Reverse Derivations With Invertible Values

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Abstract
In this paper, we will prove the following theorem, Let R be a ring with 1 having a reverse derivation $d \neq 0$ such that, for each $x \in R$, either $d(x) = 0$ or $d(x)$ is invertible in R, then R must be one of the following: (i) a division ring D, (ii) D_2, the ring of 2×2 matrices over D, (iii) $D[x]/(x^2)$ where char $D = 2$, $d(D) = 0$ and $d(x) = 1 + ax$ for some a in the center Z of D. Furthermore, if $2R \neq 0$ then $R = D_2$ is possible if and only if D does not contain all quadratic extensions of Z, the center of D.

Keywords: derivation, reverse derivation.

الاشتقاقات العكسية مع القيم العكسية

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الخلاصة
في هذا البحث سنقوم ببرهان المبرهنة التالية. لتكن R حلقة مع 1 تمتلك اشتقاق عكسي$d \neq 0$ بحيث، لكل $x \in R$، اما $d(x) = 0$ او $d(x)$ يكون لها نظير في R. فان R يجب ان تكون واحدة من الاتي : (1) حلقة القسمة D، (2) D_2 ، حلقة المصفوفات 2×2 على D ، (3) $D[x]/(x^2)$ حيث char $D =$ 2، $d(D) = 0$ و $d(x) = 1 + ax$ لبعض a في المركز Z ل D . بالاضافة الى ذلك، اذا $2R \neq 0$ فان $R = D_2$ اذا وفقط اذا D لا تحتوي على كافة توسعات الدرجة الثانية من المركز Z ل D .

Introduction

Throughout, R will represent a ring with 1. Recall that a ring R is called prime if $aRb = 0$ implies $a = 0$ or $b = 0$; and it is called semiprime if $aRa = 0$ implies $a = 0$ [1]. A ring R is said to be 2-torion free, if whenever $2a = 0$, with $a \in R$, then $a = 0$ [2]. An additive mapping d from R into itself is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x,y \in R$ [3]. Bresar and Vukman [4] have introduced the notion of a reverse derivation as an additive mapping d from R into itself satisfying $d(xy) = d(y)x + yd(x)$ holds for all $x,y \in R$. Obviously, if R is commutative, then both derivation and reverse derivation are the same. The reverse derivations on semiprime rings have been studied by Samman and Alyamani [5]. A derivation d is called inner in case there exists $a \in R$ such that $d(x) = [a, x]$ holds for all $x \in R$. In a recent paper [6] the authors proved that in case R is a prime ring with a non-zero right reverse derivation d and U be the left ideal of R then R is commutative. Some results concerning derivations in prime and semiprime rings can be found in [7-11].

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Recently Bergen, Herstein and Lanski [12] studied the structure of a ring R with a derivation $d \neq 0$ such that, for each $x \in R$, $d(x) = 0$ or $d(x)$ is invertible. They proved that, except for a special case which occurs when $2R = 0$, such a ring must be either a division ring D or the ring D_2 of 2×2 matrices over a division ring. In this paper we address ourselves to a similar problem of rings but with reverse derivations, namely:
suppose that R is a ring. If $d \neq 0$ is a reverse derivation of R such that for every $x \in R$, $d(x) = 0$ or $d(x)$ is invertible in R, what can we conclude about R? We shall prove that R must be rather special. More precisely we shall prove the following:

THEOREM. Let R be a ring with 1 and $d \neq 0$ a reverse derivation of R such that, for each $x \in R$, $d(x) = 0$ or $d(x)$ is invertible in R. Then R is either

1. a division ring D, or
2. D_2, or
3. $D[x]/(x^2)$ where char $D = 2$, $d(D) = 0$ and $d(x) = 1 + ax$ for some a in the center Z of D.

Furthermore, if $2R \neq 0$ then $R = D_2$ is possible if and only if D does not contain all quadratic extensions of Z, the center of D.

We shall also show that if $R = D_2$ then d must be inner, provided $2R \neq 0$; however, d may fail to be inner when $2R = 0$. In addition, we shall show that if $R = D[x]/(x^2)$, then d cannot be inner.

Finally, we consider a similar situation, one in which $d(x) = 0$ or is invertible not for all $x \in R$, but for all x in a suitable subset. In that context we also get results that says that $R = D$, $R = D_2$, or $R = D[x]/(x^2)$.

In what follows, R will be a ring with 1 and $d \neq 0$ will be a reverse derivation of R such that $d(x) = 0$ or is invertible, for all $x \in R$.

Preliminaries

We begin with the following

LEMMA 2.1. If $d(x) = 0$, then either $x = 0$ or x is invertible.

PROOF. Suppose that $x \neq 0$; since $d \neq 0$ there is $y \in R$ such that $d(y) \neq 0$. Hence $d(y)$ is invertible. Now $d(yx) = xd(y) \neq 0$ since $x \neq 0$ and $d(y)$ is invertible; therefore $d(yx)$ is invertible, that is, $xd(y)$ is invertible. Thus x is invertible.

LEMMA 2.2. If $L \neq R$ is a left ideal of R, then $L \cap d(R) = 0$.

PROOF. We may assume that $L \neq 0$; let $0 \neq x \in L \cap d(R)$, then $x = d(y)$ for some $y \in R$; therefore $d(y)$ is invertible, then L contains invertible element, implying that $L = R$, in contradiction to $L \neq R$.

As an easy consequence of Lemma 2.1 we have

LEMMA 2.3. If $L \neq 0$ is a one-sided ideal of R, then $d(L) \neq 0$.

PROOF. Since $d \neq 0$ the lemma is certainly true if $L = R$. Suppose that $L \neq R$, L cannot contain invertible elements. If $0 \neq a \in L$, then by Lemma 2.1, $d(a) \neq 0$ since a is not invertible. Thus $d(L) \neq 0$; in fact we saw that d is not zero on the non-zero elements of L.

Another immediate consequence of Lemma 2.1 is

LEMMA 2.4. If $2x = 0$ for some $x \neq 0$ in R, then $2R = 0$.

PROOF. Since $2x = 0$, $d(2x) = 2d(x) = 0$. If $d(x) = 0$ then, by Lemma 2.1, x is invertible and since $2x = 0$ we get $0 = (2x)x^{-1} = 2$ and so $2R = 0$. On the other hand, if $d(x) \neq 0$ then $d(x)$ is invertible and since $2d(x) = 0$ we get, once again, that $2R = 0$.

LEMMA 2.5. If L is an ideal of R, then $L + d(L)$ is also an ideal of R.

PROOF. It is clear.

LEMMA 2.6. If L is a proper ideal of R, then L is both minimal and maximal.

PROOF. It certainly suffices to show that every proper ideals of R is maximal. Let $L \subset T$ be proper ideals of R, by Lemma 2.5, $L + d(L)$ is also an ideal of R. Since, by Lemma 2.3, $d(L) \neq 0$, and so $L + d(L)$ contains invertible elements, we must have $L + d(L) = R$. Therefore if $t \in T$ there exist $a, b \in L$ such that $t = a + d(b)$. Consequently, $d(b) = t - a \in T \cap d(L) = 0$ therefore $t = a \in L$. Thus $L = T$ and L is maximal.

We can now narrow in on the structure of *R:*

LEMMA 2.7. (a) If I is a proper ideal of R, then $I^2 = 0$.

(b) If $2R \neq 0$, then R is simple.

PROOF.(a) If I is a proper ideal of R, then

$$d(I^2) \subset d(I)I + I\,d(I) \subset I,$$

hence by Lemma 2.3, $I^2 = 0$ as I cannot contain any invertible elements.

(b) Suppose $2R \neq 0$ and let $I \neq 0$ be a proper ideal of R, then by Lemma 2.3, there is a $b \in I$ such that $d(b) \neq 0$, so $d(b)$ is invertible. Now, since $b^2 = 0$ by (a)

$$0 = d^2(b^2) = b\,d^2(b) + 2d(b)^2 + d^2(b)\,b,$$

in consequence of which, $2d(b)^2 \in I$, hence

$$0 = (2d(b)^2)^2 = 4d(b)^4 .$$

Since $d(b)$ is invertible we have $2^2 = 4 = 0$, so, by Lemma 2.4, $2R = 0$, in contradiction to $2R \neq 0$. Therefore if $2R \neq 0$, R is simple.

By combining Lemmas 2.6 and 2.7we see that if $2R \neq 0$, then $R = D$ or $R = D_2$.

for any division ring D and every non-zero reverse derivation, d, of D we certainly have that $d(x) = 0$ or $d(x)$ is invertible for every $x \in R$. For D_2 , under what conditions on D, is there a non-zero reverse derivation d with this property ? to answer this question we need to analyze the reverse derivation of the 2 × 2 matrices over an arbitrary ring. In the two lemmas we assume that S is any ring with 1, R = S_2 , the ring of 2 × 2 matrices over S and d is any reverse derivation of R.

LEMMA 2.8. Let S be any ring with 1 and let $R = S_2$. If d is a reverse derivation of R, then there exists $\alpha, \beta, \gamma \in S$ such that:

$$d(e_{11}) = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix},\ d(e_{12}) = \begin{pmatrix} -\beta & \gamma \\ 0 & \beta \end{pmatrix},\ d(e_{21}) = \begin{pmatrix} -1 & 0 \\ -\gamma & 1 \end{pmatrix},\ \ d(e_{22}) = \begin{pmatrix} 0 & -\alpha \\ -\beta & 0 \end{pmatrix}$$

and, for $a \in S$,

$$d\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} f(a) & 0 \\ -(a\beta - \beta a) & f(a) - a\gamma + \gamma a \end{pmatrix} .$$

Since its proof is obtained by a straight-forward computation, we omit the proof.

We use the formulas in Lemma 2.8 to prove the following fact inter-relating d and f:

LEMMA 2.9. Let R, S, d, and f be as in Lemma 2.8. Then d is inner on R if and only if f is inner on S.

PROOF. If d is the inner derivation on R induced by $\begin{pmatrix} s & t \\ u & v \end{pmatrix}$, where $s, t, u, v \in S$, then it is immediate that $f(x) = xs - sx$ for all $x \in S$, hence f is inner on S.

Conversely, if f is the inner derivation on S defined by $f(x) = xr - rx$, where $r \in S$, then

$$d(T) = T\begin{pmatrix} r & 0 \\ -\beta & r-\gamma \end{pmatrix} - \begin{pmatrix} r & 0 \\ -\beta & r-\gamma \end{pmatrix}T$$

for all $T \in R$, where β, γ are as in Lemma 2.8. This is verified by noting that d is the inner derivation induced by $\begin{pmatrix} r & 0 \\ -\beta & r-\gamma \end{pmatrix}$ agree on all matrix units and on the elements of *S*, hence on all of *R*.

Now we return to our original situation, assuming that *R* is a ring with 1 and a reverse derivation $d \neq 0$ such that for each $x \in R$ either $d(x) = 0$ or $d(x)$ is invertible. We shall characterize those *D* for which $R = D_2$ has such a reverse derivation, at least when the characteristic of *D* is not 2. To do so we need

LEMMA 2.10.If $R = D_2$ and $2R \neq 0$, then *d* is inner.

PROOF. Given *d, f,* α, β, γ be as in Lemma 2.8. Then, by Lemma 2.9, it is enough to prove *f* is inner on *D*. If $a, b, c, e \in D$, then by Lemma 2.8 and by the multiplicative law for reverse derivation we have

$$(1) \qquad d\begin{pmatrix} a & b \\ c & e \end{pmatrix} = \begin{pmatrix} f(a) - b\beta - c & f(b) + a\alpha + b\gamma - e\alpha \\ f(c) + \beta a - e\beta - \gamma c & f(e) - e\gamma + \gamma e + \beta b + c \end{pmatrix}.$$

By (1) we have for $a \in D$ that

$$d\begin{pmatrix} a & 0 \\ f(a) & a \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ u & v \end{pmatrix}$$

Where

$u = f(f(a)) + \beta a - a\beta - \gamma f(a)$ and
$v = f(a) - a\gamma + \gamma a + f(a)$.

Since $\begin{pmatrix} 0 & 0 \\ u & v \end{pmatrix}$ is not invertible we must have $u = v = 0$.

thus $v = 0$ gives us

$$(2) \qquad 0 = v = f(a) - a\gamma + \gamma a + f(a).$$

which gives us

$2f(a) = a\gamma - \gamma a$

Since char $D \neq 0$, dividing by 2 we see that *f* is the inner derivation on *D* induced by ½(γ). The condition:"*D* does not contain all quadratic extensions of *Z*"
will come up. By this we mean that there are elements δ and σ in *Z* such that the polynomial $t^2 + \delta t + \sigma$ has no root in

LEMMA 2.11. If *D* is a division ring then $R = D_2$ has reverse inner derivation $d \neq 0$ such that for all $x \in R$ either $d(x) = 0$ or $d(x)$ is invertible if and only if *D* does not contain all quadratic extensions of *Z*.

PROOF.Suppose that *R* has such a reverse inner derivation induced by the matrix $M \in R$. We claim that *M* cannot be a diagonal matrix; for if

$M = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, where $a, b \in D$, computing

$$e_{12}M - Me_{12} = \begin{pmatrix} 0 & b-a \\ 0 & 0 \end{pmatrix}$$

we have, by our basic hypothesis, that $b = a$ Computing

$$\begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix} M - M \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} ca - ac & 0 \\ 0 & 0 \end{pmatrix} ,$$

For all $c \in D$, we get that $a \in Z$. Hence $M \in Z$, whence $d = 0$, contrary to hypothesis. Since M is not diagonal there exists an invertible matrix $T \in D_2$ such that

$$T M T^{-1} = \begin{pmatrix} 0 & 1 \\ \alpha & \beta \end{pmatrix} \quad \text{where } \alpha, \beta \in D.$$

The reverse inner derivation induced by $T M T^{-1}$ also has the property that all its values are 0 or invertible. we may assume that d is induced by $\begin{pmatrix} 0 & 1 \\ \alpha & \beta \end{pmatrix}$ $,\alpha, \beta \in D$.

If $\gamma \in D$ then

$$d\begin{pmatrix} \gamma & 0 \\ 0 & \gamma \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \gamma\alpha - \alpha\gamma & \gamma\beta - \beta\gamma \end{pmatrix}$$

which is not invertible, therefore $\alpha\gamma = \gamma\alpha$, $\beta\gamma = \gamma\beta$. In short, α and β are both in *Z.*
Since

$d\begin{pmatrix} 0 & 1 \\ \alpha & \beta \end{pmatrix} = 0$, by [Lemma 1, [12]] , we have that $\begin{pmatrix} 0 & 1 \\ \alpha & \beta \end{pmatrix}$ is invertible, hence $\alpha \neq 0$.
For $\gamma \in D$,

$$d\begin{pmatrix} 0 & 1 \\ 0 & \gamma \end{pmatrix} = \begin{pmatrix} \alpha & \beta - \gamma \\ \gamma\alpha & -\alpha \end{pmatrix}$$

cannot be 0 by [Lemma 1, [12]] , so is invertible. This gives us that

$$(\gamma^2 - \beta\gamma - \alpha)\alpha \neq 0 \quad \text{for all } \gamma \in D.$$

In other words the quadratic polynomial $t^2 - \beta t - \alpha$ over *Z* has no root in *D*, and so *D* does not contain all quadratic extensions of Z.

Conversely, if *D* does not contain all quadratic extensions of Z there exist $\alpha, \beta \in Z$, with $\alpha \neq 0$, such that $\alpha x^2 - \beta x - 1$ has no solution in *D.*

Let d be a reverse inner derivation of D_2 induced by $\begin{pmatrix} 0 & 1 \\ \alpha & \beta \end{pmatrix}$. We claim that every non-zero value of d is invertible. Let *a, b, c,* and *e* be in *D;* then

$$d\begin{pmatrix} a & b \\ c & e \end{pmatrix} = \begin{pmatrix} \alpha b - c & a - e + \beta b \\ \alpha(e - a) - \beta c & c - \alpha b \end{pmatrix}.$$

if we let $m = \alpha b - c$ and $n = a - e + \beta b$ then

$\alpha(e-a)-\beta c=-\alpha n+\beta m$, and

$$d\begin{pmatrix} a & b \\ c & e \end{pmatrix} = \begin{pmatrix} m & n \\ -\alpha n+\beta m & -m \end{pmatrix} .$$

Suppose, for the moment, that $m = 0$; in that case

$$d\begin{pmatrix} a & b \\ c & e \end{pmatrix} = \begin{pmatrix} 0 & n \\ -\alpha n & 0 \end{pmatrix}$$

which is either 0 or invertible, since $\alpha \neq 0$.
If, on the other hand, $m \neq 0$ then

$$d\begin{pmatrix} a & b \\ c & e \end{pmatrix} = \begin{pmatrix} m & n \\ -\alpha n+\beta m & -m \end{pmatrix} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} 1 & w \\ -\alpha w+\beta & -1 \end{pmatrix}$$

where $w = m^{-1}e$. Since $m \neq 0$, $d\begin{pmatrix} a & b \\ c & e \end{pmatrix}$ is invertible if and only if $\begin{pmatrix} 1 & w \\ -\alpha w+\beta & -1 \end{pmatrix}$ is invertible, that is, if and only if

$-1 - w(-\alpha w+\beta) \neq 0.$

However, by our choice of α and β, $\alpha w^2-\beta w-1 = 0$ for all $w \in D$. Thus d is a reverse inner derivation of D_2 all of whose non-zero values are invertible.
The only piece that remains in order to prove our main theorem is the case where $2R = 0$ and R is neither D nor D_2. We handle this case with

LEMMA 2.12. If R is not simple then $R = D[x]/(x^2)$, where char $D = 2$, $d(D) = 0$, $d(x) = 1 + ax$ for some a in Z, the center of D; moreover, d is not inner.
PROOF. By Lemmas 2.6 and 2.7, $2R = 0$, all proper ideals of R have square zero, and all proper ideals of R are both minimal and maximal. As a result,we easily obtain that R contains a unique (left, right, two-sided) ideal M and $M^2 = 0$. Therefore, as in the proof of Lemma 2.6, $R = M + d(M)$, hence if $r \in R$ there exist $m, n \in M$ such that $d(r) = m + d(n)$. Consequently, $d(r - n) = m \in M \cap d(R) = 0$ and so, if $D = \ker d$ then, by Lemma 2.1, D is a division ring and $R = D + M$. By the uniqueness of M, if $0 \neq x \in M$ then $R = D + Dx$ and thus $sd(x)s^{-1} = s + tx$ where $s, t \in D$ and $s \neq 0$. Since $d(D) = 0$, if we replace x by sx, we may assume $d(x) = 1 + ax$ for some $a \in D$.
If $s \in D$, we can use the facts $M = Rx$, $M^2 = 0$, $d(s) = 0$, and $d(x) = 1 + ax$ to obtain

$$0 = d((sx)^2) = d(sx)sx + sxd(sx) = (1 + ax)s^2x + sx(1 + ax)s = s^2x + sxs = s(sx + xs).$$

If $s \neq 0$, s is invertible, hence $xs = sx$ and x is in the center of R. Therefore $R = D[x]/(x^2)$.

Now, if $s \in D$ then $sx + xs = 0$, thus

$$0 = d(sx + xs) = (1 + ax)s + s(1 + ax) = axs + sax = (as + sa)x.$$

Since all non-zero elements of D are invertible in R, $as + sa = 0$, hence a is in the center of D.
Finally, since $x \in M$ and $d(x) \notin M$, it is clear that d is not inner.

Results

We can now prove our main result, which is the theorem stated at the outset, and which we record as

THEOREM 3.1. Let R be a ring with 1 and $d \neq 0$ a reverse derivation of R such that, for each $x \in R$, $d(x) = 0$ or $d(x)$ is invertible in R. Then R is either

1. a division ring D, or
2. D_2, or
3. $D[x]/(x^2)$. where char $D = 2$, $d(D) = 0$ and $d(x) = 1 + ax$ for some a in the center Z of D.

Furthermore, if $2R \neq 0$ then $R = D_2$ is possible if and only if D does not contain all quadratic extensions of Z, the center of D.

PROOF. If R is simple, then by Lemma 2.6 either $R = D$ or $R = D_2$.

Furthermore if $2R \neq 0$, by Lemma 2.10 D_2 has such a reverse derivation if and only if it has a reverse inner derivation with the special property. However Lemma 2.11 tells us that D_2 has such a reverse inner derivation if and only if D does not contain all quadratic extensions of Z.

If R is not simple, then by applying Lemma 2.12 we obtain our result.

One question concerning Theorem 3.1 remains. Namely, in the case $R = D_2$ is it necessary to assume $2R \neq 0$ in order to prove that T is inner?

We now present an example that shows if $2R = 0$ then $R = D_2$ can have a reverse outer derivation d such that $d(x) = 0$ or $d(x)$ is invertible , for all $x \in R$.

Example 3.2. Take $R = M_2(F)$ for $F = GF(2)(x)<<y>>$, the field of (finite) Laurent series with coefficients in the rational function field in one variable over $GF(2)$. Define a reverse derivation δ on F by extending the action $\delta(f(x)) = 0$ and $\delta(y) = xy$. If $a \in F$ is written $a = a_E + a_o$, where a_E is the series of even powers of y appearing in a, and $a_o = a - a_E$, then $\delta(a) = ax_o$. Let $A = \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix} \in M_2(F)$ and set $d = d_A + \bar{\delta}$ where d_A is a reverse inner derivation of $M_2(F)$ induced by A and $\bar{\delta}$ is the reverse derivation of $M_2(F)$ defined by

$$\bar{\delta}\begin{pmatrix} a & b \\ c & e \end{pmatrix} = \begin{pmatrix} xa - ax + \delta(a) & \delta(b) \\ xc - cx + \delta(c) & \delta(e) \end{pmatrix}$$

Not that d is not inner since

$$d\begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} xy & 0 \\ 0 & xy \end{pmatrix}$$

An easy computation shows

$$d\begin{pmatrix} a & b \\ c & e \end{pmatrix} = \begin{pmatrix} b + c + xa_o & a + e + xb_E \\ a + e + xc_E & b + c + xe_o \end{pmatrix}.$$

It can now be shown by a direct, if somewhat tedious computation that d has invertible values; and we omit the details.

We shall now consider a situation closely related to the one we have been discussing.

THEOREM 3.3. Let R be a ring with 1 and suppose that $d \neq 0$ is a reverse derivation of R such that $d(L) \neq 0$ for some an ideal L of R and $d(x) = 0$ or $d(x)$ is invertible for every $x \in L$. Then $R = D$, or $R = D_2$, or $R = D[x]/(x^2)$ where $2R = 0$ for some division ring D.

PROOF. Suppose that $L \neq 0$ is an ideal of R such that $d(L) \neq 0$, and such that for every $x \in L$ either $d(x) = 0$ or $d(x)$ is invertible. Since we already know the answer when $L = R$, we suppose that $L \neq R$. We wish to determine the structure of R. Since the arguments will be similar to the ones we have given earlier we give then more sketchily here.

Let $0 \neq x \in R$ be such that $d(x) = 0$ then, since $xL \subset L$ and $d(xL) = d(L)x$ we easily get the result of Lemma 2.1, namely, that x is invertible in R. This immediately implies the results of Lemmas 2.3 and 2.4, that is, that if $d(W) = 0$ for some left ideal W of R then $W = 0$, and if R have 2-torsion then $2R = 0$.

As before, from our assumptions on L, $L + d(L) = R$, hence if W is a proper ideal of R containing L and $w \in W$ then $w = a + d(b)$, for some $a, b \in L$. Once again,

$$w - a = d(b) \in W \cap T(L) = 0$$

and so, $W = L$. By this argument and our analog to Lemma 2.3, L and every non-zero ideal of R contained in L are maximal, hence L is both minimal and maximal.

We now examine $l(L) = \{x \in R \mid xL = 0\}$. Since $1 = a + d(b)$, for some $a, b \in L$, if $x \in l(L)$ then

$$x = (a + d(b))x = ax - bd(x) + d(xb) = ax - bd(x) \in L$$

and so, by the minimality of L, $l(L) = 0$ or $l(L) = L$.

Suppose $l(L) = 0$, then R is semiprime for if $I^2 = 0$ and $I \neq 0$ we obtain the contradiction $0 = I^2 L = I(IL) = IL = L$. It easily follows that R is simple, for if $I \neq 0$ then

$$0 \neq d(I^2 L) \subset d(L) \cap I,$$

hence $I = R$. By Wedderburn's theorem, $R = D$ or $R = D_2$.

On the other hand, suppose $l(L) = L$, that is $L^2 = 0$. By repeated use of the maximality and minimality of L we obtain that L is the unique ideal of R, for if $I \neq L$ is an ideal of R then $R = I + L$ and so,

$$L = LR = LI + L^2 = LI \subset I,$$

a contradiction. It is now clear that L is the unique (left, right, two-sided) ideal of R. Now, as in Lemma 2.7, if $b \in L$ such that $d(b) \neq 0$ then

$$0 = d^2(b^2) = bd^2(b) + 2d(b)^2 + d^2(b)b,$$

hence $2d(b)^2 \in L$ and so $4d(b)^4 = 0$. Once again, $2R = 0$. Let $x \in R$ and $y \in L$ such that $d(x) \in L$ and $d(y) \neq 0$; in this case

$d(xy) = d(y)x + yd(x) = d(y)x$

and so, x is 0 or invertible. Therefore $D = \{x \in R \mid d(x) \in L\}$ is a division ring and by the identical argument used in the proof of Lemma 2.12 we obtain that $R = D[x]/(x^2)$ where $d(x) = 1 + ax$ for some a in the center of D.

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