1)}, \mathcal{E}_2^{(2)}, \mathcal{E}_2^{(3)}\right)^T$ can be determined from the following equations:

$$J_1^*\ \mathcal{E}_1 = 2\tau_0 J_1. \tag{44}$$

$$J_2^*\ \mathcal{E}_2 = -\tau_0 J_2. \tag{45}$$

where,

$$J_1^* = \left(2i\omega_0^*\tau_0 I - \int_{-1}^0 d\mathfrak{T}(\vartheta)\ e^{2i\omega_0^*\tau_0\vartheta}\right),$$

$$J_2^* = \left(\int_{-1}^0 d\mathfrak{T}(\vartheta)\right),$$

$$J_1 = (\mathfrak{B}_1\ \ \mathfrak{B}_5\ \ \mathfrak{B}_9)^T,$$

$$J_2 = (\mathfrak{B}_2\ \ \mathfrak{B}_6\ \ \mathfrak{B}_{10})^T.$$

Accordingly, it is obtained that:

$$J_1^* = \begin{bmatrix} 2i\theta_0 - (f_{100}^{(1)} + f_{010}^{(1)} e^{2i\theta_0\tau_0\vartheta}) & 0 & -f_{001}^{(1)} e^{e^{2i\theta_0\tau_0\vartheta}} \\ -f_{1000}^{(2)} & 2i\theta_0 - (f_{0100}^{(2)} + f_{0010}^{(2)} e^{e^{2i\theta_0\tau_0\vartheta}}) & -f_{0001}^{(2)} e^{e^{2i\theta_0\tau_0\vartheta}} \\ -(f_{100000}^{(3)} + f_{000100}^{(3)} e^{2i\theta_0\tau_0\vartheta}) & 0 & 2iw_0 - (f_{001000}^{(3)} + f_{000001}^{(3)} e^{2i\theta_0\tau_0\vartheta}) \end{bmatrix}$$

$$J_2^* = \begin{bmatrix} -(f_{100}^{(1)} + f_{010}^{(1)}) & 0 & -f_{001}^{(1)} \\ -f_{1000}^{(2)} & -(f_{0100}^{(2)} + f_{0010}^{(2)}) & -f_{0001}^{(2)} \\ -(f_{100000}^{(3)} + f_{000100}^{(3)}) & 0 & -(f_{001000}^{(3)} + f_{000001}^{(3)}) \end{bmatrix}.$$

Thus, $\mathcal{E}_1^{(i)} = \frac{2\mathfrak{T}_i}{\mathfrak{T}}, i = 1, \dots, 3,$ where $\mathfrak{T} =$ determinant of (J_1^*) and $\mathfrak{T}_i$ is the value of the determinant v_i, where v_i is produced by substituting the i^{th} column vector of J_1^* by J_1 for $i = 1, ..,3$. In the same way, $\mathcal{E}_2^{(i)} = \frac{2\bar{\mathfrak{T}}_i}{\bar{\mathfrak{T}}}, i = 1, \dots, 3,$ where $\bar{\mathfrak{T}}$ is the determinant of (J_2^*) and $\bar{\mathfrak{T}}_i$ is the value of the determinant u_i, u_i is produced by substituting the i^{th} colum vector of J_2^* by J_2 for $i = 1, \dots, 3$.

As a result, equations (42)-(45) can be used to calculate $w_{20}(\vartheta)$ and $w_{11}(\vartheta)$. The proof is completed by determining the terms in equation (30) based on those in equation (41).

5. Numerical Simulation and Discussion .

The main outcomes are numerically depicted using the biologically feasible hypothetical parameter provided set. The objective is to validate the theoretically obtained findings and understand the impact of these parameters on the system's dynamics (3). MATLAB was used to execute all simulations. Furthermore, the values of the following hypothetical parameter set are selected as being biologically plausible.

$$\Lambda = 50, \alpha = 0.3, \square_1 = 0.001, \square_2 = 0.001, \beta_1 = 0.001\ \beta_2 = 0.001, d = 0.4,;\ \Psi = 0.5,$$
$$\tau = 6.3, n = 25$$
(46)

It is noticed that for the data given by equation (46), the trajectories of the system (3) approach to E^0 for all $\tau \geq 0$, as shown in Figure (1).

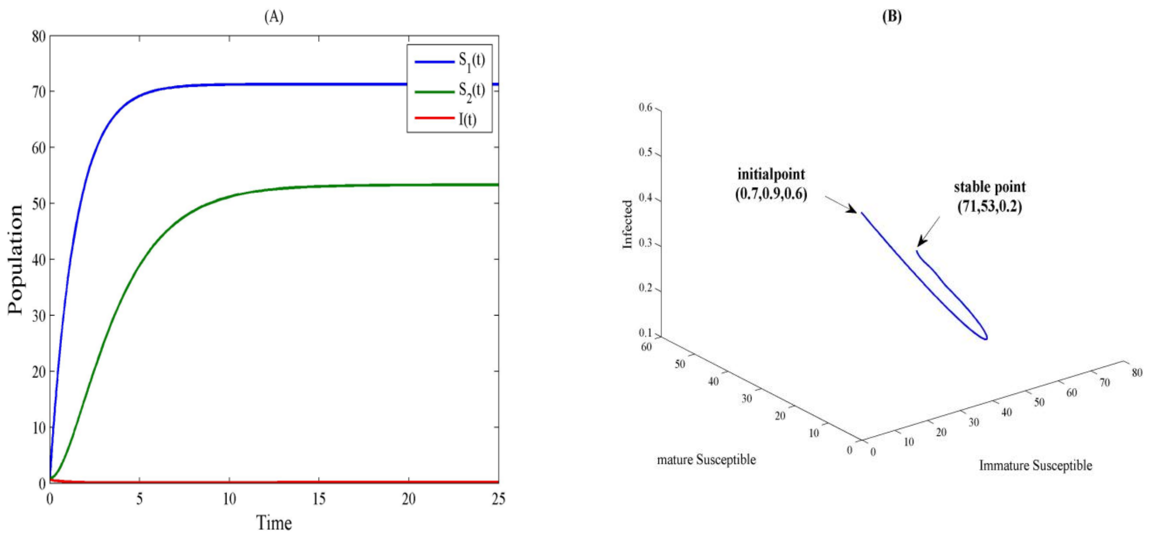

Figure 1: The system's (3) trajectories utilizing Dataset from equation (46).
(A) The system's (3) trajectories approach to E^0. (B) 3D phase plot for globally asymptotically stable E^0.

We examine the influence of time delay on the behavior of the system (3) around the E^* point.

We investigate how the time delay affects the system's (3) behavior around the E^* point.

Clearly, Theorem 2 requirements are met for the parameters in the data of equation (46) with ⍰$_2 = 0.01$and $\beta_1 = \beta_2 = 0.01$. It is noted that when $\tau = 6.3$ the E^* point is asymptotic stable as shown in Figure (2) while for $\tau = \tau_0 = 6.4$ a Hopf bifurcation occurs at E^*as shown in Figure (3).

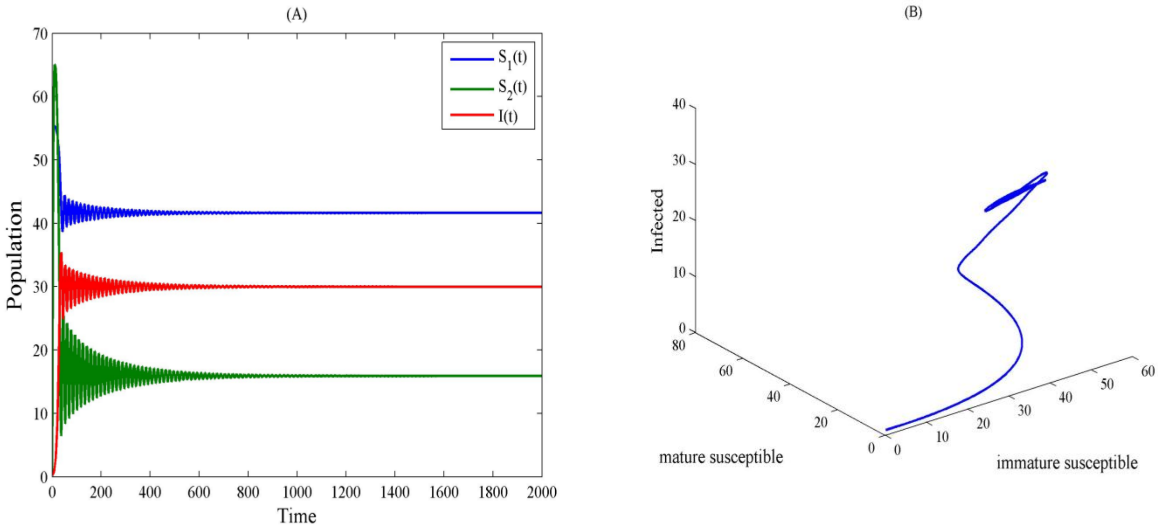

Figure 2: The system's (3) trajectories utilizing Dataset from equation (46) with . ⍰$_2 = 0.01 and\ \beta_1 = \beta_2 = 0.01$. and (A) The system's (3) trajectories approach to E^* . (B) 3D phase plot for globally asymptotically stable E^* “.

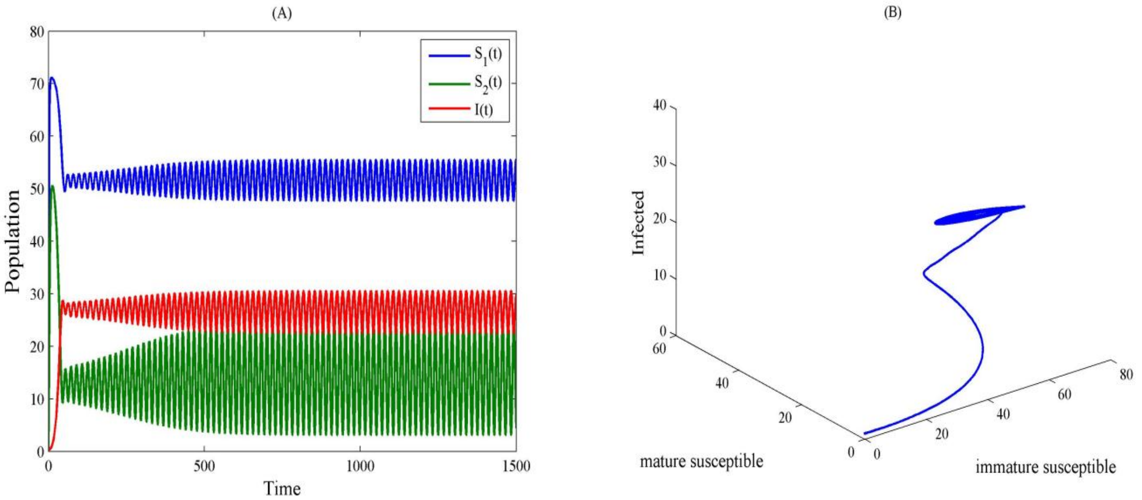

Figure 3: The system's (3) trajectories utilizing Dataset from equation (46) with ⍰$_2 = 0.01, \beta_1 = \beta_2 = 0.01.\ and\ \tau = \tau_0 = 6.4$.(A) The presence of a periodic solution near E^* “ (B) 3D periodic solution.

6. Conclusions

An epidemic model incorporating a temporal lag for the incidence rate has been formulated and examined. The introduction of a time delay into the system enhances the accuracy of representing real-world dynamics in disease spread, as it factors in the time needed for individuals who have been affected to transition into an infectious state and transmit the disease to others. With the inclusion of the SIR (Susceptible-Infectious-Removed) disease classification, the population is segregated into three categories: susceptible, infected, and removed. Within the susceptible group, a stage-based structure is presumed, resulting in two subgroups: immature susceptible and mature susceptible. The equilibrium points of the proposed system are denoted as UEP and ENEP. The boundedness of the system has been thoroughly examined. When $\tau \geq 0$, the UEP is deemed absolutely stable. Analyzing the ENEP revealed that for $\tau \in [0, \tau_0)$, it exhibits local asymptotic stability; otherwise, it becomes unstable, leading to a Hopf bifurcation occurring at $\tau = \tau_0$. Furthermore, we constructed a precise algorithm and essential criteria for analyzing the stability and direction of bifurcating periodic solutions, employing the normal form technique and the center manifold theorem. Finally, numerical simulation was employed to verify the findings and comprehend the impact of varying system parameters, utilizing a hypothetical set of parameter values.

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