 Iraqi Journal of Science, 2019, Vol. 60, No.11, pp: 2473-2477
DOI: 10.24996/ijs.2019.60.11.18

ISSN: 0067-2904

ON RICKART MODULES

Mohammed Qader Rahman, Bahar hamad Al-Bahrani
Department of mathematics, College of Science, University of Baghdad, Baghdad, Iraq

Received: 14/4/ 2019 Accepted: 17/ 7/2019

Abstract
Gangyong Lee, S.Tariq Rizvi, and Cosmin S.Roman studied Rickart modules.
The main purpose of this paper is to develop the properties of Rickart modules . We prove that each injective and prime module is a Rickart module. And we give characterizations of some kind of rings in term of Rickart modules.

Keywords: Endomorphism ring, Direct summand, Kernel of endomorphism , Rickart modules and modules with the summand intersection property.

حول مقاسات الريكارتية

محمد قادر رحمان* ,بهار حمد البحراني
قسم الرياضيات ، كلية العلوم ، جامعه بغداد ، بغداد ، العراق

الخلاصه
Gangyong Lee, S.Tariq Rizvi, and Cosmin S.Roman
درسوا المقاسات الريكارتية. الهدف الرئيسي من هذا البحث هو تطوير خواص المقاسات الريكارتية . برهنا ان كل مقاسا اغماري و اولي يكون مقاسا ريكارتيا وايضا اعطينا تعاريف مكافئه لبعض انواع الحلقات بواسطة المقاسات الريكارتية .

1 INTRODUCTION

A module M is called a Rickart module if for every $\varphi \in S = End(M)$, then $ker\varphi = eM$ for some $e^2 = e \in S$.Equivalently a module is a Rickart module if and only if for every $\varphi \in S = End(M)$, then $ker\varphi$ is a direct summand of M, See [1], [2] .

In this paper, we give some results on the Rickart modules .

In §2 ,we give characterization of the Rickart modules. Also we study the direct sum of Rickart modules. For example we prove that an R-module M is Rickart if and only if $A_M \cap T_f \leq_{\oplus} M \oplus M$, for every endomorphism $f: M \to M$, see Theorem (2.2).

In section 3, we give characterizations of certain classes of rings in term of the Rickart modules. For example we prove that a ring R is semisimple if and only if all injective R-module is Rickart , see Theorem (3.12).

Throughout this article, R is a ring with identity and M is a unital left R-module. For a left module $M, S = End_R(M)$ will denote the endomorphism ring of M. The notations $N \leq M, N \leq_{\oplus} M$ mean that N is a submodule, a direct summand of M.

2 CHARACTERIZATIONS OF RICKART MODULES

In this section , we give a characterizations for the Rickart modules. Following [1] , A module M is called a Rickart module if for every $\varphi \in S = End(M)$, $ker\varphi = eM$, for some $e^2 = e \in S$. It's known that every direct summand of a Rickart module is a Rickart module.

Remark 2.1: Let M be an R-module and $f: M \to M$ be an R-homomorphism.

*Email: Mohammed_qader_0@yahoo.com

Let $A_M = M \oplus 0$, $B_M = 0 \oplus M$ and $\bar{f}: A_M \to B_M$ be a map defined by
$\bar{f}(m,0) = (0, f(m))$, for every $m \in M$. It is clear that $M \oplus M = A_M \oplus B_M$, $\bar{f}$ is an R-homomorphism and $ker\bar{f} = kerf \oplus 0$. Let $T_f = \{x + \bar{f}(x), x \in A_M\}$.Clearly that T_f is a submodule of $M \oplus M$.

In this paper by A_M , B_M , $\bar{f}$, T_f we mean the same concepts in the previous above Remark.

Theorem 2.2: An R- module M is Rickart module if and only if for every R- homomorphism $f: M \to M$, $A_M \cap T_f$ is a direct summand of $M \oplus M$.

Proof: Let M is Rickart module and $f: M \to M$ be an R-homomorphism. Then $\ker f$is a direct summand of M and hence $ker\bar{f} = kerf \oplus 0$ is a direct summand of $M \oplus M$. Claim that $ker\bar{f} = A_M \cap T_f$. To show that ,let $(m,0) \in ker\bar{f}$. Then
$(m,0) = (m,0) + \bar{f}(m,0) \in A_M \cap T_f$. Now let $(m,0) \in A_M \cap T_f$. So there exists $m_1 \in M$ such that $(m,0) = (m_1,0) + \bar{f}(m_1,0) = (m_1,0) + (0, f(m_1)) = (m_1, f(m_1))$. Hence $m = m_1$ and $f(m_1) = 0$. Thus $(m,0) \in ker\bar{f}$.

For the converse, since $A_M \cap T_f = ker\bar{f} \leq_{\oplus} M \oplus M$ and $ker\bar{f} \leq A_M$,Then
$ker\bar{f} = kerf \oplus 0 \leq_{\oplus} M \oplus 0$. So $kerf \leq_{\oplus} M$. Thus M is a Rickart module.

Recall that An R-module M is called a prime R-module if ann(x)=ann(y), for every non zero elements x and y in M [3].

In the following proposition we give conditions under which an R-module M can be Rickart.

Propositions 2.3: Let M be an injective and prime R-module, then M is a Rickart module

Proof: Let $f: M \to M$ be an R-homomorphism. Since M is injective and prime. Then $M \oplus M$ is injective and prime. Since $A_M \leq_{\oplus} M \oplus M$, Then A_M is injective.

First claim that $M \oplus M = T_f \oplus B_M$. To show that let $(x,y) \in M \oplus M$. Hence
$(x,y) = (x,0) + (0, f(x)) - (0, f(x)) + (0,y)$. It is clear that $(x,0) + (0, f(x)) \in T_f$ and $-(0, f(x)) + (0,y) \in B_M$. So $M \oplus M = T_f + B_M$.

Now let $(m,0) + \bar{f}(m,0) \in T_f \cap B_M$. $(m, f(m)) \in B_M = 0 \oplus M$ and hence m=0. Thus $M \oplus M = T_f \oplus B_M$. Thus T_f is injective .

Let I be an ideal of R and $g: I \to A_M \cap T_f$ be a non zero homomorphism.

Let $i_1: A_M \cap T_f \to A_M$ and $i_2: A_M \cap T_f \to T_f$ be the inclusion homomorphisms.

Thus $i_1 \circ g: I \to A_M$ and $i_2 \circ g: I \to T_f$. By Baer's Criterion [4,Th(5.7.1)P.13] there exists $a \in A_M \; and \; b \in T_f$, such that $g(w) = wa \; and \; g(w) = wb, for \; each \; w \in I$. Thus $w(a-b) = 0$. Assume that $a \neq b$. Since $w \in ann(a-b) \; and$ M is prime, then $w \in ann(a)$.Thus $g = 0$, which is a contradiction, therefore $a = b \in A_M \cap T_f$ and hence $A_M \cap T_f$ is injectiv

Now consider the short exact sequence

$$0 \to A_M \cap \mathrm{T_f} \xrightarrow{i} \mathrm{M} \oplus \mathrm{M} \xrightarrow{\pi} \frac{M \oplus M}{A_M \cap \mathrm{T_f}} \to 0$$

Where i is the inclusion map and π be the natural epimorphism. The sequence splits, as shown by [4]. Hence $A_M \cap T_f \leq_{\oplus} M \oplus M$. By theorem (2.2), M is a Rickart module.

The converse of the above proposition is not always true. For example.

Consider Z_6 as Z-module.Z_6 is semismple and hence Z_6 is Rickart. But Z_6 is neither injective nor prime.

Propositions 2.4: Let M be an R-module such that for every homomorphism $f: M \to M$, $A_M + T_f$ is projective, then M is Rickart module.

Proof: Let $f: M \to M$ be an R-homomorphism, consider the following short exact sequences

$$0 \to A_M \cap \mathrm{T_f} \xrightarrow{i_1} A_M \xrightarrow{f_1} \frac{A_M}{A_M \cap \mathrm{T_f}} \to 0$$

$$0 \to \mathrm{T_f} \xrightarrow{i_2} A_M + \mathrm{T_f} \xrightarrow{f_2} \frac{A_M + \mathrm{T_f}}{\mathrm{T_f}} \to 0$$

Where i_1, i_2 are the inclusion homomorphisms and f_1, f_2 are the natural epimorphisms.

By the second isomorphism theorem$\frac{A_M}{A_M \cap T_f} \simeq \frac{A_M + T_f}{T_f}$. Since T_f is a direct summand of $M \oplus M$and $T_f \subseteq A_M + T_f$, then T_f is a summand of $A_M + T_f$. Thus the second squence splits. But $A_M + T_f$ is projective, thus $\frac{A_M}{A_M \cap T_f} \simeq \frac{A_M+T_f}{T_f}$ is projective. Hence the first squence splits. Thus $A_M \cap T_f$ is a summand of A_M. Since A_M is a summand of $M \oplus M$, then $A_M \cap T_f$ is a summand of $M \oplus M$. By the same argument of the prove of theorem 2.2, $kerf \oplus 0 = ker\bar{f} = T_f \cap A_M$, therefor $kerf \oplus 0$ is a summand of $M \oplus M$. Since $kerf \oplus 0 \subseteq A_M$, Then $kerf \oplus 0$ is a summand of A_M and hence $kerf$ is a summad of M. Thus M is a Rickart module

The converse is not true as the following example shows:

consider Z_6 as Z-module, Z_6 is semisimple and hence is a Rickart. It's is known that Z_6 is not projective. Let $f: Z \to Z$ be an R-homomorphism. One can easily show that

$A_{Z_6} + T_f = A_{Z_6} \oplus Im\bar{f}$.

Now assume that $A_{Z_6} + T_f$ is projective, then $A_{Z_6} \simeq Z_6$ is projective which is a contradiction.

However, we have the following

Theorem 2.5: Let M be a projective R-module, Then M is a Rickart module if and only if for every R-homomorphism $f: M \to M$, $A_M + T_f$ is a projective R-module.

Proof: Suppose that M is a Rickart module and let $f: M \to M$ be an R-homomorphism. Now consider the following short exact sequences

$$0 \to A_M \cap \mathrm{T_f} \xrightarrow{i_1} A_M \xrightarrow{\pi_1} \frac{A_M}{A_M \cap \mathrm{T_f}} \to 0$$

$$0 \to \mathrm{T_f} \xrightarrow{i_2} A_M + \mathrm{T_f} \xrightarrow{\pi_2} \frac{A_M + \mathrm{T_f}}{\mathrm{T_f}} \to 0$$

Where i_1, i_2 are the inclusion homomorphisms and $\pi_1, \pi_2$are the natural epimorphisms. Since M is a Rickart module, then $kerf$ is a summand of M. By the same argument of the prove of theorem 2.2 $erf \oplus 0 = ker\bar{f} = A_M \cap T_f$, hence $A_M \cap T_f$ is a summand of A_M. Thus the first squence splits. But A_M= $M \oplus 0 \simeq M$ and M is projective , there for A_M is projective. Then $\frac{A_M}{A_M \cap T_f}$ is projective. By the second isomorphism theorem

$\frac{A_M}{A_M \cap T_f} \simeq \frac{A_M+T_f}{T_f}$ .Thus $\frac{A_M+T_f}{T_f}$ is projective. Hence the second squence splits. But

$M \oplus M = T_f \oplus B_M$, therefor T_f is projective. Thus $A_M + T_f$ is projective,

(where $A_M + T_f \simeq T_f \oplus \frac{A_M+T_f}{T_f}$).

Let M and N be two R-modules. Recall that M is called N-Rickart (or relatively Rickart to N) if, for every R-homomorphism $f: M \to N$, $kerf$ is a summand of M [1].

Before giving our next resulte, we need the following.

Propositions 2.6.[1]: The following are equivalent for a module M

(1) M is a Rickart module ;

(2) For every submodule N of M, every direct summand L of M is N-Rickart.

Propositions 2.7: Let M ba an indecomposable R-module and let N be any R-module if M is N-Rickart, then either

(1) $Hom(M, N)$=0 or

(2) Every nonzero R-homomorphism from M to N is a monomorphism

Proof: Assume that $Hom(M, N) \neq 0$ and let $f: M \to M$ be a nonzero R –homomorphism, Since M is N-Rickart, then $kerf$ is a summand of M. But M is indecomposable. So $kerf = \{0\}$ and f is a monomorphism.

Recall that an R-module M is called a Quasi-Dedekind R-module if every nonzero endomorphism of M is a monomorphism [6 , Th(1.5) , CH2].

Corollary 2.8: Let M be an indecomposable R-module and let N be any R-module such that $Hom(M, N)$≠ 0 .If M is N-Rickart ,then M is Quasi-Dededkind. In particular if M is Rickart, then M is Quasi-Dedekind

Proof: By Prop.2.7, there is a monomorphism $f: M \to N$. Assume M is not Quasi-Dedekind R-module, therefore there exists a nonzero endomorphism $g: M \to M$ such that $kerg \neq 0$. Since f is a monomorphism ,then $\ker(f \circ g) = kerg \neq 0$. Since M is N-Rickart, then $kerg \leq_{\oplus} M$ and hence $kerg = M$, where M indecomposable. Thus

$g = 0$, which is a contradiction. Thus M is a Quasi-Dedekind R-module.

3 CHARACTERIZATIONS OF RINGS BY MEANS OF RICKART MODULES

It's known The direct sum of the Rickart modules need not be a Rickart module, see[1], [2].

In this section, we give a conditions under which a direct sum of Rickart modules is a Rickart module.

Proposition3.1: Let M be an R-module, If R is M-Rickart, then every cyclic submodule of M is projective. In particular if R is an R-Rickart module, then every Principale ideal is projective ideal, i.e. , R is a p.p.ring

Proof: Let $m \in M$, consider the following short exact sequence

$$0 \rightarrow kerf \overset{i}{\rightarrow} R \overset{f}{\rightarrow} Rm \rightarrow 0$$

where i is the inclusion homomorphism and f is defined as follows $f(r) = rm, \forall r \in R$. Let $i_2 : Rm \rightarrow M$ be the inclusion homomorphism. Now consider $i_2 \circ f : R \rightarrow M$. Since R is M-Rickart. Then $\ker(i_2 \circ f)\ is$ a summand of R. But i_2 is a monomorphism, therefore $kerf = \ker(i_2 \circ f) =$. Thus the sequence is split and hence Rm is summad of R, since R is a projective R-module. Then Rm is projective.

Recall that an R-module M is called dualizable if $Hom(M, R) \neq 0$, [5]

Corollary3.2: Let M be a dualizable indecomposable R-module and M is R-Rickart, then M is isomorphic to an ideal of R. Hence if R has no nonzero nilpotent elements, then $E(M)$ is commutative ,where $E(M)$ is the ring of R- endomorphism of M.

Proof: Since $Hom(M, R) \neq 0$, then by Prop (2.7) M is isomorphic to an ideal I of R and hence $E(M) \cong E(I)$. For the second part, since R has no nilpotent elements and I is an ideal in R, Then $E(I)$ is commutative[7,prop (2.1)CH1]. Thus $E(M)$ is commutative

Corollary3.3: Let M be a projective indecomposable R-module and R has no nonzero nilpoten element. If M is an R-Rickart module and $Hom(M, R) \neq 0$, then M is a multiplication module.

Proof: By the same argument of the proof of Cor(3.2) , $E(M)$ is a commutative and hence M is multipliction [8]

Recall that an R-module M is called an SIP module if the intersection of any two direct summands of M is also a direct summand of M [9]. It is known that every Rickart module is an SIP module [1].

Before we give our next result , we need the following

Theorem 3.4.[9]: Let R be a Noetherian domain and let M be an injective R-module .If M has the SIP, then either

(1) M is torsion free or

(2) M is torsion and for any two distinct indecomposable summands A and B of M, $Hom(A, B)$=0

Now, we prove that

Theorem 3.5: Let R be a Noetherian domain and let M be an injective R-module, then the following are equivalent

(1) $M \oplus M$ is a Rickart module .

(2) M is torsion free .

(3) $\oplus_{\wedge} M$ is Rickart module, for every index set $\wedge$.

Proof: (1)⇒(2) Since M is a summand of $M \oplus M$, then M is Rickart and hence M has the SIP. By Th (3.4) M is either torsion or torsion free. Suppose M is torsion, so $M \oplus M$ is torsion. Since R is noetherian domain, then by [4,Th(6.6.2),p.162], $M = \oplus_{\propto \in \wedge} M_{\propto}$ where $M_{\propto}$ is an indecomposable submodule of M, now let $\propto_{\circ} \in \Lambda$ and hence

$$\mathrm{M} \oplus \mathrm{M} \cong \mathrm{M}_{\propto_\circ} \oplus \mathrm{M}_{\propto_\circ} \oplus \left[\bigoplus_{\substack{\propto \in \wedge \\ \propto \neq \propto_\circ}} \mathrm{M}_{\propto}\right] \oplus \left[\bigoplus_{\substack{\propto \in \wedge \\ \propto \neq \propto_\circ}} \mathrm{M}_{\propto}\right]$$

mmmmmNow $M_{\propto_\circ} \oplus M_{\propto_\circ}$ is Rickart and injective ,thus by Cor (2.8) $M_{\propto_\circ}$is Quasi-Dedekind and hence $M_{\propto_\circ}$ is prime by [6,prop (1.7),p.26] which is a contradiction.To verify this suppose $M_{\propto_\circ}$ is prime. Since M is torsion ,then $M_{\propto_\circ}$ is torsion. But $M_{\propto_\circ}$ is injective over integral domain ,therefor $M_{\propto_\circ}$ is divisible. Now let $0 \neq x \in M_{\propto_\circ}$ and let 0≠r ∈ annx . Since $M_{\propto_\circ}$ is divisible , then x=ry for some $y \in M_{\propto_\circ}$.Thus $M_{\propto_\circ}$ is not prime.

(2)⇒(1) since M is torsion free, then $M \oplus M$ is torsion free. Hence $M \oplus M$ is prime and injective. Thus $M \oplus M$ is a Rickart module ,by Prop (2.3).

(2)⇒ (3) Since M is torsion free, then M is prime and hence $\bigoplus_{\wedge} M$ is prime for every index set $\wedge$.But $\bigoplus_{\wedge} M$ is injective, then by Prop (2.3), $\bigoplus_{\wedge} M$ is a Rickart module.
Recall that A ring R is a left semihereditary if every finitely generated left ideal is projective[10].
Befor we give our next result, we need the following
Theorem 3.6.[2]: A ring R is a left semihereditary if and only if every finitely generated projective (free) R-module is a Rickart module.
Theorem 3.7: The following statements are equivalent for a commutative ring R
(1) R is a semiheredatary ring.
(2) $\bigoplus_I R$ is Rickart for every finite index set I.
(3) $R \oplus R \oplus R$ is a Rickart R-module
Proof:(1)⇔(2) Clear by Th 3.6
(2)⇒ (3) Clear
(3)⇒(1) Let $I = Ra + Rb$ be two generated ideal in R.
Define $f: R \oplus R \to Ra + Rb$ by $f(r_1, r_2) = r_1 a + r_2 b$ .It is clear that f is an epimorphism. Let $i: Ra + Rb \to R$ be the inclusion map. Since $i \circ f: R \oplus R \to R$ and $R \oplus R \oplus R$ is Rickart, then $\ker(i \circ f)$ is a summand by Prop(2.6) .It is clear that i is a monomorphism, therefore $\ker(i \circ f) = kerf$ is a summand of R, Thus $Ra + Rb$ is a projective R-module. One can show that R is semihereditary[11].
We end this section by the following
Theorem 3.8: The following conditions are equivalent for a ring R
(1)R is semisimple.
(2)All R-modules are Rickart.
(3)All injective R-modules are are Rickart.
Proof: (1)→(2)→(3) It is clear.
(3)→ (1) Let Mbe any R-module, there is an injective R-module E_1 and a monomorphism $g_1: M \to E_1$, by [4] Likewise, there is an a monomorphism $g_2: \frac{E_1}{Img_1} \to E_2$, for some injective R-module E_2. Let $f: E_1 \to \frac{E_1}{Img_1}$ be the natural epimorphism. Now consider $g_2 \circ f: E_1 \to E_2$. Note that $E_1 \oplus E_2$ is injective and hence by assumption $E_1 \oplus E_2$ is Rickart, then by Prop(2.6) E_1 is E_2-Rickart.Thus $kerg_2 \circ f$ is a summand of E_1. But g_2 is a monomorphism, then $\ker(g_2 \circ f) = kerf = Img_1$ is summand of E_1, Thus Img_1 is injective. Since $M \cong Img_1$. Then M is injective. Then by [4.cor(8.2.2)P.196] R is semisimple.

References

1. Lee, G., SRiziv, S. and Roman, C.S. **2010.** Rickart modules , *Comm in Algebra*, **38**: 4005-4027.
2. Lee, G., Riziv, S.T. and Roman, C.S. **2012**. Direct sum of Rickart modules , *Journal of Algebra*, **353**: 62-78.
3. Desale, G. and Nicholson, W.K. **1981**. Endoprimitive rings , *J. Algebra*, **70**: 548-560
4. Kasch, F.**1982.** *Modules and Rings*, Acad .press, London.
5. Hamdouni, A. and Harmanci, A. **2005**. Characterization of modules and rings by the summand intersection property and the summand sum property, *JP Journal of Algebra, Number Theory and applications*.
6. Ghawi. Th.Y. **2010.** Some Generalizations of Quasi-Dedekind Modules, M.Sc.Thesis , Collage of Education Ibn AL-Haitham, University of Baghdad.
7. .AL-Aubaidy**,** W.K. **1993**. The ring of endomorphism of multiplications modules ,M.SC. Thesis, University of Baghdad.
8. Naoum**,** A.G. **1991**. A note on projective module and multiplications modules , *Beitrage Zur Algebra and Geometry*, **32**: 27-32.
9. Wilson**,** G.V. **1986**.*Modules with the summand intersection property, Comm.Algebra*. **14**: 21-38.
10. Clark, J., Lomp, C., Vanaja, N. and Wisbauer**,** R. **2006**. *Lifting modules,* Frontiers in Mathematics Birkhauser.

A. Naoum, G. **1985.** On finitly generated projective and flat ideals on commutative rings ,*Periodica mathematica* ,251-260.