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Weak Forms of Fuzzy N- Open Sets and Fuzzy $\hat{D}_N$ - Sets in Fuzzy Topological Spaces

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Abstract

In this article we introduce a new type of fuzzy open sets in fuzzy topological spaces called fuzzy N-open sets and we prove that the family of all fuzzy N-open sets in a fuzzy topological space $(X, \hat{T})$ forms a fuzzy topology $\hat{T}_N$ on X. Also we use fuzzy N-open sets to define and study new types of fuzzy sets called weak fuzzy N-open sets and weak fuzzy $\hat{D}_N$ - sets in fuzzy topological spaces. Moreover we investigate the relation between the fuzzy open sets and each of fuzzy N-open sets, weak fuzzy N-open sets and weak fuzzy $\hat{D}_N$ - sets.

Keywords: fuzzy N-open set, fuzzy β-N-open set, fuzzy b-N-open set, fuzzy pre-N-open set, fuzzy α-N-open set, fuzzy $\hat{D}_N$ -set, fuzzy $\hat{D}_{\beta-N}$ -set, fuzzy $\hat{D}_{b-N}$ -set, fuzzy $\hat{D}_{pre-N}$ -set and fuzzy $\hat{D}_{\alpha-N}$ -set.

الصيغ الضعيفة للمجموعات المفتوحة الضبابية من النمطN والمجموعات الضبابية من النمط – $\hat{D}_N$ في الفضاءات التبولوجية الضبابية

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الخلاصة

في هذة المقالة نحن قدمنا نوع جديد من المجموعات المفتوحة الضبابية في الفضاءات التبولوجية الضبابية أسميناهابالمجموعات المفتوحة الضبابية من النمط –N ثم اثبتنا ان عائلة كل المجموعات الجزئية المفتوحة الضبابية من النمط– Nفي الفضاء التبولوجي الضبابية $(X, \hat{T})$ تشكل تبولوجيا ضبابية $\hat{T}_N$ على X . كذلك استخدمنا المجموعات المفتوحة الضبابية من النمط– N في تعريف ودراسة انواع جديدة من المجموعات الضبابية اسميناها بالمجموعات المفتوحة الضبابية الضعيفة من النمط–N والمجموعات الضبابية الضعيفة من النمط – $\hat{D}_N$ في الفضاءات التبولوجية الضبابية .علاوة على ذلك تحرينا العلاقة بين المجموعات المفتوحة الضبابية وكل من المجموعات المفتوحة الضبابية من النمط– N والمجموعات المفتوحة الضبابية الضعيفة من النمط–N والمجموعات الضبابية الضعيفة من النمط. $\hat{D}_N$ –

Introduction

The concept of fuzzy set was first introduced by Zadeh [1] in 1965 as an extension of the classical notion sets. Chang [2] in 1968 introduced the notion of fuzzy topological spaces. Bin Shahna [3] in 1991, Benchalli and Jenifer [4] in 2010, and Thakur and Singh [5] in 1998 introduced and investigated

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fuzzy α-open sets, fuzzy pre-open sets, fuzzy b-open sets and fuzzy β-open sets respectively. The main purpose of this paper is to introduce a new class of fuzzy open sets in fuzzy topological spaces called fuzzy N-open sets and we show that the family of all fuzzy N-open sets in a fuzzy topological space $(\mathbf{X},\hat{\mathbf{T}})$ forms a fuzzy topology $\hat{\mathbf{T}}_{\mathbf{N}}$ on X. Furthermore we use fuzzy N-open sets to define and study new classes of fuzzy sets called weak fuzzy N-open sets and weak fuzzy $\hat{\mathbf{D}}_{\mathbf{N}}$- sets in fuzzy topological spaces. Finally, we investigate the relation between the fuzzy open sets and each of fuzzy N-open sets, weak fuzzy N-open sets and weak fuzzy $\hat{\mathbf{D}}_{\mathbf{N}}$- sets.

1. Preliminaries:

Firstly we recall the following definitions:

Definition (1.1)[1]: Let X be a non-empty set. A fuzzy set in X is a function $\hat{\mathbf{A}}$ from X to the unit interval $\mathbf{I}=[0,1]$. The image $\hat{\mathbf{A}}(\mathbf{x})$ of $\mathbf{x}\in\mathbf{X}$ is called the grade of membership of x in $\hat{\mathbf{A}}$.

The family of all fuzzy subsets of X will be denoted by $\mathbf{I}^{\mathbf{X}}$.

Definition (1.2)[1],[2],[6]: Let $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$ be two fuzzy sets in X. Then:

(i) $\hat{\mathbf{A}}\subseteq\hat{\mathbf{B}}\Leftrightarrow\hat{\mathbf{A}}(\mathbf{x})\le\hat{\mathbf{B}}(\mathbf{x}),\ \forall\ \mathbf{x}\in\mathbf{X}.$

(ii) $\hat{\mathbf{A}}=\hat{\mathbf{B}}\Leftrightarrow\hat{\mathbf{A}}(\mathbf{x})=\hat{\mathbf{B}}(\mathbf{x}),\ \forall\ \mathbf{x}\in\mathbf{X}.$

(iii) $\hat{\varphi}\in\mathbf{I}^{\mathbf{X}}\Leftrightarrow\hat{\varphi}(\mathbf{x})=0,\ \forall\ \mathbf{x}\in\mathbf{X}.$

(iv) $\hat{\mathbf{X}}\in\mathbf{I}^{\mathbf{X}}\Leftrightarrow\hat{\mathbf{X}}(\mathbf{x})=1,\ \forall\ \mathbf{x}\in\mathbf{X}.$

(v) $\hat{\mathbf{C}}=\hat{\mathbf{A}}\bigcap\hat{\mathbf{B}}\in\mathbf{I}^{\mathbf{X}}\Leftrightarrow\hat{\mathbf{C}}(\mathbf{x})=\min\{\hat{\mathbf{A}}(\mathbf{x}),\hat{\mathbf{B}}(\mathbf{x})\},\forall\,\mathbf{x}\in\mathbf{X}.$

(vi) $\hat{\mathbf{D}}=\hat{\mathbf{A}}\bigcup\hat{\mathbf{B}}\in\mathbf{I}^{\mathbf{X}}\Leftrightarrow\hat{\mathbf{D}}(\mathbf{x})=\max\{\hat{\mathbf{A}}(\mathbf{x}),\hat{\mathbf{B}}(\mathbf{x})\},\forall\,\mathbf{x}\in\mathbf{X}.$

(vii) $\hat{\mathbf{E}}=\hat{\mathbf{A}}^{\mathbf{c}}\Leftrightarrow\hat{\mathbf{E}}(\mathbf{x})=1-\hat{\mathbf{A}}(\mathbf{x}),\ \forall\,\mathbf{x}\in\mathbf{X}.$

(viii) $(\hat{\mathbf{A}}\bigcap\hat{\mathbf{B}})^{\mathbf{c}}=\hat{\mathbf{A}}^{\mathbf{c}}\bigcup\hat{\mathbf{B}}^{\mathbf{c}}.$

(ix) $\hat{\mathbf{A}}\subseteq\hat{\mathbf{B}}\Leftrightarrow\hat{\mathbf{B}}^{\mathbf{c}}\subseteq\hat{\mathbf{A}}^{\mathbf{c}}.$

(x) $\hat{\mathbf{A}}-\hat{\mathbf{B}}=\hat{\mathbf{A}}\bigcap\hat{\mathbf{B}}^{\mathbf{c}}.$

In general, if $\{\hat{\mathbf{A}}_{\alpha}:\alpha\in\Lambda\}$ **is a family of fuzzy subsets of X, then:**

(xi) $\hat{\mathbf{C}}=\bigcap_{\alpha\in\Lambda}\hat{\mathbf{A}}_{\alpha}\in\mathbf{I}^{\mathbf{X}}\Leftrightarrow\hat{\mathbf{C}}(\mathbf{x})=\inf\{\hat{\mathbf{A}}_{\alpha}(\mathbf{x}),\alpha\in\Lambda\},\forall\,\mathbf{x}\in\mathbf{X}.$

(xii) $\hat{\mathbf{D}}=\bigcup_{\alpha\in\Lambda}\hat{\mathbf{A}}_{\alpha}\in\mathbf{I}^{\mathbf{X}}\Leftrightarrow\hat{\mathbf{D}}(\mathbf{x})=\sup\{\hat{\mathbf{A}}_{\alpha}(\mathbf{x}),\alpha\in\Lambda\},\forall\,\mathbf{x}\in\mathbf{X}.$

Definition (1.3)[7]: The support of a fuzzy set $\hat{\mathbf{A}}$ in X will be denoted by $\mathbf{S}(\hat{\mathbf{A}})$ and is defined by $\mathbf{S}(\hat{\mathbf{A}})=\{\mathbf{x}\in\mathbf{X}:\hat{\mathbf{A}}(\mathbf{x})>0\}$.

Definition (1.4)[8]: Let X be a non-empty set. A fuzzy point $\mathbf{P}_{\mathbf{x}_0}^{\lambda}$ in X is a fuzzy set with membership function defined by:

$$\mathbf{P}_{\mathbf{x}_0}^{\lambda}(\mathbf{y})=\begin{cases}\lambda & \mathbf{if}\,\mathbf{x}_0=\mathbf{y}\\ 0 & \mathbf{if}\,\mathbf{x}_0\ne\mathbf{y}\end{cases}$$

Where $0<\lambda<1$. $\mathbf{P}_{\mathbf{x}_0}^{\lambda}$ is said to have support $\mathbf{x}_0$ and value λ.

Definition (1.5) [7]: A fuzzy point $\mathbf{P}_{\mathbf{x}_0}^{\lambda}$ is called belong to $\hat{\mathbf{A}}$, denoted by $\mathbf{P}_{\mathbf{x}_0}^{\lambda}\in\hat{\mathbf{A}}$ iff $\lambda<\hat{\mathbf{A}}(\mathbf{x}_0)$.

Definition (1.6) [9]: A fuzzy set $\hat{\mathbf{A}}$ is called finite if $\mathbf{S}(\hat{\mathbf{A}})$ is a finite set.

Definition (1.7) [7]: Let $\mathbf{P}_{\mathbf{x}}^{\lambda}$ be a fuzzy point in X and $\hat{\mathbf{A}}\in\mathbf{I}^{\mathbf{X}}$. Then $\mathbf{P}_{\mathbf{x}}^{\lambda}$ is called quasi-coincident with a fuzzy set $\hat{\mathbf{A}}$, denoted by $\mathbf{P}_{\mathbf{x}}^{\lambda}\mathbf{q}\hat{\mathbf{A}}$ if $\lambda>\hat{\mathbf{A}}^{\mathbf{c}}(\mathbf{x})$ or $\lambda+\hat{\mathbf{A}}(\mathbf{x})>1$.

Definition (1.8) [7]: A fuzzy set $\hat{\mathbf{A}}$ is called quasi-coincident with a fuzzy set $\hat{\mathbf{B}}$, denoted by $\hat{\mathbf{A}}\mathbf{q}\hat{\mathbf{B}}$ if there exists $\mathbf{x} \in \mathbf{X}$ such that $\hat{\mathbf{A}}(\mathbf{x}) > \hat{\mathbf{B}}(\mathbf{x})^{\mathbf{c}}$ or $\hat{\mathbf{A}}(\mathbf{x}) + \hat{\mathbf{B}}(\mathbf{x}) > 1$. We will use the notation $\hat{\mathbf{A}}\overline{\mathbf{q}}\hat{\mathbf{B}}$ if $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$ are not quasi coincident.

Definition (1.9)[2]: If $\hat{\mathbf{T}}$ is a family of fuzzy subsets of X. Then $\hat{\mathbf{T}}$ is called a fuzzy topology on X if $\hat{\mathbf{T}}$ has the following properties:

(i) $\hat{\varphi}, \hat{\mathbf{X}} \in \hat{\mathbf{T}}$.

(ii) If $\hat{\mathbf{V}}_1, \hat{\mathbf{V}}_2 \in \hat{\mathbf{T}}$, then $\hat{\mathbf{V}}_1 \cap \hat{\mathbf{V}}_2 \in \hat{\mathbf{T}}$.

(iii) If $\hat{\mathbf{V}}_\alpha \in \hat{\mathbf{T}}, \forall \alpha \in \Lambda$, then $\bigcup_{\alpha \in \Lambda} \hat{\mathbf{V}}_\alpha \in \hat{\mathbf{T}}$.

The pair $(\mathbf{X}, \hat{\mathbf{T}})$ is called a fuzzy topological space. The members of $\hat{\mathbf{T}}$ are called fuzzy open sets in X.

The complement of a fuzzy open set is called fuzzy closed.

Definition (1.10)[10]: If $\hat{\mathbf{A}}$ is a fuzzy subset of a fuzzy topological space $(\mathbf{X}, \hat{\mathbf{T}})$. Then:

(i) $\mathbf{cl}(\hat{\mathbf{A}}) = \bigcap \{ \hat{\mathbf{F}} : \hat{\mathbf{A}} \subseteq \hat{\mathbf{F}}$ and $\hat{\mathbf{F}}$ is a fuzzy closed set in $\mathbf{X}\}$ is called the fuzzy closure of $\hat{\mathbf{A}}$.

(ii) $\mathrm{int}(\hat{\mathbf{A}}) = \bigcup \{ \hat{\mathbf{V}} : \hat{\mathbf{V}} \subseteq \hat{\mathbf{A}}$ and $\hat{\mathbf{V}}$ is a fuzzy open set in $\mathbf{X}\}$ is called the fuzzy interior of $\hat{\mathbf{A}}$.

Definitions (1.11): A fuzzy subset $\hat{\mathbf{A}}$ of a fuzzy topological space $(\mathbf{X}, \hat{\mathbf{T}})$ is called:

(i) A fuzzy α-open set if $\hat{\mathbf{A}} \subseteq \mathrm{int}(\mathbf{cl}(\mathrm{int}(\hat{\mathbf{A}})))$ [3].

(ii) A fuzzy pre-open set if $\hat{\mathbf{A}} \subseteq \mathrm{int}(\mathbf{cl}(\hat{\mathbf{A}}))$ [3].

(iii) A fuzzy b-open set if $\hat{\mathbf{A}} \subseteq \mathrm{int}(\mathbf{cl}(\hat{\mathbf{A}})) \cup \mathbf{cl}(\mathrm{int}(\hat{\mathbf{A}}))$ [4].

(iv) A fuzzy β-open set if $\hat{\mathbf{A}} \subseteq \mathbf{cl}(\mathrm{int}(\mathbf{cl}((\hat{\mathbf{A}}))$ [5].

2. Fuzzy N-Open Sets

In this section we introduce a new type of fuzzy open sets in fuzzy topological spaces called fuzzy N-open sets, and we show that the family of all fuzzy N-open sets in $(\mathbf{X}, \hat{\mathbf{T}})$ forms a fuzzy topology $\hat{\mathbf{T}}_{\mathbf{N}}$ on X. Also, we study the basic properties of fuzzy N-open sets.

Definition (2.1): A fuzzy subset $\hat{\mathbf{N}}$ of a fuzzy topological space $(\mathbf{X}, \hat{\mathbf{T}})$ is called fuzzy N-open if for each $\mathbf{P}_{\mathbf{x}}^{\lambda} \in \hat{\mathbf{N}}$, there exists $\hat{\mathbf{V}} \in \hat{\mathbf{T}}$ such that $\mathbf{P}_{\mathbf{x}}^{\lambda} \in \hat{\mathbf{V}}$ and $\hat{\mathbf{V}} - \hat{\mathbf{N}}$ is a finite fuzzy set. The complement of a fuzzy N-open set is called fuzzy N-closed. The family of all fuzzy N-open subsets of $(\mathbf{X}, \hat{\mathbf{T}})$ is denoted by $\hat{\mathbf{T}}_{\mathbf{N}}$.

Remark (2.2): Fuzzy open sets and fuzzy N-open sets are independent we can see by the following examples:

Example (2.3): Let $X = \{a, b, c\}$ and $\hat{\mathbf{T}} = \{\hat{\varphi}, \hat{\mathbf{X}}\} = \hat{\mathbf{I}}$ be the indiscrete fuzzy topology on X. It is clear that $\hat{\mathbf{N}} = \{(\mathbf{a},0.1),(\mathbf{b},0.2),(\mathbf{c},0.3)\}$ is a fuzzy N-open set, but is not fuzzy open.

Example (2.4): Let $\mathbf{X} = \mathfrak{R}$ and $\hat{\mathbf{T}} = \{\hat{\varphi}, \hat{\mathfrak{R}}, \hat{\mathbf{V}}\}$ be a fuzzy topology on X, where $\hat{\mathbf{V}}$ is a fuzzy set in X with membership function defined by: $\hat{\mathbf{V}}(\mathbf{x}) = 0.3, \forall \mathbf{x} \in \mathfrak{R}$. It is clear that $\hat{\mathbf{V}}$ is a fuzzy open set, but is not fuzzy N-open.

Theorem (2.5): The family of all fuzzy N-open sets in $(\mathbf{X}, \hat{\mathbf{T}})$ forms a fuzzy topology $\hat{\mathbf{T}}_{\mathbf{N}}$ on X.

Proof: (i) It is clear that $\hat{\mathbf{X}}, \hat{\varphi} \in \hat{\mathbf{T}}_{\mathbf{N}}$.

(ii) Assume that $\hat{\mathbf{N}}_1, \hat{\mathbf{N}}_2 \in \hat{\mathbf{T}}_{\mathbf{N}}$. To show that $\mathbf{N}_1 \cap \hat{\mathbf{N}}_2 \in \hat{\mathbf{T}}_{\mathbf{N}}$. Let $\mathbf{P}_{\mathbf{x}}^{\lambda} \in (\mathbf{N}_1 \cap \hat{\mathbf{N}}_2) \Rightarrow \mathbf{P}_{\mathbf{x}}^{\lambda} \in \mathbf{N}_1$ and $\mathbf{P}_{\mathbf{x}}^{\lambda} \in \hat{\mathbf{N}}_2$. Since $\hat{\mathbf{N}}_1$ is a fuzzy N-open set $\Rightarrow \exists\ \hat{\mathbf{V}}_1 \in \hat{\mathbf{T}}$ such that $\mathbf{P}_{\mathbf{x}}^{\lambda} \in \hat{\mathbf{V}}_1$ and $\hat{\mathbf{V}}_1 - \hat{\mathbf{N}}_1$ is finite. Since $\hat{\mathbf{N}}_2$ is a fuzzy N-open set $\Rightarrow \exists\ \hat{\mathbf{V}}_2 \in \hat{\mathbf{T}}$ such that $\mathbf{P}_{\mathbf{x}}^{\lambda} \in \hat{\mathbf{V}}_2$ and $\hat{\mathbf{V}}_2 - \hat{\mathbf{N}}_2$ is finite. Since $\mathbf{P}_{\mathbf{x}}^{\lambda} \in \hat{\mathbf{V}}_1$ and $\mathbf{P}_{\mathbf{x}}^{\lambda} \in \hat{\mathbf{V}}_2 \Rightarrow \mathbf{P}_{\mathbf{x}}^{\lambda} \in (\hat{\mathbf{V}}_1 \cap \hat{\mathbf{V}}_2)$. To show that $(\hat{\mathbf{V}}_1 \cap \hat{\mathbf{V}}_2) - (\hat{\mathbf{N}}_1 \cap \hat{\mathbf{N}}_2)$ is a finite fuzzy set. Hence $(\hat{\mathbf{V}}_1 \cap \hat{\mathbf{V}}_2) - (\hat{\mathbf{N}}_1 \cap \hat{\mathbf{N}}_2) = (\hat{\mathbf{V}}_1 \cap \hat{\mathbf{V}}_2) \cap (\hat{\mathbf{N}}_1 \cap \hat{\mathbf{N}}_2)^{\mathbf{c}} = (\hat{\mathbf{V}}_1 \cap \hat{\mathbf{V}}_2) \cap (\hat{\mathbf{N}}_1{}^{\mathbf{c}} \cup \hat{\mathbf{N}}_2{}^{\mathbf{c}})$

$=[(\hat{V}_1\cap\hat{V}_2)\cap\hat{N}_1^{c}]\cup[(\hat{V}_1\cap\hat{V}_2)\cap\hat{N}_2^{c}]=[(\hat{V}_1\cap\hat{V}_2)-\hat{N}_1]\cup[(\hat{V}_1\cap\hat{V}_2)-\hat{N}_2]$.

But $(\hat{V}_1\cap\hat{V}_2)-\hat{N}_1$ and $(\hat{V}_1\cap\hat{V}_2)-\hat{N}_2$ are finite fuzzy sets, then so is $[(\hat{V}_1\cap\hat{V}_2)-\hat{N}_1]\cup[(\hat{V}_1\cap\hat{V}_2)-\hat{N}_2]$. Thus $(\hat{V}_1\cap\hat{V}_2)-(\hat{N}_1\cap\hat{N}_2)$ is a finite fuzzy set. Hence $N_1\cap\hat{N}_2\in\hat{T}_N$.

(iii) Let $\hat{N}_\alpha\in\hat{T}_N, \forall\alpha\in\Lambda$. To show that $\bigcup_{\alpha\in\Lambda}\hat{N}_\alpha\in\hat{T}_N$. Let $P_x^\lambda\in\bigcup_{\alpha\in\Lambda}\hat{N}_\alpha \Rightarrow P_x^\lambda\in\hat{N}_{\alpha_0}$ for some $\alpha_0\in\Lambda$. But $\hat{N}_{\alpha_0}\in\hat{T}_N \Rightarrow \exists\ \hat{V}\in\hat{T}$ such that $P_x^\lambda\in\hat{V}$ and $\hat{V}-\hat{N}_{\alpha_0}$ is a finite fuzzy set.

Since $\hat{N}_{\alpha_0}\subseteq\bigcup_{\alpha\in\Lambda}\hat{N}_\alpha \Rightarrow (\bigcup_{\alpha\in\Lambda}\hat{N}_\alpha)^c\subseteq\hat{N}_{\alpha_0}^c \Rightarrow \hat{V}\cap(\bigcup_{\alpha\in\Lambda}\hat{N}_\alpha)^c\subseteq\hat{V}\cap\hat{N}_{\alpha_0}^c \Rightarrow$

$\hat{V}-(\bigcup_{\alpha\in\Lambda}\hat{N}_\alpha)\subseteq\hat{V}-\hat{N}_{\alpha_0}$. But $\hat{V}-\hat{N}_{\alpha_0}$ is a finite fuzzy set, then so is $\hat{V}-(\bigcup_{\alpha\in\Lambda}\hat{N}_\alpha) \Rightarrow$

$\bigcup_{\alpha\in\Lambda}\hat{N}_\alpha\in\hat{T}_N \Rightarrow (X,\hat{T}_N)$ is a fuzzy topological space.

Definition (2.6): Let $\hat{A}$ be a fuzzy subset of a fuzzy topological space $(X,\hat{T})$. Then:

(i) $cl_N(\hat{A})=\cap\{\hat{F}:\hat{A}\subseteq\hat{F}$ and $\hat{F}$ is a fuzzy N-closed set in X} is called the fuzzy N-closure of $\hat{A}$.

(ii) $int_N(\hat{A})=\cup\{\hat{N}:\hat{N}\subseteq\hat{A}$ and $\hat{N}$ is a fuzzy N-open set in **X**} is called the fuzzy N-interior of $\hat{A}$.

Definition (2.7): A fuzzy subset $\hat{A}$ of a fuzzy topological space $(X,\hat{T})$ is called a fuzzy quasi N-neighborhood of a fuzzy point P_x^λ if there exists a fuzzy N-open set $\hat{N}$ in X such that $P_x^\lambda q\hat{N}\subseteq\hat{A}$.

Theorem (2.8): Let $\hat{A}$ and $\hat{B}$ be fuzzy subsets of a fuzzy topological space $(X,\hat{T})$. Then:

(i) $int_N(\hat{A})\subseteq\hat{A}$ and $\hat{A}\subseteq cl_N(\hat{A})$.

(ii) If $\hat{A}_\alpha$ is a fuzzy N-open set in X for each $\alpha\in\Lambda$, then so is $\bigcup_{\alpha\in\Lambda}\hat{A}_\alpha$.

(iii) If $\hat{A}_\alpha$ is a fuzzy N-closed set in X for each $\alpha\in\Lambda$, then so is $\bigcap_{\alpha\in\Lambda}\hat{A}_\alpha$.

(iv) $int_N(\hat{A})$ is a fuzzy N-open set in X and $cl_N(\hat{A})$ is a fuzzy N-closed set in X.

(v) $\hat{A}$ is a fuzzy N-open set in X iff $int_N(\hat{A})=\hat{A}$ and $\hat{A}$ is a fuzzy N-closed set in X iff $cl_N(\hat{A})=\hat{A}$.

(vi) $int_N(int_N(\hat{A}))=int_N(\hat{A})$ and $cl_N(cl_N(\hat{A}))=cl_N(\hat{A})$.

(vii) $[int_N(\hat{A})]^c=cl_N(\hat{A}^c)$ and $[cl_N(\hat{A})]^c=int_N(\hat{A}^c)$.

(viii) $P_x^\lambda\in int_N(\hat{A})$ iff there is a fuzzy N-open set $\hat{N}$ in X s.t $P_x^\lambda\in\hat{N}\subseteq\hat{A}$.

(ix) If $\hat{A}\subseteq\hat{B}$, then $int_N(\hat{A})\subseteq int_N(\hat{B})$ and $cl_N(\hat{A})\subseteq cl_N(\hat{B})$.

(x) $int_N(\hat{A}\cap\hat{B})=int_N(\hat{A})\cap int_N(\hat{B})$ and $cl_N(\hat{A}\cup\hat{B})=cl_N(\hat{A})\cup cl_N(\hat{B})$

(xi) If $P_x^\lambda\in cl_N(\hat{A})$, then every fuzzy quasi N-neighborhood of P_x^λ is quasi coincident with $\hat{A}$.

(xii) $\bigcup_{\alpha\in\Lambda}cl_N(\hat{A}_\alpha)\subseteq cl_N(\bigcup_{\alpha\in\Lambda}\hat{A}_\alpha)$ and $\bigcup_{\alpha\in\Lambda}int_N(\hat{A}_\alpha)\subseteq int_N(\bigcup_{\alpha\in\Lambda}\hat{A}_\alpha)$.

Proof:(vii) Since $int_N(\hat{A})=\cup\{\hat{N}:\hat{N}\subseteq\hat{A},\hat{N}\in\hat{T}_N\} \Rightarrow [int_N(\hat{A})]^c=[\cup\{\hat{N}:\hat{N}\subseteq\hat{A},\hat{N}\in\hat{T}_N\}]^c$ $=\cap\{\hat{N}^c:\hat{N}\subseteq\hat{A},\hat{N}\in\hat{T}_N\}=\cap\{\hat{N}^c:\hat{A}^c\subseteq\hat{N}^c,\hat{N}^c\in\hat{T}_N{}^c\}=cl_N(\hat{A}^c)$.

(x) Since $\hat{A}\cap\hat{B}\subseteq\hat{A}$ and $\hat{A}\cap\hat{B}\subseteq\hat{B}$, then by (ix), we get $int_N(\hat{A}\cap\hat{B})\subseteq int_N(\hat{A})$ and $int_N(\hat{A}\cap\hat{B})\subseteq int_N(\hat{B}) \Rightarrow int_N(\hat{A}\cap\hat{B})\subseteq int_N(\hat{A})\cap int_N(\hat{B})$. To prove that $int_N(\hat{A})\cap int_N(\hat{B})\subseteq int_N(\hat{A}\cap\hat{B})$. Let $P_x^\lambda\in int_N(\hat{A})\cap int_N(\hat{B}) \Rightarrow P_x^\lambda\in int_N(\hat{A})$ & $P_x^\lambda\in int_N(\hat{B}) \Rightarrow \exists\ \hat{N}_1\in\hat{T}_N$ s.t $P_x^\lambda\in\hat{N}_1\subseteq\hat{A}$ & $\exists\ \hat{N}_2\in\hat{T}_N$ s.t $P_x^\lambda\in\hat{N}_2\subseteq\hat{B} \Rightarrow$

$\exists\ \hat{N}_1 \cap \hat{N}_2 \in \hat{T}_N$ s.t $P_x^\lambda \in (\hat{N}_1 \cap \hat{N}_2) \subseteq \hat{A} \cap \hat{B} \Rightarrow P_x^\lambda \in int_N(\hat{A} \cap \hat{B})$. Hence $int_N(\hat{A}) \cap int_N(\hat{B}) \subseteq int_N(\hat{A} \cap \hat{B})$. Therefore $int_N(\hat{A} \cap \hat{B}) = int_N(\hat{A}) \cap int_N(\hat{B})$.

(xi) Assume that $P_x^\lambda \in cl_N(\hat{A})$ and $\hat{N}$ is a fuzzy quasi N-neighborhood of P_x^λ. To prove that $\hat{N}q\hat{A}$. Since $\hat{N}$ is a fuzzy quasi N-neighborhood of $P_x^\lambda \Rightarrow \exists$ a fuzzy N-open set $\hat{V}$ in X such that $P_x^\lambda q\hat{V} \subseteq \hat{N}$. Assume that $\hat{N}\bar{q}\hat{A} \Rightarrow \hat{N}(x) + \hat{A}(x) \le 1, \forall x \in X$. Since $\hat{V} \subseteq \hat{N} \Rightarrow$ $\hat{V}(x) + \hat{A}(x) \le \hat{N}(x) + \hat{A}(x) \le 1, \forall x \in X \Rightarrow \hat{V}(x) + \hat{A}(x) \le 1, \forall x \in X$. Hence $\hat{A} \subseteq \hat{V}^c$. Since $\hat{V}$ is a fuzzy N-open set in X $\Rightarrow \hat{V}^c$ is a fuzzy N-closed set in X which contains $\hat{A}$. Since $P_x^\lambda q\hat{V}$ $\Rightarrow \lambda + \hat{V}(x) > 1 \Rightarrow \lambda > 1 - \hat{V}(x)$. Hence $P_x^\lambda \notin \hat{V}^c \Rightarrow P_x^\lambda \notin cl_N(\hat{A})$ this is a contradiction.

3. Weak Forms of Fuzzy N-Open Sets

In this section we introduce and study new concepts of fuzzy N-open sets called fuzzy β-N-open sets, fuzzy b-N-open sets, fuzzy pre-N-open sets, fuzzy semi N-open sets and fuzzy α-N-open sets which are weaker than fuzzy N-open sets. The basic properties and characteristics of these fuzzy open sets also have been studied.

Definitions (3.1): A fuzzy subset $\hat{A}$ of a fuzzy topological space $(X, \hat{T})$ is called:

(i) A fuzzy α-N-open set if $\hat{A} \subseteq int_N(cl(int_N(\hat{A})))$.

(ii) A fuzzy pre-N-open set if $\hat{A} \subseteq int_N(cl(\hat{A}))$.

(iii) A fuzzy semi N-open set if $\hat{A} \subseteq cl(int_N(\hat{A}))$.

(iv) A fuzzy b-N-open set if $\hat{A} \subseteq int_N(cl(\hat{A})) \cup cl(int_N(\hat{A}))$.

(v) A fuzzy β-N-open set if $\hat{A} \subseteq cl(int_N(cl(\hat{A})))$.

Proposition (3.2): Let $(X, \hat{T})$ be a fuzzy topological space. Then:

(i) Every fuzzy N-open (resp. fuzzy open) set is fuzzy α-N-open (resp. fuzzy α-open)

(ii) Every fuzzy α-N-open (resp. fuzzy α-open) set is fuzzy pre-N-open (resp. fuzzy pre-open).

(iii) Every fuzzy pre-N-open (resp. fuzzy pre-open) set is fuzzy b-N-open (resp. fuzzy b-open).

(iv) Every fuzzy b-N-open (resp. fuzzy b-open) set is fuzzy β-N-open (resp. fuzzy β-open).

Proof: (i) Let $\hat{A}$ be a fuzzy N-open set in X, then $\hat{A} = int_N(\hat{A})$. Since $\hat{A} \subseteq cl(\hat{A})$, then $\hat{A} \subseteq cl(int_N(\hat{A}))$ and $\hat{A} \subseteq int_N(cl(int_N(\hat{A})))$. Hence $\hat{A}$ is a fuzzy α-N-open set in X.

(ii) Let $\hat{A}$ be a fuzzy α-N-open set in X, then $\hat{A} \subseteq int_N(cl(int_N(\hat{A}))) \subseteq int_N(cl(\hat{A}))$. Thus $\hat{A}$ is a fuzzy pre-N-open set in X.

(iii) Let $\hat{A}$ be a fuzzy pre-N-open set in X, then $\hat{A} \subseteq int_N(cl(\hat{A})) \subseteq int_N(cl(\hat{A})) \cup cl(int_N(\hat{A}))$. Hence $\hat{A}$ is a fuzzy b-N-open set in X.

(v) Let $\hat{A}$ be a fuzzy b-N-open set in X, then $\hat{A} \subseteq int_N(cl(\hat{A})) \cup cl(int_N(\hat{A})) \subseteq cl(int_N(cl(\hat{A}))) \cup$ $cl(int_N(cl(\hat{A}))) = cl(int_N(cl(\hat{A})))$. Thus $\hat{A}$ is a fuzzy β-N-open set in X.

The converse of proposition (3.2) may not be true in general we can see by the following examples.

Example (3.3): Let $X = N$ and $\hat{V}, \hat{A}$ be fuzzy subsets of X defined by:

$$\hat{V}(x) = \begin{cases} 1 & x = 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \hat{A}(x) = \begin{cases} 1 & x = 1,2 \\ 0 & \text{otherwise} \end{cases}.$$

Then $\hat{T} = \{\hat{X}, \hat{\varphi}, \hat{V}\}$ is a fuzzy topology on X, and $\hat{A}$ is a fuzzy α-N-open (fuzzy α-open) set in X, but is not fuzzy N-open (resp. fuzzy open).

Example (3.4): Let $X = N$ and $\hat{T} = \{\hat{X}, \hat{\varphi}\} = \hat{I}$ be the indiscrete fuzzy topology on X. Then $\hat{A}(x) = \begin{cases} 1 & x = 1 \\ 0 & \text{otherwise} \end{cases}$ is a fuzzy pre-N-open (resp. fuzzy pre-open) set, but is not fuzzy α-N-open (resp. fuzzy α-open).

Remark (3.5): Fuzzy N-open (resp. fuzzy α-N-open, fuzzy pre-N-open, fuzzy b-N-open, fuzzy β-N-open) sets and fuzzy open (resp. fuzzy α-open, fuzzy pre-open, fuzzy b-open, fuzzy β-open) sets are independent as shown by the following examples.

Example (3.6): Let $X = \{a, b, c, d\}$ and $\hat{T} = \{\hat{\varphi}, \hat{X}\} = \hat{I}$ be the indiscrete fuzzy topology on X. It is clear that $\hat{N} = \{(a,0.4),(b,0.6),(c,0.1),(d,0.5)\}$ is a fuzzy N-open (resp. fuzzy α-N-open) set, but is not fuzzy α-open. Also, in example (2.4), $\hat{V}$ is a fuzzy open (resp. fuzzy α-open) set, but is not fuzzy α-N-open.

Example (3.7): Let $X = N$ and $\hat{V}_1, \hat{V}_2, \hat{V}_3, \hat{A}$ be fuzzy subsets of X defined by:

$$\hat{V}_1(x) = \begin{cases} 1 & x = 1 \\ 0 & \text{otherwise} \end{cases}, \ \hat{V}_2(x) = \begin{cases} 1 & x = 2 \\ 0 & \text{otherwise} \end{cases}, \ \hat{V}_3(x) = \begin{cases} 1 & x = 1,2 \\ 0 & \text{otherwise} \end{cases}, \text{ and}$$

$$\hat{A}(x) = \begin{cases} 1 & x \in N - \{1\} \\ 0 & x = 1 \end{cases}$$

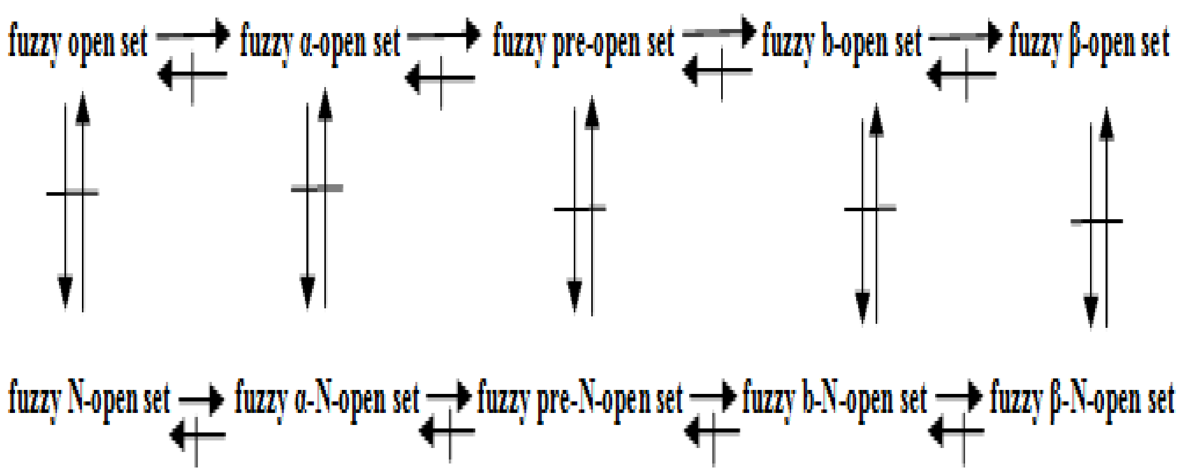
Then $\hat{T} = \{\hat{X}, \hat{\varphi}, \hat{V}_1, \hat{V}_2, \hat{V}_3\}$ is a fuzzy topology on X, and $\hat{A}$ is a fuzzy pre-N-open set, but is not fuzzy pre-open. Also, in example (2.4), $\hat{V}$ is a fuzzy pre-open set, but is not fuzzy pre-N-open.

Example (3.8): Let $X = N$ and $\hat{V}_1, \hat{V}_2, \hat{A}$ be fuzzy subsets of X defined by:

$$\hat{V}_1(x) = \begin{cases} 1 & x = 1 \\ 0 & \text{otherwise} \end{cases}, \ \hat{V}_2(x) = \begin{cases} 1 & x = 1,2 \\ 0 & \text{otherwise} \end{cases}, \text{ and } \hat{A}(x) = \begin{cases} 1 & x \in N - \{1\} \\ 0 & x = 1 \end{cases}$$

Then $\hat{T} = \{\hat{X}, \hat{\varphi}, \hat{V}_1, \hat{V}_2\}$ is a fuzzy topology on X, and $\hat{A}$ is a fuzzy b-N-open (resp. fuzzy β-N-open) set, but is not fuzzy β-open. Also, in example (2.4), $\hat{V}$ is a fuzzy b-open (resp. fuzzy β-open) set, but is not fuzzy β-N-open set.

The following diagram shows the relation between the types of fuzzy open sets and the types of weak fuzzy N-open sets in fuzzy topological spaces



Proposition (3.9): If $\hat{\mathbf{A}}$ is a fuzzy pre-N-open set in a fuzzy topological space $(\mathbf{X},\hat{\mathbf{T}})$ such that $\hat{\mathbf{B}} \subseteq \hat{\mathbf{A}} \subseteq \mathbf{cl}(\hat{\mathbf{B}})$ for any fuzzy subset $\hat{\mathbf{B}}$ of X, then $\hat{\mathbf{B}}$ is also a fuzzy pre-N-open set in X.

Proof: Since $\hat{\mathbf{A}} \subseteq \mathbf{cl}(\hat{\mathbf{B}}) \Rightarrow \mathbf{cl}(\hat{\mathbf{A}}) \subseteq \mathbf{cl}(\mathbf{cl}(\hat{\mathbf{B}})) = \mathbf{cl}(\hat{\mathbf{B}}) \Rightarrow \mathrm{int}_{\mathbf{N}}(\mathbf{cl}(\hat{\mathbf{A}})) \subseteq \mathrm{int}_{\mathbf{N}}(\mathbf{cl}(\hat{\mathbf{B}}))$. But $\hat{\mathbf{B}} \subseteq \hat{\mathbf{A}}$ and $\hat{\mathbf{A}} \subseteq \mathrm{int}_{\mathbf{N}}(\mathbf{cl}(\hat{\mathbf{A}})) \Rightarrow \hat{\mathbf{B}} \subseteq \mathrm{int}_{\mathbf{N}}(\mathbf{cl}(\hat{\mathbf{B}}))$. Thus $\hat{\mathbf{B}}$ is a fuzzy pre-N-open set in X.

Theorem (3.10): A fuzzy subset $\hat{\mathbf{A}}$ of a fuzzy topological space $(\mathbf{X},\hat{\mathbf{T}})$ is fuzzy semi N-open iff $\hat{\mathbf{A}}$ is a fuzzy β-N-open set and $\mathrm{int}_{\mathbf{N}}(\mathbf{cl}(\hat{\mathbf{A}})) \subseteq \mathbf{cl}(\mathrm{int}_{\mathbf{N}}(\hat{\mathbf{A}}))$.

Proof: Let $\hat{\mathbf{A}}$ be a fuzzy semi N-open set in X, then $\hat{\mathbf{A}} \subseteq \mathbf{cl}(\mathrm{int}_{\mathbf{N}}(\hat{\mathbf{A}})) \subseteq \mathbf{cl}(\mathrm{int}_{\mathbf{N}}(\mathbf{cl}(\hat{\mathbf{A}})))$ and hence $\hat{\mathbf{A}}$ is a fuzzy β-N-open set. Also, since $\hat{\mathbf{A}} \subseteq \mathbf{cl}(\mathrm{int}_{\mathbf{N}}(\hat{\mathbf{A}})) \Rightarrow \mathbf{cl}(\hat{\mathbf{A}}) \subseteq \mathbf{cl}(\mathrm{int}_{\mathbf{N}}(\hat{\mathbf{A}})) \Rightarrow \mathrm{int}_{\mathbf{N}}(\mathbf{cl}(\hat{\mathbf{A}})) \subseteq \mathrm{int}_{\mathbf{N}}(\mathbf{cl}(\mathrm{int}_{\mathbf{N}}(\hat{\mathbf{A}}))) \Rightarrow \mathrm{int}_{\mathbf{N}}(\mathbf{cl}(\hat{\mathbf{A}})) \subseteq \mathbf{cl}(\mathrm{int}_{\mathbf{N}}(\hat{\mathbf{A}}))$. Conversely, if $\hat{\mathbf{A}}$ is a fuzzy β-N-open set and $\mathrm{int}_{\mathbf{N}}(\mathbf{cl}(\hat{\mathbf{A}})) \subseteq \mathbf{cl}(\mathrm{int}_{\mathbf{N}}(\hat{\mathbf{A}}))$. Then $\hat{\mathbf{A}} \subseteq \mathbf{cl}(\mathrm{int}_{\mathbf{N}}(\mathbf{cl}(\hat{\mathbf{A}}))) \subseteq \mathbf{cl}(\mathbf{cl}(\mathrm{int}_{\mathbf{N}}(\hat{\mathbf{A}}))) = \mathbf{cl}(\mathrm{int}_{\mathbf{N}}(\hat{\mathbf{A}}))$. Thus $\hat{\mathbf{A}}$ is a fuzzy semi N-open set in X.

Remark (3.11): The intersection of two fuzzy pre-N-open (resp. fuzzy α-N-open, fuzzy b-N-open, fuzzy β-N-open) sets need not be fuzzy pre-N-open (resp. fuzzy α-N-open, fuzzy b-N-open, fuzzy β-N-open) we can see by the following examples:

Example (3.12): Let $\mathbf{X} = \mathfrak{R}$ and $\hat{\mathbf{V}}, \hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}$ be fuzzy subsets of X defined by:

$$\hat{\mathbf{V}}(\mathbf{x}) = \begin{cases} 1 & \mathbf{x} \in \mathfrak{R} - \{1\} \\ 0 & \mathbf{otherwise} \end{cases}, \quad \hat{\mathbf{A}}(\mathbf{x}) = \begin{cases} 1 & \mathbf{x} \in \mathbf{Q} \\ 0 & \mathbf{otherwise} \end{cases}, \quad \hat{\mathbf{B}}(\mathbf{x}) = \begin{cases} 1 & \mathbf{x} \in \mathbf{Q}^{\mathbf{c}} \cup \{1\} \\ 0 & \mathbf{otherwise} \end{cases}, \text{ and}$$

$$\hat{\mathbf{C}}(\mathbf{x}) = \begin{cases} 1 & \mathbf{x} = 1 \\ 0 & \mathbf{otherwise} \end{cases}.$$

Then $\hat{\mathbf{T}} = \{\hat{\mathbf{X}}, \hat{\varphi}, \hat{\mathbf{V}}\}$ is a fuzzy topology on X, and $\hat{\mathbf{A}}, \hat{\mathbf{B}}$ are fuzzy pre-N-open (resp. fuzzy b-N-open, fuzzy β-N-open) sets, since $\hat{\mathbf{A}} \subseteq \mathrm{int}_{\mathbf{N}}(\mathbf{cl}(\hat{\mathbf{A}})) = \mathrm{int}_{\mathbf{N}}(\hat{\mathbf{X}}) = \hat{\mathbf{X}}$ and $\hat{\mathbf{B}} \subseteq \mathrm{int}_{\mathbf{N}}(\mathbf{cl}(\hat{\mathbf{B}})) = \mathrm{int}_{\mathbf{N}}(\hat{\mathbf{X}}) = \hat{\mathbf{X}}$,

but $\hat{\mathbf{A}} \cap \hat{\mathbf{B}} = \hat{\mathbf{C}}$ is not fuzzy β-N-open, since $\hat{\mathbf{C}} \not\subset \mathbf{cl}(\mathrm{int}_{\mathbf{N}}(\mathbf{cl}(\hat{\mathbf{C}}))) = \mathbf{cl}(\mathrm{int}_{\mathbf{N}}(\hat{\mathbf{C}})) = \mathbf{cl}(\{\hat{\varphi}\}) = \hat{\varphi}$.

Example (3.13): Let $\mathrm{X} = \mathrm{N}$ and $\hat{\mathbf{V}}_1, \hat{\mathbf{V}}_2, \hat{\mathbf{V}}_3, \hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}$ be fuzzy subsets of X defined by:

$$\hat{\mathbf{V}}_1(\mathbf{x}) = \begin{cases} 1 & \mathbf{x} = 1 \\ 0 & \mathbf{otherwise} \end{cases}, \quad \hat{\mathbf{V}}_2(\mathbf{x}) = \begin{cases} 1 & \mathbf{x} = 2 \\ 0 & \mathbf{otherwise} \end{cases}, \quad \hat{\mathbf{V}}_3(\mathbf{x}) = \begin{cases} 1 & \mathbf{x} = 1,2 \\ 0 & \mathbf{otherwise} \end{cases},$$

$$\hat{\mathbf{A}}(\mathbf{x}) = \begin{cases} 1 & \mathbf{x} \in \mathbf{N} - \{1\} \\ 0 & \mathbf{x} = 1 \end{cases}, \hat{\mathbf{B}}(\mathbf{x}) = \begin{cases} 1 & \mathbf{x} = 1,3 \\ 0 & \mathbf{otherwise} \end{cases}, \text{ and } \hat{\mathbf{C}}(\mathbf{x}) = \begin{cases} 1 & \mathbf{x} = 3 \\ 0 & \mathbf{otherwise} \end{cases}.$$

Then $\hat{\mathbf{T}} = \{\hat{\mathbf{X}}, \hat{\varphi}, \hat{\mathbf{V}}_1, \hat{\mathbf{V}}_2, \hat{\mathbf{V}}_3\}$ is a fuzzy topology on X, and $\hat{\mathbf{A}}, \hat{\mathbf{B}}$ are fuzzy α-N-open sets, but $\hat{\mathbf{A}} \cap \hat{\mathbf{B}} =$

$\hat{\mathbf{C}}$ is not fuzzy α-N-open, since $\hat{\mathbf{C}} \not\subset \mathrm{int}_{\mathbf{N}}(\mathbf{cl}(\mathrm{int}_{\mathbf{N}}(\hat{\mathbf{C}}))) = \mathrm{int}_{\mathbf{N}}(\mathbf{cl}(\hat{\varphi})) = \hat{\varphi}$.

Theorem (3.14): If $\{\hat{\mathbf{A}}_\alpha : \alpha \in \Lambda\}$ is a family of fuzzy b-N-open (resp. fuzzy α-N-open, fuzzy pre-N-open, fuzzy β-N-open) sets of a fuzzy topological space $(\mathbf{X},\hat{\mathbf{T}})$, then $\bigcup_{\alpha \in \Lambda} \hat{\mathbf{A}}_\alpha$ is also fuzzy b-N-open (resp. fuzzy α-N-open, fuzzy pre-N-open, fuzzy β-N-open).

Proof: Since $\hat{\mathbf{A}}_\alpha$ is fuzzy b-N-open $\forall\ \alpha \in \Lambda \Rightarrow \hat{\mathbf{A}}_\alpha \subseteq \text{int}_{\mathbf{N}}(\mathbf{cl}(\hat{\mathbf{A}}_\alpha)) \cup \mathbf{cl}(\text{int}_{\mathbf{N}}(\hat{\mathbf{A}}_\alpha))\ \ \forall\ \alpha \in \Lambda$

Therefore $\bigcup_{\alpha\in\Lambda} \hat{\mathbf{A}}_\alpha \subseteq \bigcup_{\alpha\in\Lambda} [\text{int}_{\mathbf{N}}(\mathbf{cl}(\hat{\mathbf{A}}_\alpha)) \cup \mathbf{cl}(\text{int}_{\mathbf{N}}(\hat{\mathbf{A}}_\alpha))]$

$= [\bigcup_{\alpha\in\Lambda} \text{int}_{\mathbf{N}}(\mathbf{cl}(\hat{\mathbf{A}}_\alpha))] \cup [\bigcup_{\alpha\in\Lambda} \mathbf{cl}(\text{int}_{\mathbf{N}}(\hat{\mathbf{A}}_\alpha))]$

$\subseteq [\text{int}_{\mathbf{N}}(\bigcup_{\alpha\in\Lambda} \mathbf{cl}(\hat{\mathbf{A}}_\alpha))] \cup [\mathbf{cl}(\bigcup_{\alpha\in\Lambda} \text{int}_{\mathbf{N}}(\hat{\mathbf{A}}_\alpha))]$ (By theorem (2.8),(xii))

$\subseteq [\text{int}_{\mathbf{N}}(\mathbf{cl}(\bigcup_{\alpha\in\Lambda}(\hat{\mathbf{A}}_\alpha)))] \cup [\mathbf{cl}(\text{int}_{\mathbf{N}}(\bigcup_{\alpha\in\Lambda}(\hat{\mathbf{A}}_\alpha)))]$ (By theorem (2.8),(xii))

Thus $\bigcup_{\alpha\in\Lambda} \hat{\mathbf{A}}_\alpha$ is a fuzzy b-N-open set in X. By the same way we can prove the other cases.

Proposition (3.15): If $\hat{\mathbf{A}}$ is a fuzzy b-N-open set in a fuzzy topological space $(\mathbf{X}, \hat{\mathbf{T}})$ such that $\text{int}_{\mathbf{N}}(\hat{\mathbf{A}}) = \hat{\varphi}$, then $\hat{\mathbf{A}}$ is a fuzzy pre-N-open set in X.

Proof: Since $\hat{\mathbf{A}}$ is a fuzzy b-N-open set, then $\hat{\mathbf{A}} \subseteq \text{int}_{\mathbf{N}}(\mathbf{cl}(\hat{\mathbf{A}})) \cup \mathbf{cl}(\text{int}_{\mathbf{N}}(\hat{\mathbf{A}}))$. Since $\text{int}_{\mathbf{N}}(\hat{\mathbf{A}}) = \hat{\varphi}$, then $\mathbf{cl}(\text{int}_{\mathbf{N}}(\hat{\mathbf{A}})) = \hat{\varphi}$, hence $\hat{\mathbf{A}} \subseteq \text{int}_{\mathbf{N}}(\mathbf{cl}(\hat{\mathbf{A}}))$. Thus $\hat{\mathbf{A}}$ is a fuzzy pre-N-open set in X.

Definitions (3.16): A fuzzy subset $\hat{\mathbf{A}}$ of a fuzzy topological space $(\mathbf{X}, \hat{\mathbf{T}})$ is called:

(i) Fuzzy N-$\hat{\mathbf{t}}$-set if $\text{int}_{\mathbf{N}}(\hat{\mathbf{A}}) = \text{int}_{\mathbf{N}}(\mathbf{cl}(\hat{\mathbf{A}}))$.

(ii) Fuzzy N-$\hat{\mathbf{B}}$-set if $\hat{\mathbf{A}} = \hat{\mathbf{U}} \cap \hat{\mathbf{V}}$, where $\hat{\mathbf{U}} \in \hat{\mathbf{T}}_{\mathbf{N}}$ and $\hat{\mathbf{V}}$ is a fuzzy N-$\hat{\mathbf{t}}$-set.

Proposition (3.17): If $\hat{\mathbf{A}}_1$ and $\hat{\mathbf{A}}_2$ are fuzzy N-$\hat{\mathbf{t}}$-sets in a fuzzy topological space $(\mathbf{X}, \hat{\mathbf{T}})$, then so is $\hat{\mathbf{A}}_1 \cap \hat{\mathbf{A}}_2$.

Proof: Let $\hat{\mathbf{A}}_1$ and $\hat{\mathbf{A}}_2$ be fuzzy N-$\hat{\mathbf{t}}$-sets. Then:

$\text{int}_{\mathbf{N}}(\mathbf{cl}(\hat{\mathbf{A}}_1 \cap \hat{\mathbf{A}}_2)) \subseteq \text{int}_{\mathbf{N}}(\mathbf{cl}(\hat{\mathbf{A}}_1) \cap \mathbf{cl}(\hat{\mathbf{A}}_2))$

$= \text{int}_{\mathbf{N}}(\mathbf{cl}(\hat{\mathbf{A}}_1)) \cap \text{int}_{\mathbf{N}}(\mathbf{cl}(\hat{\mathbf{A}}_2))$

$= \text{int}_{\mathbf{N}}(\hat{\mathbf{A}}_1) \cap \text{int}_{\mathbf{N}}(\hat{\mathbf{A}}_2) = \text{int}_{\mathbf{N}}(\hat{\mathbf{A}}_1 \cap \hat{\mathbf{A}}_2)$.

Since $\text{int}_{\mathbf{N}}(\hat{\mathbf{A}}_1 \cap \hat{\mathbf{A}}_2) \subseteq \text{int}_{\mathbf{N}}(\mathbf{cl}(\hat{\mathbf{A}}_1 \cap \hat{\mathbf{A}}_2))$, then $\text{int}_{\mathbf{N}}(\hat{\mathbf{A}}_1 \cap \hat{\mathbf{A}}_2) = \text{int}_{\mathbf{N}}(\mathbf{cl}(\hat{\mathbf{A}}_1 \cap \hat{\mathbf{A}}_2))$. Thus $\hat{\mathbf{A}}_1 \cap \hat{\mathbf{A}}_2$ is a fuzzy N-$\hat{\mathbf{t}}$-set.

Proposition (3.18): Let $(\mathbf{X}, \hat{\mathbf{T}})$ be a fuzzy topological space and $\hat{\mathbf{A}}$ be a fuzzy subset of X. Then the following statements are equivalent:

(i) $\hat{\mathbf{A}}$ is a fuzzy N-open set in X.

(ii) $\hat{\mathbf{A}}$ is a fuzzy pre-N-open and a fuzzy N-$\hat{\mathbf{B}}$-set in X.

Proof: $(\text{i}) \Rightarrow (\text{ii})$. Let $\hat{\mathbf{A}}$ be a fuzzy N-open set in X, then $\hat{\mathbf{A}} = \text{int}_{\mathbf{N}}(\hat{\mathbf{A}}) \subseteq \text{int}_{\mathbf{N}}(\mathbf{cl}(\hat{\mathbf{A}}))$, thus $\hat{\mathbf{A}}$ is a fuzzy pre-N-open set in X. Also, $\hat{\mathbf{A}} = \hat{\mathbf{A}} \cap \hat{\mathbf{X}}$, where $\hat{\mathbf{A}} \in \hat{\mathbf{T}}_{\mathbf{N}}$ and $\hat{\mathbf{X}}$ is a fuzzy N-$\hat{\mathbf{t}}$-set. Hence $\hat{\mathbf{A}}$ is a fuzzy N-$\hat{\mathbf{B}}$-set in X.

$(\text{ii}) \Rightarrow (\text{i})$. If $\hat{\mathbf{A}}$ is a fuzzy N-$\hat{\mathbf{B}}$-set, then $\hat{\mathbf{A}} = \hat{\mathbf{U}} \cap \hat{\mathbf{V}}$, where $\hat{\mathbf{U}} \in \hat{\mathbf{T}}_{\mathbf{N}}$ and $\hat{\mathbf{V}}$ is a fuzzy N-$\hat{\mathbf{t}}$-set. By hypothesis, $\hat{\mathbf{A}}$ is a fuzzy pre-N-open set, then $\hat{\mathbf{A}} \subseteq \text{int}_{\mathbf{N}}(\mathbf{cl}(\hat{\mathbf{A}})) = \text{int}_{\mathbf{N}}(\mathbf{cl}(\hat{\mathbf{U}} \cap \hat{\mathbf{V}})) \subseteq$ $\text{int}_{\mathbf{N}}(\mathbf{cl}(\hat{\mathbf{U}}) \cap \mathbf{cl}(\hat{\mathbf{V}})) = \text{int}_{\mathbf{N}}(\mathbf{cl}(\hat{\mathbf{U}})) \cap \text{int}_{\mathbf{N}}(\mathbf{cl}(\hat{\mathbf{V}})) = \text{int}_{\mathbf{N}}(\mathbf{cl}(\hat{\mathbf{U}})) \cap \text{int}_{\mathbf{N}}(\hat{\mathbf{V}})$. Thus $\hat{\mathbf{A}} = \hat{\mathbf{U}} \cap \hat{\mathbf{V}} =$ $(\hat{\mathbf{U}} \cap \hat{\mathbf{V}}) \cap \hat{\mathbf{U}} \subseteq (\text{int}_{\mathbf{N}}(\mathbf{cl}(\hat{\mathbf{U}})) \cap \text{int}_{\mathbf{N}}(\hat{\mathbf{V}})) \cap \hat{\mathbf{U}} = (\text{int}_{\mathbf{N}}(\mathbf{cl}(\hat{\mathbf{U}})) \cap \hat{\mathbf{U}}) \cap \text{int}_{\mathbf{N}}(\hat{\mathbf{V}}) = \hat{\mathbf{U}} \cap \text{int}_{\mathbf{N}}(\mathbf{V}) =$ $\text{int}_{\mathbf{N}}(\hat{\mathbf{U}}) \cap \text{int}_{\mathbf{N}}(\hat{\mathbf{V}}) = \text{int}_{\mathbf{N}}(\hat{\mathbf{U}} \cap \hat{\mathbf{V}}) = \text{int}_{\mathbf{N}}(\hat{\mathbf{A}})$. Hence $\hat{\mathbf{A}} = \text{int}_{\mathbf{N}}(\hat{\mathbf{A}})$, therefore $\hat{\mathbf{A}}$ is a fuzzy N-open set in X.

Definitions (3.19): A fuzzy subset $\hat{\mathbf{A}}$ of a fuzzy topological space $(\mathbf{X}, \hat{\mathbf{T}})$ is called:

(i) Fuzzy N-$\hat{\mathbf{t}}_\alpha$-set if $\text{int}_{\mathbf{N}}(\hat{\mathbf{A}}) = \text{int}_{\mathbf{N}}(\mathbf{cl}(\text{int}_{\mathbf{N}}(\hat{\mathbf{A}})))$.

(ii) Fuzzy N-$\hat{\mathbf{B}}_\alpha$-set if $\hat{\mathbf{A}} = \hat{\mathbf{U}} \cap \hat{\mathbf{V}}$, where $\hat{\mathbf{U}} \in \hat{\mathbf{T}}_\mathbf{N}$ and $\hat{\mathbf{V}}$ is a fuzzy N-$\hat{\mathbf{t}}_\alpha$-set.

Proposition (3.20): If $\hat{\mathbf{A}}_1$ and $\hat{\mathbf{A}}_2$ are fuzzy N-$\hat{\mathbf{t}}_\alpha$-sets in a fuzzy topological space $(\mathbf{X}, \hat{\mathbf{T}})$, then so is $\hat{\mathbf{A}}_1 \cap \hat{\mathbf{A}}_2$.

Proof: Let $\hat{\mathbf{A}}_1$ and $\hat{\mathbf{A}}_2$ be fuzzy N-$\hat{\mathbf{t}}_\alpha$-sets. Then

$\text{int}_\mathbf{N}(\mathbf{cl}(\text{int}_\mathbf{N}(\hat{\mathbf{A}}_1 \cap \hat{\mathbf{A}}_2))) = \text{int}_\mathbf{N}(\mathbf{cl}(\text{int}_\mathbf{N}(\hat{\mathbf{A}}_1) \cap \text{int}_\mathbf{N}(\hat{\mathbf{A}}_2)))$
$\subseteq \text{int}_\mathbf{N}(\mathbf{cl}(\text{int}_\mathbf{N}(\hat{\mathbf{A}}_1)) \cap \mathbf{cl}(\text{int}_\mathbf{N}(\hat{\mathbf{A}}_2))) = \text{int}_\mathbf{N}(\mathbf{cl}(\text{int}_\mathbf{N}(\hat{\mathbf{A}}_1))) \cap \text{int}_\mathbf{N}(\mathbf{cl}(\text{int}_\mathbf{N}(\hat{\mathbf{A}}_2)))$
$= \text{int}_\mathbf{N}(\hat{\mathbf{A}}_1) \cap \text{int}_\mathbf{N}(\hat{\mathbf{A}}_2) = \text{int}_\mathbf{N}(\hat{\mathbf{A}}_1 \cap \hat{\mathbf{A}}_2)$.

But $\text{int}_\mathbf{N}(\hat{\mathbf{A}}_1 \cap \hat{\mathbf{A}}_2) \subseteq \text{int}_\mathbf{N}(\mathbf{cl}(\text{int}_\mathbf{N}(\hat{\mathbf{A}}_1 \cap \hat{\mathbf{A}}_2)$, then $\text{int}_\mathbf{N}(\hat{\mathbf{A}}_1 \cap \hat{\mathbf{A}}_2) = \text{int}_\mathbf{N}(\mathbf{cl}(\text{int}_\mathbf{N}(\hat{\mathbf{A}}_1 \cap \hat{\mathbf{A}}_2)))$. Hence $\hat{\mathbf{A}}_1 \cap \hat{\mathbf{A}}_2$ is a fuzzy N-$\hat{\mathbf{t}}_\alpha$-set.

Proposition (3.21): Let $(\mathbf{X}, \hat{\mathbf{T}})$ be a fuzzy topological space and $\hat{\mathbf{A}}$ be a fuzzy subset of X. Then the following statements are equivalent:

(i) $\hat{\mathbf{A}}$ is a fuzzy N-open set in X.

(ii) $\hat{\mathbf{A}}$ is a fuzzy α-N-open and a fuzzy N-$\hat{\mathbf{B}}_\alpha$-set in X.

Proof: (i) $\Rightarrow$ (ii). Let $\hat{\mathbf{A}}$ be a fuzzy N-open set in X, then $\hat{\mathbf{A}} = \text{int}_\mathbf{N}(\hat{\mathbf{A}}) \subseteq \mathbf{cl}(\text{int}_\mathbf{N}(\hat{\mathbf{A}})) \subseteq$ $\text{int}_\mathbf{N}(\mathbf{cl}(\text{int}_\mathbf{N}(\hat{\mathbf{A}})))$, hence $\hat{\mathbf{A}}$ is fuzzy α-N-open set in X. Also, $\hat{\mathbf{A}} = \hat{\mathbf{A}} \cap \hat{\mathbf{X}}$, where $\hat{\mathbf{A}} \in \hat{\mathbf{T}}_\mathbf{N}$ and $\hat{\mathbf{X}}$ is a fuzzy N-$\hat{\mathbf{t}}_\alpha$-set , thus $\hat{\mathbf{A}}$ is a fuzzy N-$\hat{\mathbf{B}}_\alpha$-set in X.

(ii) $\Rightarrow$ (i). If $\hat{\mathbf{A}}$ is a fuzzy N-$\hat{\mathbf{B}}_\alpha$-set, then $\hat{\mathbf{A}} = \hat{\mathbf{U}} \cap \hat{\mathbf{V}}$, where $\hat{\mathbf{U}} \in \hat{\mathbf{T}}_\mathbf{N}$ and $\hat{\mathbf{V}}$ is a fuzzy N-$\hat{\mathbf{t}}_\alpha$-set. By hypothesis, $\hat{\mathbf{A}}$ is a fuzzy α-N-open set, then $\hat{\mathbf{A}} \subseteq \text{int}_\mathbf{N}(\mathbf{cl}(\text{int}_\mathbf{N}(\hat{\mathbf{A}}))) = \text{int}_\mathbf{N}(\mathbf{cl}(\text{int}_\mathbf{N}(\hat{\mathbf{U}} \cap \hat{\mathbf{V}}))) =$
$\text{int}_\mathbf{N}(\mathbf{cl}(\text{int}_\mathbf{N}(\hat{\mathbf{U}}) \cap \text{int}_\mathbf{N}(\hat{\mathbf{V}}))) \subseteq \text{int}_\mathbf{N}(\mathbf{cl}(\text{int}_\mathbf{N}(\hat{\mathbf{U}}) \cap \mathbf{cl}(\text{int}_\mathbf{N}(\hat{\mathbf{V}})))$
$= \text{int}_\mathbf{N}(\mathbf{cl}(\text{int}_\mathbf{N}(\hat{\mathbf{U}}))) \cap \text{int}_\mathbf{N}(\mathbf{cl}(\text{int}_\mathbf{N}(\hat{\mathbf{V}}))) \subseteq \text{int}_\mathbf{N}(\mathbf{cl}(\hat{\mathbf{U}})) \cap \text{int}_\mathbf{N}(\hat{\mathbf{V}})$. Therefore $\hat{\mathbf{A}} = \hat{\mathbf{U}} \cap \hat{\mathbf{V}} = (\hat{\mathbf{U}} \cap \hat{\mathbf{V}})$ $\cap \hat{\mathbf{U}} \subseteq (\text{int}_\mathbf{N}(\mathbf{cl}(\hat{\mathbf{U}})) \cap \text{int}_\mathbf{N}(\hat{\mathbf{V}})) \cap \hat{\mathbf{U}} = (\text{int}_\mathbf{N}(\mathbf{cl}(\hat{\mathbf{U}})) \cap \hat{\mathbf{U}}) \cap \text{int}_\mathbf{N}(\hat{\mathbf{V}}) = \hat{\mathbf{U}} \cap \text{int}_\mathbf{N}(\mathbf{V})$
$= \text{int}_\mathbf{N}(\hat{\mathbf{U}}) \cap \text{int}_\mathbf{N}(\hat{\mathbf{V}}) = \text{int}_\mathbf{N}(\hat{\mathbf{U}} \cap \hat{\mathbf{V}}) = \text{int}_\mathbf{N}(\hat{\mathbf{A}})$. Hence $\hat{\mathbf{A}} = \text{int}_\mathbf{N}(\hat{\mathbf{A}})$, thus $\hat{\mathbf{A}}$ is a fuzzy N-open in X.

Definitions (3.22): A fuzzy topological space $(\mathbf{X}, \hat{\mathbf{T}})$ is said to satisfy:

(i) The fuzzy N-$\hat{\mathbf{B}}_\alpha$-condition if every fuzzy α-N-open set is fuzzy N-$\hat{\mathbf{B}}_\alpha$-set.

(ii) The fuzzy N-$\hat{\mathbf{B}}$-condition if every fuzzy pre-N-open set is fuzzy N-$\hat{\mathbf{B}}$-set.

4. Weak Forms of Fuzzy $\hat{\mathbf{D}}_\mathbf{N}$-Sets

In this section we introduce and study new concepts in fuzzy topological spaces $(\mathbf{X}, \hat{\mathbf{T}})$ called fuzzy $\hat{\mathbf{D}}_\mathbf{N}$-sets, fuzzy $\hat{\mathbf{D}}_{\alpha-\mathbf{N}}$-sets, fuzzy $\hat{\mathbf{D}}_{\mathbf{pre}-\mathbf{N}}$-sets, fuzzy $\hat{\mathbf{D}}_{\mathbf{b}-\mathbf{N}}$-sets and fuzzy $\hat{\mathbf{D}}_{\beta-\mathbf{N}}$-sets which are weaker than fuzzy $\hat{\mathbf{D}}_\mathbf{N}$-sets. Furthermore we discussed the relation between weak fuzzy N-open sets and weak fuzzy $\hat{\mathbf{D}}_\mathbf{N}$-sets.

Definition (4.1): A fuzzy subset $\hat{\mathbf{A}}$ of a fuzzy topological space $(\mathbf{X}, \hat{\mathbf{T}})$ is called a fuzzy $\hat{\mathbf{D}}_\mathbf{N}$-set (resp. fuzzy $\hat{\mathbf{D}}_{\alpha-\mathbf{N}}$-set, fuzzy $\hat{\mathbf{D}}_{\mathbf{pre}-\mathbf{N}}$-set, fuzzy $\hat{\mathbf{D}}_{\mathbf{b}-\mathbf{N}}$-set, fuzzy $\hat{\mathbf{D}}_{\beta-\mathbf{N}}$-set) if there exists two fuzzy N-open (resp. fuzzy α-N-open, fuzzy pre-N-open, fuzzy b-N-open, fuzzy β-N-open) sets $\hat{\mathbf{A}}_1$ and $\hat{\mathbf{A}}_2$ in X such that $\hat{\mathbf{A}} = \hat{\mathbf{A}}_1 \setminus \hat{\mathbf{A}}_2$.

Remark (4.2): In definition (4.1), if $\hat{\mathbf{A}}_2 = \hat{\varphi}$, then every fuzzy N-open (resp. fuzzy α-N-open, fuzzy pre-N-open, fuzzy b-N-open, fuzzy β-N-open) set in X is a fuzzy $\hat{\mathbf{D}}_\mathbf{N}$-set (resp. fuzzy $\hat{\mathbf{D}}_{\alpha-\mathbf{N}}$-set, fuzzy $\hat{\mathbf{D}}_{\mathbf{pre}-\mathbf{N}}$-set, fuzzy $\hat{\mathbf{D}}_{\mathbf{b}-\mathbf{N}}$-set, fuzzy $\hat{\mathbf{D}}_{\beta-\mathbf{N}}$-set).

The converse of Remark (4.2) may not be true in general as shown by the following examples.

Example (4.3): Let $X = N$ and $\hat{T} = \{\hat{X}, \hat{\varphi}\}$ be a fuzzy topology on X. Then $\hat{A}(x) = \begin{cases} 1 & x = 1 \\ 0 & \text{otherwise} \end{cases}$ is a fuzzy $\hat{D}_{\alpha-N}$-set and a fuzzy $\hat{D}_N$-set, but is not fuzzy α-N-open set.

Example (4.4): Let $X = N$ and $\hat{V}, \hat{A}$ be fuzzy subsets of X defined by:

$$\hat{V}(x) = \begin{cases} 1 & x \in N - \{1\} \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \hat{A}(x) = \begin{cases} 1 & x = 1 \\ 0 & \text{otherwise} \end{cases}.$$

Then $\hat{T} = \{\hat{X}, \hat{\varphi}, \hat{V}\}$ is a fuzzy topology on X, and $\hat{A}$ is a fuzzy $\hat{D}_{pre-N}$-set (resp. fuzzy $\hat{D}_{b-N}$-set, fuzzy $\hat{D}_{\beta-N}$-set), but is not fuzzy β-N-open set.

Proposition (4.5): Let $(X, \hat{T})$ be a fuzzy topological space. Then:

(i) Every fuzzy $\hat{D}_N$-set is a fuzzy $\hat{D}_{\alpha-N}$-set.

(ii) Every fuzzy $\hat{D}_{\alpha-N}$-set is a fuzzy $\hat{D}_{pre-N}$-set.

(iii) Every fuzzy $\hat{D}_{pre-N}$-set is a fuzzy $\hat{D}_{b-N}$-set.

(iv) Every fuzzy $\hat{D}_{b-N}$-set is a fuzzy $\hat{D}_{\beta-N}$-set.

Proof: Follows from proposition (3.2).

The converse of proposition (4.5) number (i) and (ii) may not be true in general we can see in the following examples.

Example (4.6): Let $X = N$ and $\hat{V}, \hat{A}, \hat{A}_1, \hat{A}_2$ be fuzzy subsets of X defined by:

$$\hat{V}(x) = \begin{cases} 1 & x = 1 \\ 0 & \text{otherwise} \end{cases}, \hat{A}(x) = \begin{cases} 1 & x \in N - \{1,2\} \\ 0 & \text{otherwise} \end{cases}, \quad \hat{A}_1(x) = \begin{cases} 1 & x \in N - \{1,2\} \\ 0 & \text{otherwise} \end{cases}, \quad \text{and}$$

$$\hat{A}_2(x) = \begin{cases} 1 & x = 1,2 \\ 0 & \text{otherwise} \end{cases}.$$

Then $\hat{T} = \{\hat{X}, \hat{\varphi}, \hat{V}\}$ is a fuzzy topology on X, and $\hat{A}$ is a fuzzy $\hat{D}_{\alpha-N}$-set, since $\exists$ $\hat{A}_1$ and $\hat{A}_2$ are fuzzy α-N-open sets such that $\hat{A} = \hat{A}_1 \setminus \hat{A}_2$, but $\hat{A}$ is not fuzzy $\hat{D}_N$-set.

Example (4.7): Let $X = N$ and $\hat{A}, \hat{A}_1, \hat{A}_2$ be fuzzy subsets of X defined by:

$$\hat{A}(x) = \begin{cases} 1 & x \in N - \{1,2\} \\ 0 & \text{otherwise} \end{cases}, \ \hat{A}_1(x) = \begin{cases} 1 & x \in N - \{1\} \\ 0 & \text{otherwise} \end{cases}, \text{ and } \hat{A}_2(x) = \begin{cases} 1 & x = 2 \\ 0 & \text{otherwise} \end{cases}.$$

Then $\hat{T} = \{\hat{X}, \hat{\varphi}\}$ is a fuzzy topology on X and $\hat{A}$ is a fuzzy $\hat{D}_{pre-N}$-set, since $\exists$ $\hat{A}_1$ and $\hat{A}_2$ are fuzzy pre-N-open sets such that $\hat{A} = \hat{A}_1 \setminus \hat{A}_2$, but $\hat{A}$ is not fuzzy $\hat{D}_{\alpha-N}$-set.

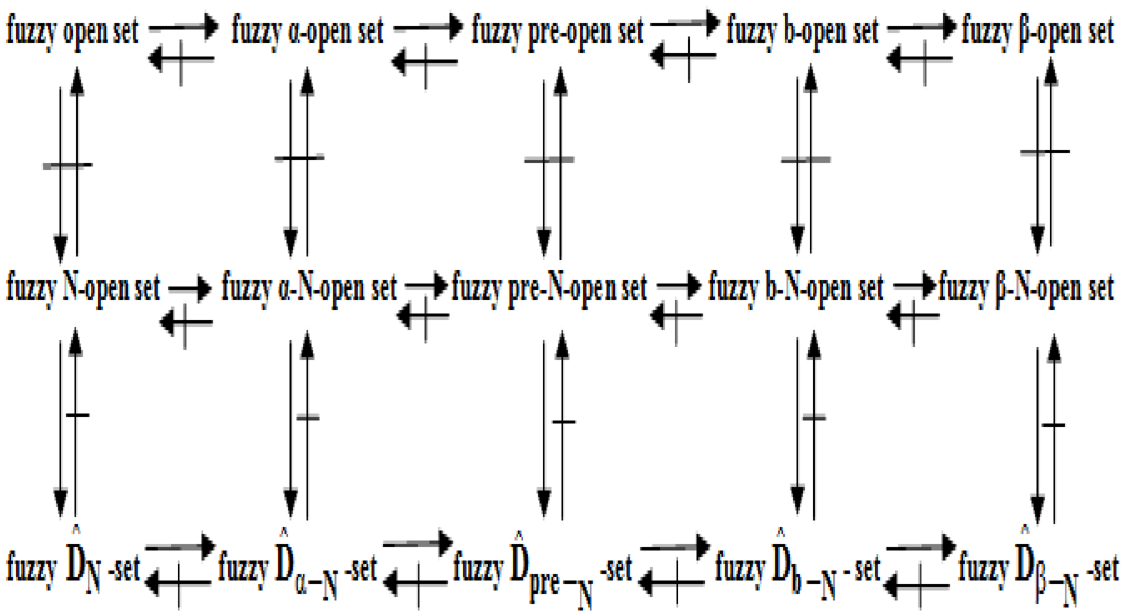
Proposition (4.8): In any fuzzy topological space satisfies fuzzy N-$\hat{B}$-condition fuzzy $\hat{D}_{pre-N}$-set is a fuzzy $\hat{D}_N$-set.

Proof: Follows from Proposition (3.18).

Proposition (4.9): In any fuzzy topological space satisfies fuzzy N- $\hat{\mathbf{B}}_{\alpha}$ -condition fuzzy $\hat{\mathbf{D}}_{\alpha-\mathbf{N}}$ -set is a fuzzy $\hat{\mathbf{D}}_{\mathbf{N}}$ -set.

Proof: Follows from Proposition (3.21).

The following diagram shows the relation between the fuzzy open sets and each of fuzzy N-open sets, weak fuzzy N-open sets and weak fuzzy $\hat{\mathbf{D}}_{\mathbf{N}}$ -set.



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