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New Type of Grill Topological Spaces

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Abstract
The primary objective of this work pertains to the examination of a new class of grill topological spaces denominated as AE-space. This nomenclature is rooted in the delineation of said spaces upon a non-empty set, upon which a myriad of topological structures is imposed via an arbitrary grill. The construction of these grill topologies precedes the establishment of our nascent topological construct through the intersection of these individualized topologies. In this discourse, we introduce and rigorously define the notions of AE-open sets and AE-closed sets. Furthermore, we establish the result that the AE-space exhibits the fundamental attributes of a topological space, assuming the absence of pairwise disjoint non-empty sets within the intersection of the grill topologies. Illustrative instances of such AE-spaces are furnished, accompanied by a comprehensive demonstration of their pertinent properties.

Keywords: Grill, AE-open set, AE -closed set, AE -space, AE -interior operator and AE -closure operator.

نوع جديد من الفضاءات التبولوجية الگرلية

رنا بهجت اسماعيل
الرياضيات, كلية التربية للعلوم الصرفة – ابن الهيثم, جامعة بغداد, بغداد, العراق

الخلاصة
الهدف الأساسي من هذا العمل يتمثل في تعريف نوع جديد من الفضاءات التبولوجية الگرلية المعروفة باسم "فضاء AE ". ينبع هذا المصطلح من وصف هذه الفضاءات على مجموعة غير خالية، حيث يتم فرض مجموعة متنوعة من الفضاءات التوبولوجية الگرلية على مجموعه. يسبق بناء هذه التبولوجيات الگرلية إنشاء بنية توبولوجية جديدة من خلال تقاطع هذه التوبولوجيات الگرلية. في هذا السياق، نقوم بتقديم وتعريف دقيق لمفاهيم المجموعات المفتوحة "AE" والمجموعات المغلقة "AE". علاوة على ذلك، نثبت النتيجة التي تفيد بأن الفضاء "AE" يتمتع بالسمات الأساسية لفضاء توبولوجي، على أن نفترض عدم وجود مجموعات غير خالية داخل تقاطع التبولوجيات الگرلية. نقدم أمثلة توضيحية للفضاءات "AE"، مرفقة بشرح شامل للخصائص ذات الصلة بالمفهوم قيد الدراسة.

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1. Introduction

Grill topological spaces are a given set with a topology based on the grill concept, where the open set is generalized to topological spaces, using the grill concept [1].

In grill topological spaces, topology is defined using a set of functions that cover the space and that determine how points are close together, in addition to the definition of grill-open sets. This allows the emergence of different patterns for defining topology that are more precise than the traditional definition of topology and more general. Simply put, grill topological spaces provide a different perspective for understanding topology which is useful in studying certain properties of topological spaces [2], [3].

The notion of grill was first introduced by Choquet [1], later many studies appeared on this concept. For example, we mention the most important studies of the present time. In 2021, Mustafa and Esmaeel [2], presented new generalizations of separation axioms using the concept of grill. In 2021, Mustafa and Esmaeel [3], studied new properties of open and closed sets in grill-topological space. In 2022, Suliman and Esmaeel [4], popularized the notion of α-open set via grill concept and defined new types of continuous functions using this notion.

2. Preliminaries

Definition 2.1 [5]

Consider a given topological space denoted as (X, τ). Within this context, a grill structure on the set X is defined as an assemblage G of non-empty subsets of X that adheres meticulously to the following stipulations:

- The null set ∅ is conspicuously excluded from the collection G.
- For any set A belonging to G, if A is a subset of another set B, then B must also be a constituent of G.
- Whenever two sets, A and B, both of which are absent from G, are encountered, the union of these sets, symbolized as A ∪ B, is rigorously precluded from membership in G.

When we conjoin a preexisting topological space (X, τ) with a concomitant grill structure G imposed upon the set X, a distinctive form of topological space is engendered, designated herein as a grill topological space. Notably, this unique spatial construct is succinctly denoted as (X, τ, G).

Definition 2.2 [6]

Suppose we have a grill G defined on a topological space (L, τ). The operator Φ: P(L) → P(L), on subsets of L. For any subset A ⊆ L, the operator Φ(A) is the collection of elements I∈ L such that for all sets S∈ τ containing I. The intersection of S with A belongs to the G.

Now, we define another mapping Ψ: P(L) → P(L), where for any subset A of L, then Ψ(A) is the union of A with Φ(A).

The significant part here is that the map Ψ induces what known as Kuratowski's closure operator. This, in turn, generates a finer topology on L compared to the original τ. This finer topology is denoted as τ_G and is defined as follows: a subset A of L is in τ_G if and only if the complement of the set obtained by applying Ψ to the complement of A (i.e., Ψ(L-A)) is equal to the complement of A itself (i.e., L-A).

Definition 2.3 [7]

Imagine we have a grill G established on a topological space (X, τ), and say (X, τ_G) represents the grill topological space that emerges from this grill G. In this context, the assortment B (G, τ) is created, which consists of sets of the form V-A, where V belongs to the original topology τ, and A is not a part of the grill G. This collection, B (G, τ), essentially acts as the foundation for the new grill-induced topology τ_G.

Scientific research has witnessed many studies conducted to study generalizations of weak sets in the grill topological space. These studies centred on studying the characteristics and behaviour of these weak sets, while presenting some important relationships and theorems. The study included the concept of continuity mostly according to the different types of vulnerable sets, in addition to other topological concepts (see, [8-20]).

What distinguishes our current research is the innovative introduction of a new topological space using the concept of grill. This new space opens new doors to explore the properties of weak sets in a new topological context, contributing to the development of our understanding of this field and making a contribution to the field of study.

Notation 2.4

Suppose X stands as a non-empty set, and consider an assortment of topologies be $\{\tau_k\}_{k\epsilon J}$, where the index k ranges from 2 onwards, all defined on the set X. Additionally, assume we have a grill G associated with X. In such scenario, we denote $\{\tau_{k_G}, k\epsilon J \text{ and } k \geq 2\}$ by the collection of grill topologies formed on X using the grill G.

3. **AE-space**

Definition 3.1

Consider a non-empty set L and consider an assortment of grill topologies $\{\tau_{k_G}\}_{k\epsilon J}$, where the index k ranges from 2 onwards. Now, we delve into the concept of AE- open sets within this context. A subset W of L is deemed to be AE- open if either of the following conditions holds: there exists a set T, belonging to the intersection of all the grill topologies $\{\tau_{k_G}\}_{k\epsilon J}$, such that T is non-empty and T is contained within W; or W itself is an empty set.

This leads us to the concept of AE- closed sets, which are simply the complements of AE-open sets. We denote the collection of all AE- closed subsets of L as AEC_X. Consequently, the pair (L, AEO_X) can be referred as an AE-space, where AEO_X represents the family of AE-open sets which are defined as: $AEO_X = \{W \subseteq L: W = \emptyset$ or there exists a set $T \in \bigcap_{k\epsilon J} \tau_{k_G}$ such that T is non-empty and contained within W$\}$.

Example 3.2

Let $L = \{a, b, c, d\}$, and let $\{\tau_i\}_{i=1}^3$ be a family of topologies on L , where $\tau_1 = \mathbb{P}(L)$,
$\tau_2 = \{L, \emptyset, \{a\}, \{a, b\}, \{c\}, \{a, c\}, \{a, b, c\}\}$ and $\tau_3 = \{L, \emptyset, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}\}$.
Let $G = \{ \{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, L\}$ be a grill on L, then
$\tau_{1_G} = \mathbb{P}(L), \tau_{2_G} = \{L, \emptyset, \{a\}, \{a, b\}, \{c\}, \{a, c\}, \{a, b, c\}, \{a, d\}, \{a, c, d\}, \{a, b, d\}\}$
$\tau_{3_G} = \mathbb{P}(L)$, it follows that $\bigcap_{k=1}^3 \tau_{k_G} = \tau_{2_G}$, hence
$AEO_X = \{L, \emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, d\}, \{a, c, d\}, \{a, b, d\}, \{b, c\}, \{c, d\}, \{b, c, d\}\}$.
$AEC_X = \{L, \emptyset, \{b, c, d\}, \{a, b, d\}, \{c, d\}, \{b, d\}, \{d\}, \{b, c\}, \{b\}, \{c\}, \{a, d\}, \{a, b\}, \{a\}\}$

Proposition 3.4

The arbitrary union of AE- open sets is an AE- open set.

Proof:

Let $\{W_i\}_{i\in I}$ be a subcollection of AEO_X. If $W_i = \emptyset$ for each $i \in I$, then $\bigcup_{i\in I} W_i = \emptyset$, thus $\bigcup_{i\in I} W_i \in AEO_X$. If $W_i \neq \emptyset$ for each $i \in I$ (or for some $i \in I$), then there exits $\mathcal{T}_i \in \bigcap_{k\epsilon J} \tau_{k_G} \ni \emptyset \neq \mathcal{T}_i \subseteq W_i$, it follows that $\bigcup_{i\in I} \mathcal{T}_i \subseteq \bigcup_{i\in I} W_i$, but $\bigcup_{i\in I} \mathcal{T}_i \in \bigcap_{k\epsilon J} \tau_{k_G}$, hence $\bigcup_{i\in I} W_i \in AEO_X$.

Remark 3.5
The finite intersection of AE- open sets is not necessarily an $\boldsymbol{AE}$ −open set in general.
For example
Let $L = \{a, b, c\}$, and let $\{\tau_i\}_{i=1}^{3}$ be a family of topologies on L, where $\tau_1 = \mathbb{P}(L)$,
$\tau_2 = \{L, \emptyset, \{a\}, \{a, b\}\}$ and $\tau_3 = \{L, \emptyset, \{a\}, \{b, c\}\}$.
Let $G = \{\{b\}, \{a, b\}, \{b, c\}, L\}$ be a grill on L, then
$\tau_{1_G} = \mathbb{P}(L)$, $\tau_{2_G} = \{L, \emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$ and $\tau_{3_G} = \{L, \emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$, then $\bigcap_{k=1}^{3} \tau_{k_G} = \{L, \emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$. Hence $AEO_L = \{L, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$.
Note that, $\{a, c\}$ and $\{b, c\}$ are AE- open sets, but $\{a, c\} \cap \{b, c\} = \{c\}$ is not AE- open set.

Proposition3.6
If there are no two non-empty disjoint sets in $\bigcap_{k\epsilon J} \tau_{k_G}$, then the finite intersection of AE-open sets is an AE- open set.

Proof:
Let $\{W_i : i = 1, \dots, n\}$ be a subcollection of AEO_X. If $W_i = \emptyset$ for each $i = 1, \dots, n$ (or for some $i = 1, \dots, n$), then $\bigcap_{i=1}^{n} W_i = \emptyset$, thus $\bigcap_{i=1}^{n} W_i \in AEO_X$. If $W_i \neq \emptyset$ for each $i = 1, \dots, n$, then $\exists \mathcal{T}_i \in \bigcap_{k\epsilon J} \tau_{k_G} \ni \emptyset \neq \mathcal{T}_i \subseteq W_i$, it follows that $\bigcap_{i=1}^{n} \mathcal{T}_i \subseteq \bigcap_{i=1}^{n} W_i$, but $\bigcap_{i=1}^{n} \mathcal{T}_i \in \bigcap_{k\epsilon J} \tau_{k_G}$, and $\bigcap_{i=1}^{n} \mathcal{T}_i \neq \emptyset$. Hence $\bigcap_{i=1}^{n} W_i \in AEO_X$.

Proposition3.7
X and $\emptyset$ are AE- open sets.

Proof:
An empty set $\emptyset \in AEO_X$ (From definition of AEO_X).
Since $X \in \tau_{k_G}\ \forall\ k\epsilon J$, so $X \in \bigcap_{k\epsilon J} \tau_{k_G}$, hence $X \in AEO_X$.

Theorem 3.8
If there are no two non-empty disjoint sets in $\bigcap_{k\epsilon J} \tau_{k_G}$, then the AE −space is a topological space, this means that the collection $AEO_X = \{W \subseteq X : W = \emptyset \text{ or } \exists\, \mathcal{T} \in \bigcap_{k\epsilon J} \tau_{k_G} \text{ such that } \emptyset \neq \mathcal{T} \subseteq W\}$ satisfying the following conditions:
1. $X, \emptyset \in AEO_X$.
2. $\bigcup_{i\in I} W_i \in AEO_X\ \forall\ W_i \in AEO_X, i \in I$.
3. $\bigcap_{i=1}^{n} W_i \in AEO_X\ \forall\ W_i \in AEO_X, i = 1, \dots, n$.

Proof:
It follows from Propositions 3.4, 3.6 and 3.7.

Theorem 3.9
Consider the family AEC_L of all AE − closed subsets of an AE −space (L, AEO_L). If there are no two non-empty disjoint sets in $\bigcap_{k\epsilon J} \tau_{k_G}$, then this family satisfying the following conditions:
1. $L, \emptyset \in AEC_L$.
2. $\bigcap_{i\in I} F_i \in AEC_L\ \forall\ F_i \in AEC_L, i \in I$.
3. $\bigcup_{i=1}^{n} F_i \in AEC_L\ \forall\ F_i \in AEC_L, i = 1, \dots, n$.

Proof:
1. Since $L, \emptyset \in AEO_L$, so $L, \emptyset \in AEC_L$.

2. Let $\{F_i\}_{i\in I}$ be a subcollection of AEC_L, then $\{F_i{}^c\}_{i\in I}$ is a subcollection of AEO_L, so $\bigcup_{i\in I} F_i{}^c \in AEO_L$. Using De-Morgan's laws, we get $(\bigcap_{i\in I} F_i)^c \in AEO_L$. Thus $\bigcap_{i\in I} F_i \in AEC_L$.
3. Let $\{F_i: i = 1, \ldots, n\}$ be a subcollection of AEC_L, then $\{F_i{}^c: i = 1, \ldots, n\}$ is a subcollection of AEO_L, so $\bigcap_{i=1}^{n} F_i{}^c \in AEO_L$. Using De-Morgan's laws, we get $(\bigcup_{i=1}^{n} F_i)^c \in AEO_L$. Hence, $\bigcup_{i=1}^{n} F_i \in AEC_L$.

Definition 3.10
Let (L, AEO_L) be an AE −space. A subset H of L is called an AE − neighborhood of an element x in L if there exists an AE − open set W such that $x \in W \subseteq H$.

Example 3.11
Let $L = \{a, b, c\}$, and let $\tau_1 = \mathbb{P}(L), \tau_2 = I$ and $\tau_3 = \{L, \emptyset, \{a\}\}$.
Let $G = \{\{c\}, \{a, c\}, \{b, c\}, L\}$ be a grill on L, then
$\tau_{1_G} = \mathbb{P}(L)$, $\tau_{2_G} = \{L, \emptyset, \{c\}, \{a, c\}, \{b, c\}\}$ and $\tau_{3_G} = \{L, \emptyset, \{a\}, \{c\}, \{a, c\}, \{b, c\}\}$, then $\bigcap_{k=1}^{3} \tau_{k_G} = \{L, \emptyset, \{c\}, \{a, c\}, \{b, c\}\}$. Hence, $AEO_L = \{L, \emptyset, \{c\}, \{a, c\}, \{b, c\}\}$.
Note that, $\{a, c\}$ is AE −open set, so it is an AE − neighborhood of the elements a and c.

Theorem 3.12
Let (L, AEO_L) be an AE −space. A subset H of L is an AE − open set if and only if it is an AE − neighborhood for each of its elements.

Proof:
⇒) Suppose that H is an AE − open set, such that for all $h \in H, \exists\, W \in AEO_L \ni h \in W \subseteq H$, where $W = H$. Thus, H is an AE − neighborhood for each of its elements.

⇐) Suppose that H is an AE − neighborhood for each of its elements, then for all $h \in H, \exists\, W_h \in AEO_L \ni h \in W_h \subseteq H$, therefore, $H = \bigcup_{h\in H} W_h$, it follows that $H \in AEO_L$. Hence, H is AE − open set.

Definition 3.13
Let (L, AEO_L) be an AE −space. A subcollection $\mathfrak{B}$ of AEO_X is said to be AE −Bases for AEO_X if the following terms hold:
1. $\forall\, x \in L\, \exists\, S \in \mathfrak{B} \ni x \in S$.
2. If $x \in S_1 \cap S_2$, for some $S_1, S_2 \in \mathfrak{B}$, then $\exists\, S \in \mathfrak{B} \ni x \in S \subseteq S_1 \cap S_2$.

Example 3.14
Let us take the AE − space in Example 3.2, then $\mathfrak{B} = \{\{a\}, \{a, c\}, \{a, b, d\}\}$ is AE −Bases for AEO_L.

Theorem 3.15
Let (X, AEO_X) be an AE −space and let $\mathfrak{B}$ be an AE −Bases for AEO_X. If there are no two nonempty disjoint sets in $\bigcap_{k\in J} \tau_{k_G}$, then the family $\mathfrak{B}O_X = \{W \in AEO_X: \forall\, w \in W\, \exists\, S \in \mathfrak{B} \ni w \in S \subseteq W\}$ is a topology on X containing $\mathfrak{B}$.

Proof:
1. As $X, \emptyset \in \mathfrak{B}O_X$ from Proposition 3.7, we have $X, \emptyset \in AEO_X$, and from the Definition 3.13(1), we have for all $x \in X\, \exists\, S \in \mathfrak{B} \ni x \in S$, so $X \in \mathfrak{B}O_X$. Now, using the properties of logical expressions $(w \in \emptyset \Longrightarrow \exists\, S \in \mathfrak{B} \ni w \in S \subseteq \emptyset) \equiv (\text{False} \Longrightarrow \text{False})$ which is true, thus $\emptyset \in \mathfrak{B}O_X$.

2. Let $\{W_i\}_{i\in I}$ be a subcollection of $\mathfrak{B}O_X$, then $\{W_i\}_{i\in I}$ be a subcollection of AEO_X, it follows from Proposition 3.4 that $\bigcup_{i\in I} W_i \in AEO_X$. Now, let $w \in \bigcup_{i\in I} W_i$, so $w \in W_j$ for some $j \in I$, therefore, $\exists\, S \in \mathfrak{B} \ni w \in S_j \subseteq W_j \subseteq \bigcup_{i\in I} W_i$. Hence, $\bigcup_{i\in I} W_i \in \mathfrak{B}O_X$.
3. Let $\{W_i: i = 1, \dots, n\}$ be a subcollection of $\mathfrak{B}O_X$, then $\{W_i: i = 1, \dots, n\}$ be a subcollection of AEO_X, it follows from Proposition 3.6 that $\bigcap_{i=1}^{n} W_i \in AEO_X$. Now, let $w \in \bigcap_{i=1}^{n} W_i$, then $w \in W_i \ \forall i = 1, \dots, n$, therefor $\exists\, S_i \in \mathfrak{B} \ni w \in S_i \subseteq W_i \ \forall i = 1, \dots, n$, so we get that.

$w \in \bigcap_{i=1}^{n} S_i \subseteq \bigcap_{i=1}^{n} W_i$, it follows from Definition 3.13(2) that,

$\exists\, S \in \mathfrak{B} \ni x \in S \subseteq \bigcap_{i=1}^{n} S_i \subseteq \bigcap_{i=1}^{n} W_i$. Thus $\bigcap_{i=1}^{n} W_i \in \mathfrak{B}O_X$.

Definition 3.16

Let (X, AEO_X) be an AE −space and let $\subseteq X$. Let $AEO_{\mathcal{K}}$ be a family of subsets of $\mathcal{K}$ defined as $AEO_{\mathcal{K}} = \{H \subseteq \mathcal{K}: H = W \cap \mathcal{K} \ \ni W \in AEO_X\}$. Then the pair $(\mathcal{K}, AEO_{\mathcal{K}})$ is said to be an AE −subspace of the AE −space (X, AEO_X).

Theorem 3.17

Let (X, AEO_X) be an AE −space and let $\subseteq X$. If there are no two non-empty disjoint sets in $\bigcap_{k\epsilon J} \tau_{k_G}$, then $AEO_{\mathcal{K}}$ is a topology on $\mathcal{K}$.

Proof:

1. Since $\mathcal{K} = X \cap \mathcal{K}$ and $X \in AEO_X$, so $\mathcal{K} \in AEO_{\mathcal{K}}$.
Since $\emptyset = \emptyset \cap \mathcal{K}$ and $\emptyset \in AEO_X$, so $\emptyset \in AEO_{\mathcal{K}}$.

2. Let $\{H_i\}_{i\in I}$ be a subcollection of $AEO_{\mathcal{K}}$, then $H_i = W_i \cap \mathcal{K} \ \ni W_i \in AEO_X \ \forall\, i \in I$ so $\bigcup_{i\in I} H_i = \bigcup_{i\in I}(W_i \cap \mathcal{K}) = \bigcup_{i\in I} W_i \cap \mathcal{K}$, but $\bigcup_{i\in I} W_i \in AEO_X$. Thus, $\bigcup_{i\in I} H_i \in AEO_{\mathcal{K}}$.

3. Let $\{H_i: i = 1, \dots, n\}$ be a subcollection of $AEO_{\mathcal{K}}$, then $H_i = W_i \cap \mathcal{K} \ \ni W_i \in AEO_X$ $\forall\, i = 1, \dots, n$, then $\bigcap_{i=1}^{n} H_i = \bigcap_{i=1}^{n}(W_i \cap \mathcal{K}) = \bigcap_{i=1}^{n} W_i \cap \mathcal{K}$, but $\bigcap_{i=1}^{n} W_i \in AEO_X$. Hence, $\bigcap_{i=1}^{n} H_i \in AEO_{\mathcal{K}}$.

4. AE-interior operator and AE-closure operator

Definition 4.1

Let (X, AEO_X) be an AE −space. An AE − interior operator is a map $\mathcal{L}: (X, AEO_X) \longrightarrow (X, AEO_X)$ satisfying the following axioms:

1. $\mathcal{L}(X) = X$,
2. $\mathcal{L}(H) \subseteq H \ \forall H \subseteq X$,
3. If $\{H_i: i = 1, \dots, n\}$ is a collection of subsets of X, then $\mathcal{L}(\bigcap_{i=1}^{n} H_i) = \bigcap_{i=1}^{n} \mathcal{L}(H_i)$,
4. $\mathcal{L}(\mathcal{L}(H)) = \mathcal{L}(H) \ \forall H \subseteq X$.

Theorem 4.2

Let (L, AEO_X) be an AE −space and let $\mathcal{L}: (L, AEO_L) \longrightarrow (L, AEO_L)$ be an AE − interior operator and t_1, t_2 be two subsets of X, then

1. $t_1 \subseteq t_2 \Longrightarrow \mathcal{L}(t_1) \subseteq \mathcal{L}(t_2)$.
2. $\mathcal{L}(t_1) \cup \mathcal{L}(t_2) \subseteq \mathcal{L}(t_1 \cup t_2)$.
3. $\mathcal{L}(\emptyset) = \emptyset$.

Proof:
1. Suppose that $t_1 \subseteq t_2$, then $t_1 \cap t_2 = t_1$,it follows that $\mathcal{L}(t_1 \cap t_2) = \mathcal{L}(t_1)$. But $\mathcal{L}(t_1 \cap t_2) = \mathcal{L}(t_1) \cap \mathcal{L}(t_2)$, so $\mathcal{L}(t_1) = \mathcal{L}(t_1) \cap \mathcal{L}(t_2)$ which means $\mathcal{L}(t_1) \subseteq \mathcal{L}(t_2)$.

2. Since $t_1 \subseteq t_1 \cup t_2$ and $t_2 \subseteq t_1 \cup t_2$, it follows from (1) that $\mathcal{L}(t_1) \subseteq \mathcal{L}(t_1 \cup t_2)$ and $\mathcal{L}(t_2) \subseteq \mathcal{L}(t_1 \cup t_2)$. Hence, $\mathcal{L}(t_1) \cup \mathcal{L}(t_2) \subseteq \mathcal{L}(t_1 \cup t_2)$.

3. From Definition 4.1(2), we have $\mathcal{L}(\emptyset) \subseteq \emptyset$, but $\emptyset \subseteq \mathcal{L}(\emptyset)$. Thus $\mathcal{L}(\emptyset) = \emptyset$.

Theorem 4.3
Let $(\mathrm{X}, \mathrm{AEO_X})$ be an AE −space and let $\mathcal{L}: (\mathrm{X}, \mathrm{AEO_X}) \longrightarrow (\mathrm{X}, \mathrm{AEO_X})$ be an AE − interior operator. A collection $\mathcal{L}O_X = \{\mathrm{t} \subseteq \mathrm{X}: \mathcal{L}(\mathrm{t}) = \mathrm{t}\}$ is a topology on X.

Proof:
1. From Definition 4.1(1), we have $\mathcal{L}(\mathrm{X}) = \mathrm{X}$ and from Theorem 4.2(3), we have $\mathcal{L}(\emptyset) = \emptyset$. Thus, $\mathrm{X}, \emptyset \in \mathcal{L}O_X$.

2. Let $\{\mathrm{t_i}\}_{\mathrm{i}\in\mathrm{I}}$ be a subcollection of $\mathcal{L}O_X$, then $\mathcal{L}(\mathrm{t_i}) = \mathrm{t_i}\ \forall\ \mathrm{i} \in \mathrm{I}$, it follows that $\bigcup_{\mathrm{i}\in\mathrm{I}} \mathcal{L}(\mathrm{t_i}) = \bigcup_{\mathrm{i}\in\mathrm{I}} \mathrm{t_i}$. But $\bigcup_{\mathrm{i}\in\mathrm{I}} \mathcal{L}(\mathrm{t_i}) \subseteq \mathcal{L}(\bigcup_{\mathrm{i}\in\mathrm{I}} \mathrm{t_i})$, so

$$\bigcup_{\mathrm{i}\in\mathrm{I}} \mathrm{t_i} \subseteq \mathcal{L}(\bigcup_{\mathrm{i}\in\mathrm{I}} \mathrm{t_i}) \qquad \cdots\cdots\cdots (1).$$

From Definition 4.1(2) we have

$$\mathcal{L}(\bigcup_{\mathrm{i}\in\mathrm{I}} \mathrm{t_i}) \subseteq \bigcup_{\mathrm{i}\in\mathrm{I}} \mathrm{t_i} \qquad \cdots\cdots\cdots (2).$$

By (1) and (2), we get $\mathcal{L}(\bigcup_{\mathrm{i}\in\mathrm{I}} \mathrm{t_i}) = \bigcup_{\mathrm{i}\in\mathrm{I}} \mathrm{t_i}$. Hence, $\bigcup_{\mathrm{i}\in\mathrm{I}} \mathrm{t_i} \in \mathcal{L}O_X$.
3. Let $\{\mathrm{t_i}: \mathrm{i} = 1, \dots, \mathrm{n}\}$ be a subcollection of $\mathcal{L}O_X$, then $\mathcal{L}(\mathrm{t_i}) = \mathrm{t_i}\ \forall\ i = 1, \dots, \mathrm{n}$, it follows that $\bigcap_{\mathrm{i}=1}^{\mathrm{n}} \mathcal{L}(\mathrm{t_i}) = \bigcap_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{t_i}$, but $\bigcap_{\mathrm{i}=1}^{\mathrm{n}} \mathcal{L}(\mathrm{t_i}) = \mathcal{L}(\bigcap_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{t_i})$, therefore, $\mathcal{L}(\bigcap_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{t_i}) = \bigcap_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{t_i}$. Thus, $\bigcap_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{t_i} \in \mathcal{L}O_X$.

Definition 4.4
Let $(\mathrm{X}, \mathrm{AEO_X})$ be an AE −space. An AE − closure operator is a map $£: (\mathrm{X}, \mathrm{AEO_X}) \longrightarrow (\mathrm{X}, \mathrm{AEO_X})$ satisfying the following axioms:
1. $£(\emptyset) = \emptyset$,
2. $\mathrm{t} \subseteq £(\mathrm{t})\ \ \forall \mathrm{t} \subseteq \mathrm{X}$,
3. If $\{\mathrm{t_i}: \mathrm{i} = 1, \dots, \mathrm{n}\}$ is a collection of subsets of X, then $£(\bigcup_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{t_i}) = \bigcup_{\mathrm{i}=1}^{\mathrm{n}} £(\mathrm{t}_i)$,
4. $£\big(£(\mathrm{t})\big) = £(\mathrm{t})\ \forall \mathrm{t} \subseteq \mathrm{X}$.

Theorem 4.5
Let $(\mathrm{X}, \mathrm{AEO_X})$ be an AE −space and let $£: (\mathrm{X}, \mathrm{AEO_X}) \longrightarrow (\mathrm{X}, \mathrm{AEO_X})$ be an AE − closure operator and $\mathrm{t_1}, \mathrm{t_2}$ be two subsets of X, then
1. $t_1 \subseteq t_2 \Longrightarrow £(t_1) \subseteq £(t_2)$.
2. $£(t_1 \cap t_2) \subseteq £(t_1) \cap £(t_2)$.
3. $£(\mathrm{X}) = \mathrm{X}$.

Proof:

1. Suppose that $t_1 \subseteq t_2$, then $t_1 \cup t_2 = t_2$, it follows that $£(t_1 \cup t_2) = £(t_2)$. But $£(t_1 \cup t_2) = £(t_1) \cup £(t_2)$, so $£(t_2) = £(t_1) \cup £(t_2)$ which means $£(t_1) \subseteq £(t_2)$.

2. Since $t_1 \cap t_2 \subseteq t_1$ and $t_1 \cap t_2 \subseteq t_2$, it follows from (1) that $£(t_1 \cap t_2) \subseteq £(t_1)$ and $£(t_1 \cap t_2) \subseteq £(t_2)$. Hence $£(t_1 \cap t_2) \subseteq £(t_1) \cap £(t_2)$.

3. From Definition 4.4(2), we have $X \subseteq £(X)$, but $£(X) \subseteq X$. Thus $£(X) = X$.

Theorem 4.6

Let (X, AEO_X) be an AE −space and let $£: (X, AEO_X) \longrightarrow (X, AEO_X)$ be an AE − closure operator. A collection $£O_X = \{t \subseteq X: £(X - t) = X - t\}$ is a topology on X.

Proof:

1. From Definition 4.4(1), we have $£(\emptyset) = \emptyset$, that means $£(X - X) = X - X$ and from Theorem 4.5(3), we have $£(X) = X$, that means $£(X - \emptyset) = X - \emptyset$. Thus $X, \emptyset \in £O_X$.

2. Let $\{t_i\}_{i\in I}$ be a subcollection of $£O_X$, then $£(t_i{}^c) = t_i{}^c \; \forall \; i \in I$, it follows that
$\bigcap_{i\in I} £(t_i{}^c) = \bigcap_{i\in I} t_i{}^c$.
But $£(\bigcap_{i\in I} t_i{}^c) \subseteq \bigcap_{i\in I} £(t_i{}^c)$. So $£(\bigcap_{i\in I} t_i{}^c) \subseteq \bigcap_{i\in I} t_i{}^c \cdots\cdots\cdots (1)$.
From Definition 4.4(2) we have $\bigcap_{i\in I} t_i{}^c \subseteq £(\bigcap_{i\in I} t_i{}^c) \cdots\cdots\cdots (2)$.
By (1) and (2), we get $£(\bigcap_{i\in I} t_i{}^c) = \bigcap_{i\in I} t_i{}^c$. From De- Morgan Laws we get
$£((\bigcup_{i\in I} t_i)^c) = (\bigcup_{i\in I} t_i)^c$. Hence, $\bigcup_{i\in I} t_i \in £O_X$.

3. Let $\{t_i: i = 1, \dots, n\}$ be a subcollection of $£O_X$, then $£(t_i{}^c) = t_i{}^c \; \forall \; i = 1, \dots, n$, it follows that $\bigcup_{i=1}^{n} £(t_i{}^c) = \bigcup_{i=1}^{n} t_i{}^c$, but $\bigcup_{i=1}^{n} £(t_i{}^c) = £(\bigcup_{i=1}^{n} t_i{}^c)$, therefore, $£(\bigcup_{i=1}^{n} t_i{}^c) = \bigcup_{i=1}^{n} t_i{}^c$. From De- Morgan Laws we get $£((\bigcap_{i=1}^{n} t_i)^c) = (\bigcap_{i=1}^{n} t_i)^c$. Thus, $\bigcap_{i=1}^{n} t_i \in £O_X$.

5. **DE-space**

Definition 5.1

Consider a topological space denoted as (X, τ), and let $\{G_k\}_{k\epsilon J}$, k≥2 be a collection of arbitrary grills defined on X. Consequently, the set $\{\tau_{G_k}$, kϵJ and k≥2$\}$ forms an assemblage of grill topologies imposed upon X. A subset Έ of X is classified as DE-open if there exists a non-empty subset T⊆Έ, where T is an element of the intersection $\bigcap_{k\epsilon J} \{\tau_{kG}\}$, or it may be that Έ itself is empty. The complete set of all such DE-open subsets within X is symbolized as DEO_X. Furthermore, the notion of a DE-closed set corresponds to the complement of a DE-open set. The collection of all DE-closed subsets of X is designated as DEO_X. The pair (X, DEO_X) is referred to as a DE-space, wherein DEO_X is the set of all subsets Έ⊆X satisfying the conditions Έ=∅ or ∃ T∈$\bigcap_{k\epsilon J} \{\tau_{kG}\}$ such that ∅≠T⊆Έ.

Remark 5.2

In a manner congruent with prior exposition, it is of scholarly import to discern that the entirety of the antecedently expounded deliberations remains applicable in the context of the stipulated definition above.

Conclusion

The primary goal of this work is to construct and define a new type of topological space known as “AE-space”. This definition can be used to study all topological properties and compare its properties with previously studied topological properties according to the previous topological structure, especially the various applied aspects, in addition to identifying new weak sets, determining the accuracy of these sets, and presenting soft topological spaces according to the context presented in this work.

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