Asymptotic Stability of Index 2 and 3 Hesenberg Differential Algebraic Equations

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Abstract
This paper aims to study the asymptotic stability of the equilibrium points of the index 2 and index 3 Hesenberg differential algebraic equations. The problem reformulated to an equivalent explicit differential algebraic equations system, so the asymptotic stability is easily investigated. The singular points such as impasse points and singularity induced bifurcation points are identified in this kind of differential algebraic equations by using conclusion of the explicit differential algebraic equations.

Keywords: DAEs, Asymptotic stability.

الاستقرارية المطلقه للمعادلة التفاضلية الجبرية من نوع هسنبرك ذات الدليل 2و3

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الخلاصة
يهدف هذا البحث الى دراسة الاستقرارية المطلقة لنقاط الاتزان للمعادله التفاضلية الجبرية من نوع هسنبرك ذات الدليل 2 و3. تم اعادة صياغة المعادلة التفاضلية الجبرية الى معادلة تفاضلية جبرية ضمنية مكافئة وبالتالي فمن السهوله دراسة الاستقراريه المطلقه. كذلك باستخدام المعادلة التفاضلية الجبريه الضمنية المكافئه تم البحث عن النقاط المنفرده مثل النقاط المعيقه والنقاط التي تولد تفرع الحل .

1.Introduction

We study the index-2 Hesenberg differential algebraic equations of the form

$$\acute{x} = f(x,y,\lambda), \quad f \in C^1(R^n \times R^m \times R^r, R^n)$$
$$0 = g(x,\lambda), \quad g \in C^2(R^n \times R^r, R^m) \qquad (1.1)$$

Where the Jacobean, $\left(\frac{\partial g}{\partial x}\right)\left(\frac{\partial f}{\partial g}\right)$ is non singular, and the index-3 Hesenberg differential algebraic equations of the form:

$$\acute{x} = f(x,y,z,\lambda), \quad f \in C^1(R^n \times R^m \times R^k \times R^r, R^n),$$
$$\acute{y} = g(x,y,\lambda), \quad g \in C^2(R^n \times R^m \times R^r, R^m),$$
$$0 = h(y,\lambda), \quad g \in C^2(R^m \times R^r, R^k) \qquad (1.2)$$

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where the Jacobean, $\left(\frac{\partial h}{\partial g}\right)\left(\frac{\partial g}{\partial x}\right)\left(\frac{\partial f}{\partial z}\right)$ is non singular. As a shorthand notion for such kind of systems we often write Hesenberg (DAEs). This study includes investigating the asymptotic stability of the equilibrium point in this kind of systems. Also we study the case where $\left(\frac{\partial g}{\partial x}\right)\left(\frac{\partial f}{\partial g}\right)$ and $\left(\frac{\partial h}{\partial g}\right)\left(\frac{\partial g}{\partial x}\right)\left(\frac{\partial f}{\partial z}\right)$ are singular at a point (x_*, y_*, λ_*) and $(x_*, y_*, z_*, \lambda_*)$ respectively.

In [1] the asymptotic stability of Euler Lagrange equations for constrained mechanical system studied by showing the equivalence of the direct linearization of the original system to that of corresponding state space form. We follow the idea of [1] by applying it to the general DAEs of Hesenberg form of higher index. This implies the asymptotic stability of the non- linear DAEs can be studied locally near the equilibrium point via its linearization. The basic idea here is to transform the DAEs (1.1) and (1.2) to another equivalent index-1 DAE system.

$$\begin{aligned} \acute{x} &= f(x,y,\lambda), \quad f \in C^1(R^n \times R^m \times R^r, R^n) \\ 0 &= \tilde{g}(x,y,\lambda), \quad \tilde{g} \in C^2(R^n \times R^m \times R^r, R^m) \qquad (1.3) \end{aligned}$$

where $\left(\frac{\partial \tilde{g}}{\partial g}\right)$ is non-singular. Then the treatment of the asymptotic stability is more easily. In general the study of asymptotic stability in this paper depends on the linear part of the state space form of the DAEs. So first we try to obtain the formal state space of the DAEs by using implicit function theorem. Then we obtain the linearization which is a linear ODE. On the other hand the direct linearization of original DAEs is also linear ODE. Then the equivalence between these two linear ODE is obvious by using the property of the equilibrium solutions and the implicit function theorem.

For the singular case, i.e. $\left(\frac{\partial g}{\partial x}\right)\left(\frac{\partial f}{\partial y}\right)$ and $\left(\frac{\partial h}{\partial g}\right)\left(\frac{\partial g}{\partial y}\right)\left(\frac{\partial f}{\partial z}\right)$ are singular at the point (x_*, y_*, λ_*) and $(x_*, y_*, z_*, \lambda_*)$ respectively, we recognize two kinds of singular point on each of the singular surfaces

$$S_1 = \{(x,y,\lambda) \in (R^n \times R^m \times R^r) : g(x) = 0, \det[\left(\frac{\partial g}{\partial x}\right)\left(\frac{\partial f}{\partial y}\right) = 0\} \qquad (1.4)$$

and

$$S_2 = \{(x,y,z,\lambda) \in (R^n \times R^m \times R^k \times R^r) : h(y) = 0, \det[\left(\frac{\partial h}{\partial x}\right)\left(\frac{\partial g}{\partial y}\right)\left(\frac{\partial f}{\partial z}\right)] = 0\} \qquad (1.5)$$

The first one is the impasse points $(x_*, y_*) \in S_1$ and $(x_*, y_*, z_*) \in S_2$ for which $f(x_*, y_*) \neq 0$ and $f(x_*, y_*, z_*) \neq 0$ respectively. The second one is the singularity indueed bifurcation (SIB) points $(x_*, y_*, \lambda_*) \in S_1$ and $(x_*, y_*, \lambda_*, z_*) \in S_2$ for which $f(x_*, y_*, \lambda_*) = 0$ and $f(x_*, y_*, z_*, \lambda_*) = 0$ respectively.

We show that the well know result about impasse points given by Chua and Deng [2] cannot be applied directly to the Hesenberg DAE system. In other words this can be applied to the equivalent index lower one system so that we can get information about the impasse points in the DAEs (1.1) and (1.2).

This paper is organized as follows: Section 2 is devoted to study the asymptotic stability of the index 2 and index 3 Hesenberg DAE system. In Section 3 we consider the impasse points in Hesenberg DAE system. The SIB points are studied in Section 4.

2. Hesenberg DAEs with Non-Singular Case

In this section we will study the asymptotic stability of index 2 and 3 Hesenberg DAEs.

2.1 Index 2 Hesenberg DAEs

Consider the following explicit Hesenberg DAE system of the form.

$$\begin{aligned} \acute{x} &= f(x,y), \quad f \in C^1(R^n \times R^m, R^n) \\ 0 &= g(x), \quad g \in C^2(R^n, R^m) \qquad (2.1) \end{aligned}$$

where the product of the Jacobean matrix $\left(\frac{\partial g}{\partial x}\right)\left(\frac{\partial f}{\partial y}\right)$ is non-singular on $R^n \times R^m$. Then it is clear that the index of (2.1) is 2. Usually it is quite difficult to treat (2.1) directly so it is more convenient to transform (2.1) to another equivalently system with index one lower. Then the conclusion of index -1 DAEs can be applied. Now by differentiating the second equation in (2.1) one get.

$$\frac{\partial g(x)}{\partial x} f(x,y) = 0.$$

Define

$$\tilde{g}(x,y) = \frac{\partial g(x)}{\partial x} f(x,y),$$

then we obtain the DAEs:

$$\begin{aligned} \acute{x} &= f(x,y) \\ 0 &= \tilde{g}(x,y). \end{aligned} \qquad (2.2)$$

The Jacobean matrix:

$$\frac{\partial \tilde{g}}{\partial y} = \left(\frac{\partial g}{\partial x}\right)\left(\frac{\partial f}{\partial y}\right)$$

is non-singular on $R^n \times R^m$. Therefore (2.2) is an index-1 DAEs.
Now let (x_0, y_0) be the equilibrium solution of (2.1). Then

$$f(x_0,y_0) = 0 \, , \tilde{g}(x_0,y_0) = 0,$$

and by implicit function theorem there exist a neighborhood N of (x_0, y_0) and a differentiable function $\psi(x)$ such that in N we have $y = \psi(x)$ and

$$\tilde{g}(x,\psi(x)) = 0$$

is holds for any $x \in N$. Thus in N the constraint can be satisfied naturally and from (2.2) the reduced space is given by

$$\acute{x} = f(x,\psi(x)) \qquad (2.3)$$

Now without lose of the generality linearizing (2.3) along the zero equilibrium solution yields.

$$\begin{aligned} \acute{x} &= Df\begin{pmatrix} x \\ y \end{pmatrix} \\ &= \left(\frac{\partial f}{\partial x}\,\frac{\partial f}{\partial y}\,\frac{\partial \psi}{\partial x}\right)\begin{pmatrix} x \\ y \end{pmatrix} \\ &= \left(\frac{\partial f}{\partial x}x + \frac{\partial f}{\partial y}\frac{\partial \psi}{\partial x}y\right) \end{aligned} \qquad (2.4)$$

where the partial derivatives all take values at the equilibrium point. The linearization is assured to give the correct local information about the equilibrium solution.
Next we shall consider the direct linearization of (2.2) at the zero equilibrium solution which yields.

$$\acute{x} = \frac{\partial f}{\partial x}x + \frac{\partial f}{\partial y}y \qquad (2.5a)$$

$$0 = \frac{\partial \tilde{g}}{\partial x}x + \frac{\partial \tilde{g}}{\partial y}y \qquad (2.5b)$$

since $\left(\frac{\partial \tilde{g}}{\partial y}\right) \neq 0$, solving for y using (2.5b) and insert it back into (2.5a) to obtain:

$$\acute{x} = \frac{\partial f}{\partial x}x - \frac{\partial f}{\partial y}\left(\frac{\partial \tilde{g}}{\partial y}\right)^{-1}\frac{\partial \tilde{g}}{\partial x}x \qquad (2.6)$$

and since $\tilde{g}(x,\psi(x)) = 0$ in N, then we have

$$\frac{\partial \psi}{\partial x} = -\left(\frac{\partial \tilde{g}}{\partial y}\right)^{-1}\frac{\partial \tilde{g}}{\partial x}$$

and the linearization (2.6) becomes:

$$\acute{x} = \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial \psi}{\partial x}\right)x \qquad (2.7)$$

from which, we can see that the linearization (2.7) and (2.4) are same. This conclude that linearizing DAE directly along the equilibrium solution is valid and will give the same asymptotic stability information as that of the state space from ODE problem.

2.2 Index 3 DAEs

Consider the following index-3 DAEs with explicit Hesenberg form:

$$\begin{aligned} \acute{x} &= f(x,y,z), \quad f \in C^1(R^n \times R^m \times R^k, R^n) \\ \acute{y} &= g(x,y), \quad g \in C^2(R^n \times R^m, R^m) \\ 0 &= h(y), h \in C^3(R^m, R^k) \end{aligned} \qquad (2.8)$$

where the product of the Jacobean matrices:

$$(\frac{\partial h}{\partial y})(\frac{\partial g}{\partial x})(\frac{\partial f}{\partial z})$$

is non-singular on $R^n \times R^m \times R^k$.

Now differentiating the third equation in (2.8) we get:

$$\frac{\partial h(y)}{\partial y} g(x,y) = 0,$$

and denote

$$\breve{h}(x,y) = \left(\frac{\partial h(y)}{\partial y}\right) g(x,y),$$

then the result is the DAEs given by:

$$\begin{aligned} \acute{x} &= f(x,y,z), \\ \acute{y} &= g(x,y), \\ 0 &= \breve{h}(x,y). \end{aligned} \qquad (2.9)$$

We claim to transform the system (2.8) to another index-2 system in order to use the conclusion of section 2. For this purpose assume that $X=(x, y)^T$ then the DAEs (2.9) can be written as:

$$\begin{aligned} \acute{X} &= G_1(X,z), \\ 0 &= G_2(X), \end{aligned} \qquad (2.10)$$

where

$$G_1(X,z) = \begin{pmatrix} f(x,y,z) \\ g(x,y) \end{pmatrix}, G_2(X) = \breve{h}(x,y).$$

Now since

$$\frac{\partial G_1}{\partial z} = \begin{pmatrix} \frac{\partial f}{\partial z} \\ 0 \end{pmatrix},$$

and

$$\begin{aligned} \frac{\partial G_2}{\partial X} &= \left(\frac{\partial \breve{h}}{\partial x}, \frac{\partial \breve{h}}{\partial y}\right) \\ &= \left(\frac{\partial h}{\partial y}\frac{\partial g}{\partial x}, \frac{\partial^2 h}{\partial y^2} g + \frac{\partial h}{\partial y}\frac{\partial g}{\partial y}\right), \end{aligned} \qquad (2.11)$$

consequently we get:

$$\left(\frac{\partial G_2}{\partial X}\right)\left(\frac{\partial G_1}{\partial z}\right) = \left(\frac{\partial h}{\partial y}\right)\left(\frac{\partial g}{\partial x}\right)\left(\frac{\partial f}{\partial z}\right)$$

which is non singular on $R^n \times R^m \times R^k$. So the non-singularity of $\left(\frac{\partial G_2}{\partial X}\right)\left(\frac{\partial G_1}{\partial z}\right)$ is guaranteed by the non-singularity of $\left(\frac{\partial h}{\partial y}\right)\left(\frac{\partial g}{\partial x}\right)\left(\frac{\partial f}{\partial z}\right)$. Therefore the system (2.8) is of index-2 Hesenberg form and the conclusion of Section 2 can be applied.

Now consider the equivalent index-1 DAE given by:

$$\begin{aligned} \acute{X} &= G_1(X,z), \\ 0 &= G_2(X,z), \end{aligned} \qquad (2.12)$$

where

$$G_2(X,z) = \frac{\partial G_2}{\partial X} G_1(X,z)$$
$$= (\frac{\partial h}{\partial y}\frac{\partial g}{\partial x}(f+g) + \frac{\partial^2 h}{\partial y^2} g^2. \qquad (2.13)$$

And since

$$\frac{G_2(X,z)}{\partial z} = (\frac{\partial h}{\partial y})(\frac{\partial g}{\partial x})(\frac{\partial f}{\partial z})$$

is non-singular so (2.12) is index-1 DAEs. Assume that (X_0, z_0) is an equilibrium point of the DAEs (2.12) then obviously (X_0 , z_0) is also an equilibrium point to the original system (2.8).

Then by implicit function theorem there exists a neighborhood $\acute{N}$ of (X_0, z_0) and a differential function $\acute{U}(X)$ such that in $\acute{N}$ we have $z = \acute{U}(X)$ and $G_2\left(X, \acute{U}(X)\right) = 0$ holds for $X \in \acute{U}$. In other words the constraint can be satisfied in $\acute{N}$ and from (2.12) the reduced state space is given by:

$$X' = G_1\left(X, \acute{U}(X)\right). \qquad (2.14)$$

Linearizing (2.14) along the zero equilibrium solution yields:

$$X' = \left(\frac{\partial G_1}{\partial X} + \frac{\partial G_1}{\partial z}\frac{\partial \acute{U}(X)}{\partial X}\right) X$$
$$= \begin{pmatrix} \left(\frac{\partial f}{\partial z} + \frac{\partial f}{\partial z}\frac{\partial \acute{U}(x,y)}{\partial x}\right)x & \left(\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial \acute{U}(x,y)}{\partial y}\right)y \\ \frac{\partial g}{\partial x}x & \frac{\partial g}{\partial y}y \end{pmatrix}, \qquad (2.15)$$

where the partial derivatives all take values at the equilibrium solution (0, 0) .

On the other hand a similar application of the conclusion of Section 2 and direct linearization to the DAEs (2.12) one obtains:

$$\dot{X} = \frac{\partial G_1}{\partial X} X + \frac{\partial G_1}{\partial z} z, \qquad (2.16a)$$
$$0 = \frac{\partial \acute{G}_2}{\partial X} X + \frac{\partial \acute{G}_2}{\partial z} z, \qquad (2.16b)$$

since $\frac{\partial \acute{G}_2}{\partial z}$ is non-singular , solve for z in (2.16b) and insert back into (2.16a) and using the fact that $G_2\left(X, \acute{U}(x)\right) = 0 \ \forall X \in \acute{N}$ we get the direct linearization of (2.12) is given by:

$$\dot{X} = \left(\frac{\partial G_1}{\partial X} + \frac{\partial G_1}{\partial z}\frac{\partial \acute{U}}{\partial X}\right) X. \qquad (2.17)$$

From which we can see that linearizations (2.17) and (2.15) are the same and that means linearizing index -3 Hesenberg DAE directly along the equilibrium solutions will give the same asymptotic stability information.

3. Hesenberg DAE Systems with Singular Case

In this section we study the DAEs

$$\acute{x} = f(x,y,\lambda), \quad f \in C^1(R^n \times R^m \times R^r, R^n)$$
$$\acute{y} = g(x,\lambda), \quad g \in C^2(R^n \times R^r, R^m) \qquad (3.1)$$

where the product of the Jacobean matrix:

$$J = \left(\frac{\partial g}{\partial y}\right)\left(\frac{\partial f}{\partial y}\right), \qquad (3.2)$$

is singular at the point (x_*, y_*, λ_*) ∈ $R^n \times R^m \times R^r$. The point (x_*, y_*, λ_*) will belong to the singular surface:

$$S_2 = \{(x,y,\lambda) \in (R^n \times R^m \times R^r) : g(x,\lambda) = 0, \det J(x,y,\lambda) = 0\}.$$

As we mentioned it is quite difficult to deal with DAEs (3.1) directly. In particular for investigating the singular points. Hence we need to transform the system (3.1) into another equivalent system with index lower. We want to identify the singular points in the Hesenberg DAE system (3.1). In particular the so called "impasse points" [2] and "singularity induced bifurcation points" (SIB) [3] , [4]. Our goal is to transform the system (3.1) into another equivalent system. Then the theorems related to the equivalent DAE can be applied to identify those singular points. Hence firstly it is useful to show that the singular points of the original system are included in the set of singular points of the equivalent system which is given by:

$$\begin{aligned} \dot{x} &= f(x,y), \\ 0 &= \tilde{g}(x,y), \end{aligned} \qquad (3.3)$$

where $\tilde{g}(x,y) = \frac{\partial g}{\partial x} f$.

Assume that the Jacobean matrix J given by (3.2) is singular at the point (x_*, y_*, λ_*). The following lemma shows the relation between the singular points of the both two systems (3.1) and (3.3).

Lemma 3.1. *If the point* (x_*, y_*, λ_*) *is singular point of the system* (3.1) *then* (x_*, y_*, λ_*) *is singular point of the system* (3.3).

Proof : Consider the singular surface of the equivalent DAE system (3.3) defined by :

$$S_1 = \{(x,y,\lambda) \in (R^n \times R^m \times R^r) : \tilde{g}(x,y,\lambda) = 0 , \det \tilde{g}_y = 0\}$$

Now if (x_*, y_*, λ_*) is a singular point in S_2 then we have g(x_*, λ_*) =0 and det$J$(x_*, y_*, λ_*) =0. From g(x_*, λ_*) =0 we have g_x(x_* , λ_*) $f(x,y,\lambda)$ =0 . In particular g(x_*, λ_*) =0 implies $\hat{g}$(x_*, y_*, λ_*) = g_x(x_*, λ_*) f(x_*, y_*, λ_*) =0. Also since $\hat{g}_y(x, y,\lambda)$ = $J(x, y,\lambda)$ so det $\hat{g}_y$(x_*, y_*, λ_*) =0. Then (x_*, y_*, λ_*) is a singular point belongs to the singular surface S_1. In other words we conclude that $S_2 \subset S_1$.

Hence it is clear that the investigating of the singular points in the Hesenberg DAEs (3.1) can be covered by investigating of the singular points in the equivalent DAE system (3.3).

In particular the impasse points (x_*, y_*) $\in$ S_2 of the Hesenberg DAE system (3.1) is also an impasse point of the equivalent DAE system belongs to S_1. In [2] Chua and Deng found a relation between the impasse points (x_*, y_*) of the DAE (3.3) and the limit points (0, y_*) of the non-linear equation:

$$h(\lambda,y) = \tilde{g}(x_* + \lambda f(x_*,y_*),y) = 0 \qquad (3.4)$$

where f and $\hat{g}$ as defined in (3.3). That is (x_*, y_*) is an impasse point of the DAE (3.3) if and only if (0, y_*) is a limit point of (3.4). This result cannot be applied directly to the Hesenberg DAE system (3.1). Hence the transformation of the system (3.1) into the equivalent DAE system is necessary in order to identify the impasse points in the Hesenberg DAE system (3.1) by using Chua's result as given in the following theorem.

Theorem 3.2. *Consider the Hesenberg DAE system* (3.1). *The point* (x_*, y_*) $\in$ S_2 *is an impasse point if* (0, y_*) *is a limit point to the induced solution curve*

$$\begin{aligned} h(\lambda,y) &= \tilde{g}(x_* + \lambda f(x_*,y_*),y) \\ &= \frac{\partial g(x)}{\partial x}|_{x_*+\lambda f(x_*,y_*)} f(x_* + \lambda f(x_*,y_*),y) \end{aligned} \qquad (3.5)$$

Proof: Assume that (x_*, y_*) $\in$ S_2 is an impasse point to the system (3.1). Then by Lemma 4.1 (x_*, y_*) $\in$ S_1 is an impasse point to the system (3.3). And by Chua's result applied to the system (3.3) we get (0, y_*) is a limit point to the induced solution curve given by $h(\lambda,y) = \tilde{g}(x_* + \lambda f(x_* , y_*) , y)$. Then the Proof is follows by using the definition of the function $\tilde{g}$.

The inverse of Theorem 4.2 is not valid. This is because the impasse points of the equivalent DAE system (3.3) may not impasse points to the original DAE system (3.1).

4. SIB Points in Hesenberg DAEs

We consider the singularity induced bifurcation points (SIB), (x_*, y_*, λ_*) in the system (3.1), the points which are solutions to the system

$$\begin{cases} f(x,y,\lambda)=0, \\ g(x,\lambda)=0, \\ \det J(x,y,\lambda)=0. \end{cases} \quad (4.1)$$

In other words the system (3.1) will bifurcate at $\lambda = \lambda_*$ for which the conditions

$$\begin{cases} f(x_*,y_*,\lambda_*)=0, \\ g(x_*,y_*,\lambda_*)=0, \\ \Delta(x_*,y_*,\lambda_*)=0, \end{cases} \quad (4.2)$$

are satisfied. Evidently the conditions (4.2) imply $(x_*, y_*, \lambda_*) \in S_2$, the singular surface. In [3] , [4] the authors introduce the SIB theorem for the DAEs of the form (3.3). We want to apply this theorem to the system (3.1). Therefore it is needed to transform the system (3.1) into another system which is given by (3.3). Obviously identifying of the SIB point in the system (3.1) can be covered by that of the system (3.3) according to Lemma 4.1.

Now consider the following Hesenberg DAE system of the form

$$\begin{aligned} &\acute{x} = f(x,y,z,\lambda), \quad f \in C^1(R^n \times R^m \times R^k, R^n) \\ &\acute{y} = g(x,y,\lambda), \quad g \in C^2(R^n \times R^m, R^m) \\ &0 = h(y,\lambda), \quad h \in C^3(R^m, R^k) \end{aligned} \quad (4.3)$$

where the product of the Jacobean matrices:

$$H = \left(\frac{\partial h}{\partial y}\right)\left(\frac{\partial g}{\partial x}\right)\left(\frac{\partial f}{\partial z}\right) \quad (4.4)$$

is singular at the point $(x_*, y_*, \lambda_*) \in S_3$ defined by:

$$S_3 = \{(x,y,\lambda) \in (R^n \times R^m \times R^r) : h(y,\lambda)=0, \det H(x,y,\lambda)=0\}. \quad (4.5)$$

Also to identify the singular points in such system we need to transform the DAE system (4.5) into an equivalent system for which the set of singular points consist of the set of singular points of the original system. For this purpose we shall use the transformation given by (2.10).

$$\begin{aligned} &\dot{X} = G_1(X,z,\lambda), \\ &0 = G_2(X,\lambda), \end{aligned} \quad (4.6)$$

where

$$G_1(X,z,\lambda) = \begin{pmatrix} f(x,y,z,\lambda) \\ g(x,y,\lambda) \end{pmatrix}, G_2(X,\lambda) = \frac{\partial h(y,\lambda)}{\partial y} g(x,y,\lambda).$$

The following lemma shows the relation between the singular points of the system (4.5) and the system (4.6).

Lemma 4.1. *If the point* $(x_*, y_*, z_*, \lambda_*)$ *is a singular point of the system* (4.5*) then it is a singular point of the system* (4.6).

Proof. Consider the singular surface of the equivalent DAE system (4.6) defined by.

$$S_4 = \left\{(x,y,z,\lambda) \in (R^n \times R^m \times R^k \times R^r) : G_2(x,y,\lambda)=0, \det G_{2_z} = 0\right\}.$$

Now if (x_*, y_*, z, λ_*) is a singular point in S_3 then we have $h(y_*, \lambda_*) = 0$ and det $H(x_*, y_*, z, \lambda_*) = 0$. From $h(y, \lambda) = 0$ we have $h_y(y, \lambda)\, g(x, y, \lambda) = 0$. In particular $h(y_*, \lambda_*) = 0$ implies $\hat{h}(x_*, y_*, \lambda_*) = h_y(y_*, \lambda_*)\, g(x_*, y_*, \lambda_*) = 0$. Also since $\hat{h}_y(x, y, \lambda) = H(x, y, \lambda)$ so det $\hat{h}(x_*, y_*, \lambda_*) = 0$. Then (x_*, y_*, λ_*) is a singular point belongs to the singular surface S_4. In other words we conclude that $S_3 \subset S_4$

Hence the identifying of the singular points in the system (4.5) can be covered by that of the system (4.6). Obviously the system (4.6) is similar to the system (3.3) and then the conclusion of the identifying of impasse points and SIB point can be applied to the system (4.6).

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