DOI: 10.24996/ijs.2019.60.9.18

ISSN: 0067-2904

Some Geometric Properties of Generalized Class of Meromorphic Functionsassocisted with Higher Ruscheweyh Derivatives

Adnan Aziz Hussein*, Kassim Abdulhameed Jassim
Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq

Abstract
The applications of Ruscheweyh derivative are studied and discussed of class of meromorphic multivalent application. We get some interesting geometric properties, such as coefficient bound, Convex linear combination, growth and distortion bounds, radii of starlikenss , convexity and neighborhood property.

Keywords : meromorphic functions, Ruscheweyh derivative.

بعض الخصائص الهندسية لصنف الدوال الميرومورفية المعممة المرتبطة مع المشتقات العليا الرشوية

عدنان عزيز حسين*، قاسم عبدالحميد جاسم
قسم الرياضيات، كليه العلوم، جامعه بغداد، بغداد، العراق

الخلاصة
في البحث الحالي, تم دارسه تطبيقات المشتقه الرشوية ومناقشتهالصنف يتضمن دوال ميرومفية متعددةالتكافؤ, حصلناعلى بعض الخصائص الهندسية ألمثيره للاهتمام , مثل حدود المعاملات, التركيبة الخطية المحدبة , وخاصية النمو و التشوه, انصاف اقطار النجمية والتحدبية و خاصية الجوار.

1. Introduction

Let A^*denote the class of functions of the form:

$$f(w)=w^{-p}+\sum_{i=p}^{\infty} a_i w^i, \qquad (a_i \geq 0; p \in N=[1,2 \ldots .]), \tag{1}$$

which are analytic and meromorphic in the punctured unit disk
$U^*=\{w: w \in \mathbb{C}, 0<|w|<1\}=U \backslash\{0\}$.

The Hadamard product (or convolution) of two functions :

$$f(w)=w^{-p}+\sum_{i=p}^{\infty} a_i w^i, \quad g(w)=w^{-p}+\sum_{i=p}^{\infty} b_i w^i .$$

in $A^*$is defined by

$$(f * g)(w) \quad=f(w) * g(w) \;=w^{-p}+\sum_{i=p}^{\infty} b_i a_i w^i, \tag{2}$$

Definition (1)[1, 2] : The Ruscheweyh derivative of f of order $(\lambda+p-1)$,is denoted by $D^{\lambda+p-1} f$,defined as following :

$$D^{\lambda+p-1} f(\mathrm{w})=\frac{1}{w^p(1-w)^{p+\lambda}} . f(w)$$

$$w^{-p}+\sum_{\mathrm{i}=\mathrm{p}}^{\infty} \frac{\Gamma(\lambda+\mathrm{p}+\mathrm{k}+1)}{\Gamma(\lambda+\mathrm{p})(\mathrm{i}+1)!} \mathrm{a}_{\mathrm{i}} \mathrm{w}^{\mathrm{i}}, >-p, \; \mathrm{w} \in U^* .$$

In particular , we have

$$\left(D^{\lambda+p-1} f(\mathrm{w})\right)^{\prime}=-p w^{-p-1}+\sum_{\mathrm{i}=\mathrm{p}}^{\infty} \mathrm{i} \frac{\Gamma(\lambda+\mathrm{p}+\mathrm{w}+1)}{\Gamma(\lambda+\mathrm{p})(\mathrm{i}+1)!} \mathrm{a}_{\mathrm{i}} \mathrm{w}^{\mathrm{i}-1}$$

$$\left(D^{\lambda+p-1} f(\mathrm{w})\right)^{\prime\prime}=-p(-p-1) w^{-p-2}+\sum_{\mathrm{i}=\mathrm{p}}^{\infty} \mathrm{i}(\mathrm{i}-1) \frac{\Gamma(\lambda+\mathrm{p}+\mathrm{i}+1)}{\Gamma(\lambda+\mathrm{p})(\mathrm{i}+1)!} \mathrm{a}_{\mathrm{i}} \mathrm{w}^{\mathrm{i}-2} .$$

.
.

*Email: adn. altameemi@gmail.com

.

$$(D^{\lambda+p-1}f(\mathrm{w}))^{q}=\frac{(p+q-1)!}{(p-1)!}(-1)^{q}w^{-p-q}+\sum_{\mathrm{i=p}}^{\infty}\frac{\Gamma(\lambda+\mathrm{p+i}+1)}{\Gamma(\lambda+\mathrm{p})(\mathrm{i}+1)!}\frac{\mathrm{i}!}{(\mathrm{i-q})!}\mathrm{a_i w^{i-q}}.$$

Definition (2): Let $f\in A$be given by (1) . The class $\varepsilon_p(\beta,\alpha,\eta,\lambda)$ is defined

$$\varepsilon_p(\beta,\alpha,\eta,\lambda)=\left\{f\in A^*:\left|\frac{\eta w^{q+1}(D^{\lambda+p-1}f(\mathrm{w}))^{q+1}+\eta(p+1)w^{q}(D^{\lambda+p-1}f(\mathrm{w}))^{q}}{w^{q+1}(D^{\lambda+p-1}f(\mathrm{w}))^{q+1}+(\eta-\alpha)w^{q}(D^{\lambda+p-1}f(\mathrm{w}))^{q}}\right|<\beta\right\}. \quad (3)$$

$\{0<\beta\le 1, 0<\alpha\le 1, 0<\eta<1, \lambda>-p, p, i\in N=\{1,2,..\}, w\in U^*, q\in N_{0=}N\cup[0]\}$.

This kind of study was carried out by several different authors such as Cho et al [3], Atshan and Buti[4], Altintas et al [5], Liu [6] Joshi et al. [7], and Aouf and Shammaky[8], studied meromorphic univalent and multivalent functions for another class.

2. Coefficient Bounds

We get a necessary and sufficient condition for f to be in the class $\varepsilon_p(\beta,\alpha,\eta,\lambda)$.

Theorem (1): Let$f\in A^*$.Then$f\in\varepsilon_p(\beta,\alpha,\eta,\lambda)$, if and only if

$$\sum_{\mathrm{i=p}}^{\infty}\frac{\Gamma(\lambda+\mathrm{p}+\mathrm{i}+1)}{\Gamma(\lambda+\mathrm{p})(\mathrm{i}+1)!}\frac{\mathrm{i}!}{(\mathrm{i}-\mathrm{q}-1)!}\left[\eta-\beta+\frac{\eta(\mathrm{p}+1)-\beta(\eta-\alpha)}{\mathrm{i}-\mathrm{q}}\right]\mathrm{a_i}\le \frac{(p+q-1)!}{(p-1)!}\left[\beta\big((p+\alpha)+(q-\eta)\big)-\eta(1-q)\right], \quad (4)$$

for $(0\le\beta<1, 0<\alpha<\eta<1, \lambda>-p, p, i\in N=\{1,2,..\}, w\in U^*, q\in N_0=N\cup\{0\}$.
sharpness of the result following by setting

$$f(w)=w^{-p}+\frac{\frac{(p+q-1)!}{(p-1)!}\left[\beta\big((p+\alpha)+(q-\eta)\big)-\eta(1-q)\right]}{\frac{\Gamma(\lambda+\mathrm{p+i}+1)}{\Gamma(\lambda+\mathrm{p})(\mathrm{i}+1)!}\frac{\mathrm{i}!}{(\mathrm{i-q}-1)!}\left[\eta-\beta+\frac{\eta(\mathrm{p}+1)-\beta(\eta-\alpha)}{\mathrm{i-q}}\right]}w^{i}, \mathrm{i}\ge \mathrm{p}$$

proof: Assume that the inequality (4) holds true and let $|w|=1$ we have .

$$\left|\eta w^{q+1}(D^{\lambda+p-1}f(\mathrm{w}))^{q+1}+\eta(p+1)w^{q}(D^{\lambda+p-1}f(\mathrm{w}))^{q}\right| -\beta\left|w^{q+1}(D^{\lambda+p-1}f(\mathrm{w}))^{q+1}+(\eta-\alpha)w^{q}(D^{\lambda+p-1}f(\mathrm{w}))^{q}\right|$$

$$=\left|\frac{\eta(p+q)!}{(p-1)!}(-1)^{q+1}w^{-p}+\sum_{i=p}^{\infty}\frac{\Gamma(\lambda+\mathrm{p+i}+1)}{\Gamma(\lambda+\mathrm{p})(\mathrm{i}+1)!}\frac{\eta\mathrm{i}!}{(\mathrm{i-q}-1)!}\mathrm{a_i w^i}+\frac{\eta(\mathrm{p}+1)(\mathrm{p+q}-1)!}{(\mathrm{p}-1)!}(-1)^{q}w^{-p}+\sum_{i=p}^{\infty}\frac{\Gamma(\lambda+\mathrm{p+i}+1)}{\Gamma(\lambda+\mathrm{p})(\mathrm{i}+1)!}\frac{\eta(\mathrm{p}+1)\mathrm{i}!}{(\mathrm{i-q})!}\mathrm{a_i w^i}\right|-\beta\left|\frac{(p+q)!}{(p-1)!}(-1)^{q+1}w^{-p}+\sum_{i=p}^{\infty}\frac{\Gamma(\lambda+\mathrm{p+i}+1)}{\Gamma(\lambda+\mathrm{p})(\mathrm{i}+1)!}\frac{\mathrm{i}!}{(\mathrm{i-q}-1)!}\mathrm{a_i w^i}+\frac{(\eta+1)(\mathrm{p+q}-1)!}{(\mathrm{p}-1)!}(-1)^{q}w^{-p}+\sum_{i=p}^{\infty}\frac{\Gamma(\lambda+\mathrm{p+i}+1)}{\Gamma(\lambda+\mathrm{p})(\mathrm{i}+1)!}\frac{(\eta+1)\mathrm{i}!}{(\mathrm{i-q})!}\mathrm{a_i w^i}\right|$$

$$=\left|\sum_{i=p}^{\infty}\frac{\Gamma(\lambda+\mathrm{p+i}+1)}{\Gamma(\lambda+\mathrm{p})(\mathrm{i}+1)!}\frac{\eta\mathrm{i}!}{(\mathrm{i-q}-1)!}\mathrm{a_i w^i}\left(1+\frac{p+1}{i-q}\right)+\frac{\eta(\mathrm{p}+1)(\mathrm{p+q}-1)!}{(\mathrm{p}-1)!}(-1)^{q}(1-q)\right|$$

$$-\beta\left|\sum_{i=p}^{\infty}\frac{\Gamma(\lambda+\mathrm{p}+\mathrm{i}+1)}{\Gamma(\lambda+\mathrm{p})(\mathrm{i}+1)!}\frac{\mathrm{i}!}{(\mathrm{i}-\mathrm{q}-1)!}\mathrm{a_i w^i}\left(1+\frac{\eta-\alpha}{\mathrm{i}-\mathrm{q}}\right)+\frac{(\mathrm{p}+\mathrm{q}-1)!}{(\mathrm{p}-1)!}(-1)^{q}w^{-p}(-(p+q)+(\eta-\alpha))\right|$$

$$\le\sum_{\mathrm{i=p}}^{\infty}\frac{\Gamma(\lambda+\mathrm{p+i}+1)}{\Gamma(\lambda+\mathrm{p})(\mathrm{i}+1)!}\frac{\mathrm{i}!}{(\mathrm{i-q}-1)!}\left[\eta-\beta+\frac{\eta(\mathrm{p}+1)-\beta(\eta-\alpha)}{\mathrm{i-q}}\right]\mathrm{a_i}-\left[\frac{(p+q-1)!}{(p-1)!}\left[\beta\big((p+\alpha)+(q-\eta)\big)-\eta(1-q)\right]\right]\le 0,$$ by hypothesis.

Hence by maximum modulus principle , $f\in\varepsilon_p(\beta,\alpha,\eta,\lambda)$.
Conversely,suppose that $f\in\varepsilon_p(\beta,\alpha,\eta,\lambda)$.
Hence

$$\left|\frac{\eta w^{q+1}(D^{\lambda+p-1}f(\mathrm{w}))^{q+1}+\eta(p+1)z^{q}(D^{\lambda+p-1}f(\mathrm{w}))^{q}}{w^{q+1}(D^{\lambda+p-1}f(\mathrm{w}))^{q+1}+(\eta-\alpha)w^{q}(D^{\lambda+p-1}f(\mathrm{w}))^{q}}\right|=$$

$$\left|\frac{\sum_{i=p}^{\infty}\frac{\Gamma(\lambda+\mathrm{p+i}+1)}{\Gamma(\lambda+\mathrm{p})(\mathrm{i}+1)!}\frac{\eta\mathrm{i}!}{(\mathrm{i-q}-1)!}\mathrm{a_i w^i}\left(1+\frac{p+1}{i-q}\right)+\frac{\eta(\mathrm{p}+1)(\mathrm{p+q}-1)!}{(\mathrm{p}-1)!}(-1)^{q}(1-q)}{\sum_{i=p}^{\infty}\frac{\Gamma(\lambda+\mathrm{p+i}+1)}{\Gamma(\lambda+\mathrm{p})(\mathrm{i}+1)!}\frac{\mathrm{i}!}{(\mathrm{i-q}-1)!}\mathrm{a_i w^i}\left(1+\frac{\eta-\alpha}{\mathrm{i-q}}\right)+\frac{(\mathrm{p+q}-1)!}{(\mathrm{p}-1)!}(-1)^{q}w^{-p}(-(p+q)+(\eta-\alpha)}\right|<\beta.$$

Since |Re(w)|≤|w| for all w, we have

$$\mathrm{Re}\left\{\frac{\sum_{i=p}^{\infty}\frac{\Gamma(\lambda+p+i+1)}{\Gamma(\lambda+p)(i+1)!}\frac{\eta i!}{(i-q-1)!}a_i w^i\left(1+\frac{p+1}{i-q}\right)+\frac{\eta(p+1)(p+q-1)!}{(p-1)!}(-1)^q(1-q)}{\sum_{i=p}^{\infty}\frac{\Gamma(\lambda+p+i+1)}{\Gamma(\lambda+p)(i+1)!}\frac{i!}{(i-q-1)!}a_i w^i\left(1+\frac{\eta-\alpha}{i-q}\right)+\frac{(p+q-1)!}{(p-1)!}(-1)^q w^{-p}(-(p+q)+(\eta-\alpha))}\right\}<\beta.$$

We can choose the value of z on the real axis, so that $w^q(D^{\lambda+p-1}f(w))^q$ is real, let w→ 1^-. Through real values , we get the inequality (4) . sharpness of the result following by setting

$$f(w)=w^{-p}+\frac{\frac{(p+q-1)!}{(p-1)!}[\beta((p+\alpha)+(q-\eta))-\eta(1-q)]}{\frac{\Gamma(\lambda+2p+1)}{\Gamma(\lambda+p)(p+1)!}\frac{p!}{(p-q-1)!}\left[\eta-\beta+\frac{\eta(p+1)-\beta(\eta-\alpha)}{p-q}\right]}w^i\quad ,i\geq p \qquad (5)$$

the proof is complete.

Corollary(1): Let $f\in \varepsilon_p(\beta,\alpha,\eta,\lambda)$. Then

$$a_i\leq\frac{\frac{(p+q-1)!}{(p-1)!}[\beta((p+\alpha)+(q-\eta))-\eta(1-q)]}{\frac{\Gamma(\lambda+2p+1)}{\Gamma(\lambda+p)(p+1)!}\frac{p!}{(p-q-1)!}\left[\eta-\beta+\frac{\eta(p+1)-\beta(\eta-\alpha)}{p-q}\right]},$$

where

$$(0\leq\beta<1, 0<\alpha\leq 1, 0<\eta<1, \lambda>-p, p, i\in N\{1,2,..\}, w\in U^*, q\in N_0=N\cup\{0\}.$$

Theorem (2): The $f\in \varepsilon_p(\beta,\alpha,\eta,\lambda)$ is closed convex linear combination.

Proof: Suppose the function

$$f_j(w)=w^{-p}+\sum_{i=p}^{\infty}\frac{\Gamma(\lambda+p+i+1)}{\Gamma(\lambda+p)(i+1)!}a_i w^i,\quad a_{i,j}\geq 0,\ j=1,2,\ p\in N, \quad (6)$$

be in the class $f\in \varepsilon_p(\beta,\alpha,\eta,\lambda)$, Its sufficient is show that the function h(w)$\in \varepsilon_p(\beta,\alpha,\eta,\lambda)$ defined byh(w)=$(1-t)f_1(w)+f_2(w)\in\varepsilon_p(\beta,\alpha,\eta,\lambda)$, $0\leq t\leq 1$

since $h(w)=w^{-p}+\sum_{i=p}^{\infty}\left[(1-t)a_{i,1}+ta_{i,2}\right]w^i$, $0\leq t\leq 1$. (7)

By making use of theorem (1) , we see that

$$\sum_{i=p}^{\infty}\frac{\Gamma(\lambda+p+i+1)}{\Gamma(\lambda+p)(i+1)!}\frac{i!}{(i-q-1)!}\left[\eta-\beta+\frac{\eta(p+1)-\beta(\eta-\alpha)}{i-q}\right][(1-t)a_{i,1}+ta_{i,2}]$$
$$\leq(1-t)\frac{(p+q-1)!}{(p-1)!}[\beta((p+\alpha)+(q-\eta))-\eta(1-q)]$$
$$+\ t\frac{(p+q-1)!}{(p-1)!}[\beta((p+\alpha)+(q-\eta))-\eta(1-q)]$$

$$=(1-t)\sum_{i=p}^{\infty}\frac{\Gamma(\lambda+p+i+1)}{\Gamma(\lambda+p)(i+1)!}\frac{i!}{(i-q-1)!}\left[\eta-\beta+\frac{\eta(p+1)-\beta(\eta-\alpha)}{i-q}\right]a_{i,1}+t\sum_{i=p}^{\infty}\frac{\Gamma(\lambda+p+i+1)}{\Gamma(\lambda+p)(i+1)!}\frac{i!}{(i-q-1)!}\left[\eta-\beta+\frac{\eta(p+1)-\beta(\eta-\alpha)}{i-q}\right]a_{i,2}\leq\frac{(p+q-1)!}{(p-1)!}[\beta((p+\alpha)+(q-\eta))-\eta(1-q)].$$

Hence $h(w)\in \varepsilon_p(\beta,\alpha,\eta,\lambda)$.

The proof is complete.

3.Distortion theorems

In the following theorem , we obtain growth and distortion bounds for the function $f\in\varepsilon_p(\beta,\alpha,\eta,\lambda)$.

Theorem (3) : If the function$f\in\varepsilon_p(\beta,\alpha,\eta,\lambda)$,then for 0<|w|< 1 .

$$\frac{1}{|w|^p}-\frac{\frac{(p+q-1)!}{(p-1)!}[\beta((p+\alpha)+(q-\eta))-\eta(1-q)]}{\frac{\Gamma(\lambda+2p+1)}{\Gamma(\lambda+p)(p+1)!}\frac{p!}{(p-q-1)!}\left[\eta-\beta+\frac{\eta(p+1)-\beta(\eta-\alpha)}{p-q}\right]}|w|^p\leq|f(w)|$$
$$\leq\frac{1}{|w|^p}+\frac{\frac{(p+q-1)!}{(p-1)!}[\beta((p+\alpha)+(q-\eta))-\eta(1-q)]}{\frac{\Gamma(\lambda+2p+1)}{\Gamma(\lambda+p)(p+1)!}\frac{p!}{(p-q-1)!}\left[\eta-\beta+\frac{\eta(p+1)-\beta(\eta-\alpha)}{p-q}\right]}|w|^p. \qquad (8)$$

The result is sharp and attained for

$$\frac{1}{|w|^p}-\frac{\frac{(p+q-1)!}{(p-1)!}\left[\beta\big((p+\alpha)+(q-\eta)\big)-\eta(1-q)\right]}{\frac{\Gamma(\lambda+2p+1)}{\Gamma(\lambda+p)(p+1)!}\frac{p!}{(p-q-1)!}\left[\eta-\beta+\frac{\eta(p+1)-\beta(\eta-\alpha)}{p-q}\right]}|w|^p \quad (9)$$

Proof: Let $f\in\varepsilon_p(\beta,\alpha,\eta,\lambda)$, then
$|f(w)|=\left|w^{-p}+\sum_{i=p}^{\infty}a_iw^i\right|\le|w|^{-p}+\sum_{i=p}^{\infty}a_i|w^i|\le|w|^{-p}+|w|^p\sum_{i=p}^{\infty}a_i$.
By theorem (1), we have

$$\sum_{i=p}^{\infty}a_i\le\frac{\frac{(p+q-1)!}{(p-1)!}\left[\beta\big((p+\alpha)+(q-\eta)\big)-\eta(1-q)\right]}{\frac{\Gamma(\lambda+2p+1)}{\Gamma(\lambda+p)(p+1)!}\frac{p!}{(p-q-1)!}\left[\eta-\beta+\frac{\eta(p+1)-\beta(\eta-\alpha)}{p-q}\right]}.$$

Thus

$$|f(w)|\le\frac{1}{|w|^p}+\frac{\frac{(p+q-1)!}{(p-1)!}\left[\beta\big((p+\alpha)+(q-\eta)\big)-\eta(1-q)\right]}{\frac{\Gamma(\lambda+2p+1)}{\Gamma(\lambda+p)(p+1)!}\frac{p!}{(p-q-1)!}\left[\eta-\beta+\frac{\eta(p+1)-\beta(\eta-\alpha)}{p-q}\right]}|w|^p.$$

Similarly, we have
$f(w)|\ge|w|^{-p}-\sum_{i=p}^{\infty}a_i|w^i|\ \ge|w|^{-p}-|w|^p\sum_{i=p}^{\infty}a_i$

$$|f(w)|\ge\frac{1}{|w|^p}-\frac{\frac{(p+q-1)!}{(p-1)!}\left[\beta\big((p+\alpha)+(q-\eta)\big)-\eta(1-q)\right]}{\frac{\Gamma(\lambda+2p+1)}{\Gamma(\lambda+p)(p+1)!}\frac{p!}{(p-q-1)!}\left[\eta-\beta+\frac{\eta(p+1)-\beta(\eta-\alpha)}{p-q}\right]}|w|^p.$$

Hence result (8) follows . The proof is complete.

Theorem (4) : If the function $f\in\varepsilon_p(\beta,\alpha,\eta,\lambda)$, then for $0<|w|<1$,

$$\frac{p}{|w|^{p+1}}-\frac{\frac{(p+q-1)!}{(p-1)!}p\left[\beta\left(\left(p+|z|^{k+p}\alpha\right)+(q-\eta)\right)-\eta(1-q)\right]}{\frac{\Gamma(\lambda+2p+1)}{\Gamma(\lambda+p)(p+1)!}\frac{p!}{(p-q-1)!}\left[\eta-\beta+\frac{\eta(p+1)-\beta(\eta-\alpha)}{p-q}\right]}|w|^{p-1}\le\left|f'(w)\right|$$

$$\le\frac{p}{|w|^{p+1}}+\frac{\frac{(p+q-1)!}{(p-1)!}p\left[\beta\big((p+\alpha)+(q-\eta)\big)-\eta(1-q)\right]}{\frac{\Gamma(\lambda+2p+1)}{\Gamma(\lambda+p)(p+1)!}\frac{p!}{(p-q-1)!}\left[\eta-\beta+\frac{\eta(p+1)-\beta(\eta-\alpha)}{p-q}\right]}|w|^{p-1}.$$

The result is sharp and achieved for the function given by (9)

Proof: Let $f\in\varepsilon_p(\beta,\alpha,\eta,\lambda)$ then

$$\left|f'(w)\right|\le\frac{p}{|w|^{p+1}}+\frac{\frac{(p+q-1)!}{(p-1)!}p\left[\beta\big((p+\alpha)+(q-\eta)\big)-\eta(1-q)\right]}{\frac{\Gamma(\lambda+2p+1)}{\Gamma(\lambda+p)(p+1)!}\frac{p!}{(p-q-1)!}\left[\eta-\beta+\frac{\eta(p+1)-\beta(\eta-\alpha)}{p-q}\right]}|w|^{p-1}.$$

On the other hand

$$\left|f'(w)\right|\ge\frac{p}{|w|^{p+1}}-\sum_{i=p}^{\infty}ia_i|w|^{i-1}\frac{p}{|w|^{p+1}}-|w|^{p-1}\sum_{i=p}^{\infty}pa_i\frac{p}{|w|^{p+1}}$$
$$-\frac{\frac{(p+q-1)!}{(p-1)!}p\left[\beta\big((p+\alpha)+(q-\eta)\big)-\eta(1-q)\right]}{\frac{\Gamma(\lambda+2p+1)}{\Gamma(\lambda+p)(p+1)!}\frac{p!}{(p-q-1)!}\left[\eta-\beta+\frac{\eta(p+1)-\beta(\eta-\alpha)}{p-q}\right]}|w|^{p-1}.$$

The proof is complete.

4. Radii of starlikenss and convexity

In the following theorem we obtain radii of starlikeness and convexity.

Theorem (5): Let $f\in\varepsilon_p(\beta,\alpha,\eta,\lambda)$, then $f$is meromorphically multivalent starlike of order δ $(0\le\delta<p)$. In the disk $|w|<r_1$,where

$$r_1(\beta,\alpha,\eta,\lambda,\delta) = inf_i \left\{ \frac{\frac{\Gamma(\lambda+p+i+1)\ (p+\delta)i!}{\Gamma(\lambda+p)(i+1)!(i-q-1)!}\left[\eta-\beta+\frac{\eta(p+1)-\beta(\eta-\alpha)}{i-q}\right]}{\frac{(p+q-1)!}{(p-1)!}(i+2p-\delta)\left[\beta\big((p+\alpha)+(q-\eta)\big)-\eta(1-q)\right]} \right\}^{\frac{1}{i+p}}. \quad (11)$$

Proof : It is sufficient to show that

$$\left|\frac{z\acute{f}(w)}{f(w)}+p\right| \le p-\delta, \qquad (0 \le p < \delta) \quad (12)$$

for |w|< $r_1(\beta,\alpha,\eta,\lambda,\delta)$, we have

$$\left|\frac{w\acute{f}(w)}{f(w)}+p\right| = \left|\frac{w\acute{f}(w)+pf(w)}{f(w)}\right| = \left|\frac{-pw^{-p}+\sum_{i=p}^{\infty} ia_iw^i+pw^{-p}+\sum_{i=p}^{\infty} pa_iw^i}{w^{-p}+\sum_{i=p}^{\infty} a_iw^i}\right|$$

$$= \left|\frac{\sum_{i=p}^{\infty}(i+p)a_iw^i}{w^{-p}+\sum_{i=p}^{\infty} a_iw^i}\right| \le \frac{\sum_{i=p}^{\infty}(i+p)a_i|w|^{i+p}}{1-\sum_{i=p}^{\infty} a_iw|^{i+p}}.$$

The last idiom above is bounded by $p-\delta$ if

$$\sum_{i=p}^{\infty} \frac{i+2p-\delta}{p-\delta} a_i|w|^{i+p} \le 1. \quad (13)$$

In view of (12),that follows (13) is right if

$$\frac{i+2p-\delta}{p-\delta}|w|^{i+p} \le$$

$$\frac{\frac{\Gamma(\lambda+p+i+1)\quad i!}{\Gamma(\lambda+p)(i+1)!(i-q-1)!}\left[\eta-\beta+\frac{\eta(p+1)-\beta(\eta-\alpha)}{i-q}\right]}{\frac{(p+q-1)!}{(p-1)!}\left[\beta\big((p+\alpha)+(q-\eta)\big)-\eta(1-q)\right]} |w| \le \left\{ \frac{\frac{\Gamma(\lambda+p+i+1)\ (p+\delta)i!}{\Gamma(\lambda+p)(i+1)!(i-q-1)!}\left[\eta-\beta+\frac{\eta(p+1)-\beta(\eta-\alpha)}{i-q}\right]}{\frac{(p+q-1)!}{(p-1)!}(i+2p-\delta)\left[\beta\big((p+\alpha)+(q-\eta)\big)-\eta(1-q)\right]} \right\}^{\frac{1}{i+p}} .(14)$$

Theorem (5) follows easily from (14).

Theorem (6): Let $f \in \varepsilon_p(\beta,\alpha,\eta,\lambda)$, then f is meromorphically multivalent convex of order δ $(0 \le \delta < p)$. In the disk |w|< r_2,where

$$r_2(\beta,\alpha,\eta,\lambda,\delta) = inf_i \left\{ \frac{\frac{\Gamma(\lambda+p+i+1)\ (p+\delta)i!}{\Gamma(\lambda+p)(i+1)!(i-q-1)!}\left[\eta-\beta+\frac{\eta(p+1)-\beta(\eta-\alpha)}{i-q}\right]}{\frac{(p+q-1)!}{(p-1)!}(i+2p-\delta)\left[\beta\big((p+\alpha)+(q-\eta)\big)-\eta(1-q)\right]} \right\}^{\frac{1}{i+p}}. \quad (15)$$

Proof: it is sufficient to show that

$$\left|1+\frac{\acute{\acute{f}}(w)}{\acute{f}(w)}+p\right| \le p-\delta\,, \qquad (0 \le p < \delta\,) \quad (16)$$

for |w|< $r_2(\beta,\alpha,\eta,\lambda,\delta)$, we have, we have

$$\left|1+\frac{\acute{\acute{f}}(w)}{\acute{f}(w)}+p\right| = \left|\frac{w\acute{\acute{f}}(w)+(1+p)\acute{f}(w)}{\acute{f}(w)}\right| = \left|\frac{p(p+1)w^{-(p+1)}+\sum_{i=p}^{\infty} i(i-1)a_iw^{i-1}-p(p+1)w^{-(p+1)}+\sum_{i=p}^{\infty} i(p+1)a_iw^{i-1}}{-pw^{-(p+1)}+\sum_{i=p}^{\infty} ia_iw^{i-1}}\right|$$

$$= \left|\frac{\sum_{i=p}^{\infty} i(i+p)a_iw^{i-1}}{-pw^{-(p+1)}+\sum_{i=p}^{\infty} ia_iw^{i-1}}\right| \le \frac{\sum_{i=p}^{\infty} i(i+p)a_i|w|^{i+p}}{p-\sum_{i=p}^{\infty} a_i|w|^{i+p}}.$$

The last expression above is bounded by $p-\delta$ if

$$\sum_{i=p}^{\infty} \frac{i(i+2p-\delta)}{p(p-\delta)} a_i|w|^{i+p} \le 1. \quad (17)$$

In view (16) , it follows that (17) is true if

$$\frac{(i+2p-\delta)}{p-\delta} \quad |w|^{i+p} \leq \frac{\frac{\Gamma(\lambda+p+i+1)}{\Gamma(\lambda+p)(i+1)!}\frac{i!}{(i-q-1)!}\left[\eta-\beta+\frac{\eta(p+1)-\beta(\eta-\alpha)}{i-q}\right]}{\frac{(p+q-1)!}{(p-1)!}\left[\beta\big((p+\alpha)+(q-\eta)\big)-\eta(1-q)\right]}|w|$$

$$\leq \left\{\frac{\frac{\Gamma(\lambda+p+i+1)}{\Gamma(\lambda+p)(i+1)!}\frac{(p+\delta)i!}{(i-q-1)!}\left[\eta-\beta+\frac{\eta(p+1)-\beta(\eta-\alpha)}{i-q}\right]}{\frac{(p+q-1)!}{(p-1)!}(i+2p-\delta)\left[\beta\big((p+\alpha)+(q-\eta)\big)-\eta(1-q)\right]}\right\}^{\frac{1}{i+p}} \tag{18}$$

Theorem (6)follows easily from (18).

5. Neighborhoods Property

We defined the (i, δ)-neighborhoods of functions$f(w)\in \varepsilon_p(\beta,\alpha,\eta,\lambda)$.

Definition(3)[9]: For $f\in\in \varepsilon_p(\beta,\alpha,\eta,\lambda)$, of the from (1), and $\delta \geq 0$. We define.

$$N_{i,\delta}(g;f) = \left\{\begin{matrix} g: g\in \varepsilon_p: g(w) = w^{-p}+\sum_{i=p}^{\infty} b_i w^i and \sum_{i=p}^{\infty} i|a_i-b_i| \leq \delta \\ 0\leq \delta < 1 \end{matrix}\right\}, \tag{19}$$

and private, for the identity function

$e(w) = w^{-p}$,

we immediately have

$$N_{i,\delta}(e;f) = \left\{\begin{matrix} g: g\in \varepsilon_p: g(w) = w^{-p}+\sum_{i=p}^{\infty} b_i w^i and \sum_{i=p}^{\infty} i|b_i| \leq \delta \\ 0\leq \delta < 1 \end{matrix}\right\}, \tag{20}$$

We will study neighborhood result of the $\varepsilon_\varphi\ (\beta,\alpha,\eta,\lambda)$,due to Goodman[10] and Ruscheweyh[11]

Definition (4) : A function $f\in \mathcal{M}$ is saidto be in the class $\varepsilon_{\varphi}\ (\beta,\alpha,\eta,\lambda)$ if there exist a function$g\in \varepsilon_p(\beta,\alpha,\eta,\lambda)$ such that

$$\left|\frac{f(w)}{g(w)}-1\right| < 1-\varphi, \qquad (w\in U, 0\leq\varphi<1).$$

Theorem (7): If $g\in \varepsilon_p(\beta,\alpha,\eta,\lambda)$ and

$$\varphi = 1-\frac{\delta\frac{\Gamma(\lambda+2p+1)}{\Gamma(\lambda+p^2+p)}\cdot\frac{p!}{(p-q-1)!}\left[\eta-\beta+\frac{\eta(p+1)(\beta(\eta-\alpha)}{p-q}\right]}{\frac{(p+q-1)!}{(p-1)!}[\beta(p+\alpha)-(q-\eta)-\eta(1-q)]}. \tag{21}$$

Then $N_{i,\delta}(g)\subset \varepsilon_\varphi\ (\beta,\alpha,\eta,\lambda)$

Proof : Let $f\in N_{i,\delta}(g)$. Then we find from (1) that

$$\sum_{i=p}^{\infty} i|a_i-b_i| \leq \delta,$$

which implies the coefficient inequality

$\sum_{i=p}^{\infty}|a_i-b_i| \leq \delta,$ $\qquad (i\in \mathbb{N})$.

Since $g\in \varepsilon_p(\beta,\alpha,\eta,\lambda)$, then by using theorem (1), we get

$$\sum_{i=p}^{\infty} b_i \leq \frac{\frac{(p+q-1)!}{(p-1)!}[\beta(p+\alpha)-(q-\eta)-\eta(1-q)]}{\frac{\Gamma(\lambda+2p+1)}{\Gamma(\lambda+p^2+p)}\cdot\frac{p!}{(p-q-1)!}\left[\eta-\beta+\frac{\eta(p+1)(\beta(\eta-\alpha)}{p-q}\right]}$$

$$\left|\frac{f(w)}{g(w)}-1\right| < \frac{\sum_{i=p}^{\infty}|a_i-b_i|}{1-\sum_{i=p}^{\infty} b_i} \leq \frac{\delta\frac{\Gamma(\lambda+2p+1)}{\Gamma(\lambda+p^2+p)}\cdot\frac{p!}{(p-q-1)!}\left[\eta-\beta+\frac{\eta(p+1)(\beta(\eta-\alpha)}{p-q}\right]}{\frac{(p+q-1)!}{(p-1)!}[\beta(p+\alpha)-(q-\eta)-\eta(1-q)]} = 1-\varphi.$$

By definition (3), $f\in \varepsilon_p(\beta,\alpha,\eta,\lambda)$ for φ given by (21).

This completes the proof.

References

1. Ruscheweyh, S. **1975.** New criteria for univalent functions , *Proc. Amer. Math.,Soc*., **49**: 109 115.
2. Srivastava, H.M. and Owa, S. **1992.** (fds), *Corrent topics inanalytic functions theory*, world scientific publishing company, Sigapore.
3. Cho, N.E. and leeand, H. **1987.** A class of meromorphic univalent functions with positive cofficionts, *Koebe Math*, **4**: 43-50.
4. Atshan, W.G. and Buti, R.H. **2011.** Fractioal calculus of univalent function with negative coefficient defind by Hdamard product with Rafid operator , *European Journal of pure of appl. Math*. **4**(2): 162-173.
5. Altinas, O., limak, H. and Sirvastava, H.M. **1995.** A family of meromorphically univalent with positive Coefficitient, panamer. *Math. J*. **5**(1): 75-81.
6. Liu, JL. **2000.** Propertles of same families of meromorphic p-valent functions, *Japanica*. 52: 425-439.
7. Joshi, S.B., Kulkarni, S.R. and Sirvastava, H.M. **1995.** Certain class of meromorphic univalent functions with positive cofficionts. *K.Math, Anal.Appl*, **193**: 1-4.
8. Aouf, M.K. and A.E Shamaky, A.E. **2005.** A Certain subclass of meromorphically p-valent convex functions with negative coefficients, *G.Appl*, **1**(2): 123-143.
9. Altintas, O. and Owa, S. **1996.** Neighborhoods of Certain analytic functions with negative coefficients. *J. Math. and Math. Sci*. **19**: 797-800.
10. Goodman, A.W. **1975.** Univalent functions and non-analytic curves, *Proc. Amer. Math. Soc*., **8**: 598-601.
11. Ruscheweyh, S. **1981.** Neighboors of univalent functions, Proc. Amer. Math. Soc. 81, 521-527.