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Solving the Nonlinear Time-Fractional Zakharov-Kuznetsov Equation with the Chebyshev Spectral Method

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Abstract

In this study, we introduce a new application to a spectral approach for solving two-dimensional (2D) time-fractional Zakhrov-Kuznetsov equations (TFZKEs) with initial conditions (ICs) and boundary conditions (BCs). When the regular magnetic domain is present, this equation represents a model that illustrates the conduct of weakly nonlinear ionic phonetic waves in a plasma that provides cool ions and an electronically isothermal environment. The fundamental qualifiers of fractional derivatives are characterized in the Caputo concept. We propose a new numerical approach that relies on shifted Chebyshev polynomials (SCPs) as test functions and uniformly grid points for time and space. In the field of fractional calculus, we have to introduce several schemes to evaluate a solution to the nonlinear fractional problems. This new technique is a preferable attempt. The results show that the current method is quite effective, and robust which provides excellent accuracy, and is appropriate for implementation to solve many significant fractional differential equations.

Keywords: Spectral Approach, Time-Fractional Zakhrov-Kuznetsov Equations, Shifted Chebyshev Polynomials, Maximum Error, Accuracy.

حل معادلة زاخاروف-كوزنتسوف الكسرية الزمانية غير الخطية باستخدام طريقة شبيشيف الطيفية

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الخلاصة

في هذه الدراسة، قدمنا تطبيقًا جديدًا للنهج الطيفي لحل معادلات Zakhrov–Kuznetsov ثنائية الأبعاد الكسرية (TFZKEs) مع الشروط الأبتدائية والحدودية، تمثل هذه المعادلة نموذجًا يوضح إجراء موجات صوتية ضعيفة غير خطية أيونية في البلازما توفر أيونات باردة وبيئة متساوية الحرارة إلكترونيًا عند وجود المجال المغناطيسي العادي. تمّ البحث بالأعتماد على المفاهيم الأساسية للمشتقات الجزئية بمفهوم Caputo الأسلوب المقترح يعتبر نهجًا عدديًا جديدًا يعتمد على متعددات الحدود Chebyshev المُزاحة كدوال اختبار ونقاط الشبكة متساوية للزمان والمكان في مجال حساب التفاضل والتكامل الكسري من الضروري تقديم عدة

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أساليب لحل المسائل الكسرية غير الخطية. أظهرت نتائج هذه التقنية المقترحة أن الطريقة الحالية فعالة للغاية، وتوفر دقة ممتازة، وهي مناسبة للتطبيق لحل المعادلات التفاضلية الكسرية .

1. Introduction

Fractional calculus has been taken more attention and appreciation recently due to its effective contributions to the modeling of natural phenomena and its great role to extend the basis of its applications in the various fields such as solids mechanics, diffusion issues, control theory, biomedical engineering, viscoelasticity, and economics [1, 2, 3]. The majority of fractional differential equations (FDEs) are based on linear and nonlinear mathematical models that are absent or extremely complicated to fix analytically. As a result, numerical and approximation methods must be employed [4]. We will go through one of these numerical methods and strategies to solve these kinds of equations.

The spectral method is a potent mathematical technique for numerically resolving differential equations of fractional or integer order. It becomes one of the most recently used approximate methods because it requires fewer grid points and more accurate than other numerical methods.
The calculation of weighting factors for the discretization of fractional or integer order spatial or temporal derivatives is one of the most important aspects of the spectral approach. The spectral approach utilizes a variety of orthogonal polynomials as test functions, which play an important role in determining the validity of numerical solutions and mesh points for evenly and unequally spacing grid points.

There are numerous studies that have employed collocation approach for various sorts of purpose and have used many formulas for solving FDEs that relate to large current research. Spline collocation methodologies are based on the Lagrange basic polynomials were employed by Pedas and Tamme [5] in 2011 to solve linear multi-term FDEs. To solve the fractional nonlinear Langevin equation, Bhrawy and Alghamdi [6] created shifted Jacobi-Gauss-Lobatto polynomials in 2012. Khader [7] developed generalized Laguerre polynomials to find a rough solution for fractional delay differential equations (FDDEs) in (2013). For solving the fractional advection–diffusion equation, Tian et al. [8] and Saeed with Rehman [3] in (2014) exploited the Legendre-Gauss-Lobatto and Chebyshev-Gauss-Lobatto polynomials in matrix notation, and suggested the Hermite wavelet polynomials to solve linear and nonlinear FDDEs. With regard to the operational matrix of fractional derivatives of shifted Jacobi polynomials, Bhrawy [9] in 2015 suggested an efficient method to solve nonlinear fractional sub-diffusion and response sub-diffusion equations. Jaleb with Adibi [10]. In (2016), Alshbool and Hashim [11] utilized shifted Legendre and Bernstein polynomials to obtain rigid numerical solutions to space fractional diffusion equations, fractional Riccati differential equations, and fractional-order systems. The operational matrix was proposed by Rahimkhani et al. [12] in (2017), who studied Bernoulli wavelet polynomial for discover numerical solution of FDDEs. In (2018), Agarwal as well as El-Sayed [13] and M. Bahmanpour et al. [14] used shifted Chebyshev polynomials of the second kind to solve the fractional order diffusion equation, and introduced Müntz wavelets by using Müntz Legendre and Jacobi polynomials as a test function to solve FDEs. In (2019), Ali et al. [15] applied the spectral collocation formula to propose solutions for FDDEs with Chebyshev operational form. Al-Humedi with Al-Saadawi [16] and Al-Humedi with Al-Saadawi [17] in (2020-2021) solved 1D and 2D time-space fractional bioheat models adopting shifted Jacobi-Gauss-Lobatto polynomials and fractional shifted Legendre polynomials as test functions and spectral collocation methods.

Spectral approaches are developed to present a new numerical solution for 2D nonlinear TFZKEs that relies on Chebyshev polynomials as test functions in this study.

2. Basic Equation

Many observations demonstrate that the wave moves irregularly in time, and various characteristics of a nonlinear wave should be examined over different time scales, so the ZKE is a model defining the isotropic growth of a nonlinear ion wavefront. The primary flow feature of a tube flow can be described by the flow theory, however, the vortices near the boundary must be explained on a smaller time scale necessitating the use of a fractional model [18].

The time-fractional form of 2D unsteady state nonlinear TFZKE $(\beta_1, \beta_2, \beta_3)$ with an arbitrary positive real order derivative $\alpha \in (0,1]$ has been studied:

$$D_t^{\alpha} u + \gamma_1 \left(u^{\beta_1}\right)_x + \gamma_2 \left(u^{\beta_2}\right)_{xxx} + \gamma_3 \left(u^{\beta_3}\right)_{xyy} = 0, \quad 0 \le x \le N_1, 0 \le y \le N_2, 0 \le t \le N_3 \quad (1)$$

subject to the following ICs and BCs

$$u(x, y, 0) = f_0(x, y), \quad 0 \le x \le N_1, \; 0 \le y \le N_2, \quad (2)$$

$$u(0, y, t) = f_1(y, t), \quad 0 \le y \le N_2, \; 0 \le t \le N_3, \quad (3)$$

$$u(N_1, y, t) = f_2(y, t), \quad 0 \le y \le N_2, \; 0 \le t \le N_3, \quad (4)$$

$$u(x, 0, t) = f_3(x, t), \quad 0 \le x \le N_1, \; 0 \le t \le N_3, \quad (5)$$

$$u(x, N_2, t) = f_4(x, t), \quad 0 \le x \le N_1, \; 0 \le t \le N_3. \quad (6)$$

Where $u = u(x, y, t)$, γ_1, γ_2 and γ_3 are random constants. In the systematic electric field, the integers β_1, β_2 and β_3 are responsible for the conduct of weak nonlinear acoustic ion vibrations in a plasma containing cool ions and hot exothermic electrons [19].

3. Notes and Prelims

This part focuses mostly on the fundamental definitions of fractional calculus that might be employed in our study.

Definition 3.1. The Riemann-Liouville fractional integral operator of order $\alpha > 0$ is defined by [20]:

$$I^{\alpha} \varphi(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} \varphi(s) ds, \quad \alpha > 0, \qquad I^0 \varphi(x) = \varphi(x). \quad (7)$$

Definition 3.2. The Riemann-Liouville fractional differential operator (FDO) is given as follows [21]:

$$D^{\alpha} \varphi(x) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \dfrac{d^n}{dx^n} \displaystyle\int_0^x \dfrac{\varphi(s)}{(x-s)^{\alpha-n+1}} ds, & \alpha > 0, n-1 \le \alpha < n, \\ \dfrac{d^n \varphi(x)}{dx^n} & , \alpha = n. \end{cases} \quad (8)$$

Definition 3.3. The Caputo FDO is defined as follows [22]:

$$D^{\alpha} \varphi(x) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{\varphi^{(n)}(s)}{(x-s)^{\alpha-n+1}} ds, & n-1 \le \alpha < n, \\ \frac{d^n \varphi(x)}{dx^n} & , \alpha = n. \end{cases} \quad (9)$$

The expressions describe the relationship between the Riemann-Liouville fractional integral and Caputo differential operators [23]:

$$D^{\alpha} I^{\alpha}\ \varphi(x) = \varphi(x),$$
$$I^{\alpha} D^{\alpha}\ \varphi(x) = \varphi(x) - \sum_{k=0}^{n-1} \varphi^{(k)}(0^{+}) \frac{x^k}{k!} \tag{10}$$

for $\beta \geq -1$, $\alpha \geq 0$, and C is constant. The Caputo FDO possesses several important features that are required in this situation. They are as follows [24]:

$i)\ D^{\alpha} C = 0,$

$$ii)\ D^{\alpha} x^{\beta} = \begin{cases} 0 & \text{for } \beta \in N_0 \text{ and } \beta < \lceil \alpha \rceil, \\ \dfrac{\Gamma(\beta + 1)}{\Gamma(\beta + 1 - \alpha)} x^{\beta-\alpha}, & \text{for } \beta \in N_0 \text{ and } \beta \geq \lceil \alpha \rceil, \end{cases} \tag{11}$$

$$iii)\ D^{\alpha}\left(\sum_{i=0}^{n} c_i\, \varphi_i(x)\right) = \sum_{i=0}^{n} c_i D^{\alpha}\, \varphi_i(x), \text{ where } \{c_i\}_{i=0}^{n} \text{ are constants.}$$

4. The shifted Chebyshev Polynomials for Fractional and Ordinary Derivatives

The first kind of Chebyshev polynomials $T_i(t)$ are orthogonal polynomials of degree i in t defined on $[-1,1]$

$$T_i(t) = \cos i\theta,$$

where $t = \cos\theta$ and $\theta \in [0, \pi]$. By using the following recurrence relation, the polynomials $T_i(t)$ can be constructed [25]

$$T_{i+1}(t) = 2tT_i(t) - T_{i-1}(t), \quad i = 1, 2, \ldots, \ T_0(t) = 1, \ T_1(t) = t. \tag{12}$$

To employ these polynomials on interval $x \in [0, N_1]$, we defined SCPs by applying the variable change $t = \frac{2x}{N_1} - 1$. Let SCPs, $T_i(\frac{2x}{N_1} - 1)$ which is characterized by $T_{N_1,i}(x)$. Then $T_{N_1,i}(x)$ can be generated by using equation (12) such that [26]:

$$T_{N_1,i+1}(x) = 2\left(\frac{2x}{N_1} - 1\right) T_{N_1,i}(x) - T_{N_1,i-1}(x), \quad i = 1, 2, \ldots,$$

where $T_{N_1,0}(x) = 1$ and $T_{N_1,1}(x) = \frac{2x}{N_1} - 1$. The analytic form of SCPs, $T_{N_1,i}(x)$ of degree i is given by

$$T_{N_1,i}(x) = i \sum_{k=0}^{i} (-1)^{i-k} \frac{(i+k-1)!\, 2^{2k}}{{N_1}^{k}(i-k)!\,(2k)!} x^k, \tag{13}$$

where $T_{N_1,i}(0) = (-1)^i$ and $T_{N_1,i}(N_1) = 1$. The requirement for orthogonality is

$$\int_0^{N_1} T_{N_1,j}(x)\, T_{N_1,k}(x) \mathcal{w}_{N_1}(x) dx = \delta_{jk} h_k, \tag{14}$$

where $\mathcal{w}_{N_1} = (N_1 x - x^2)^{-\frac{1}{2}}$ and $h_k = \frac{\mathcal{C}_k}{2}\pi$, with $\mathcal{C}_0 = 2$, $\mathcal{C}_k = 1$, $k \geq 1$.

The first order derivative depends on SCPs of the vector

$\psi(x) = [T_{N_1,0}(x), T_{N_1,1}(x), \ldots, T_{N_1,M_1}(x)]'$ can be expressed by

$$\frac{d\psi(x)}{dx} = \mathcal{D}^{(1)}\psi(x) \tag{15}$$

$\mathcal{D}^{(1)}$ is the $(M_1 + 1) \times (M_1 + 1)$ shifted Chebyshev operational matrix (SCOM) of derivative [27]:

$$\mathcal{D}^{(1)} = d_{ij} = \begin{cases} \dfrac{4i}{\mathcal{C}_j N_1}, & j = 0,1, \ldots, i = j + k, \\ 0, & othewise, \end{cases} \quad \begin{cases} k = 1,3,5, \ldots, M_1, & if\ N_1\ is\ odd, \\ k = 1,3,5, \ldots, M_1 - 1, & if\ N_1\ is\ even, \end{cases}$$

when N_1 is even, $\mathcal{D}^{(1)}$ can be given as follows

$$\mathcal{D}^{(1)} = \frac{2}{N_1}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 3 & 0 & 6 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 8 & 0 & 8 & 0 & \cdots & 0 & 0 \\ 5 & 0 & 10 & 0 & 10 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ M_1-1 & 0 & 2M_1-2 & 0 & 2M_1-2 & \cdots & 0 & 0 \\ 0 & 2M_1 & 0 & 2M_1 & 0 & & 2M_1 & 0 \end{bmatrix}$$

We utilize equation (15) to generalize SCOM of derivatives for fractional calculus

$$\frac{d^n\psi(x)}{dx^n} = \left(\mathcal{D}^{(1)}\right)^n \psi(x), \qquad (16)$$

where $n \in \mathbb{N}$ represents the matrix powers for integer calculus. Thus

$$\mathcal{D}^{(n)} = \left(\mathcal{D}^{(1)}\right)^n, \qquad n = 1,2,\ldots \qquad (17)$$

Lemma 4.1. Assume $T_{N_1,i}(x)$ is an SCPs, then we have $\mathcal{D}^{(\alpha)}T_{N_1,i}(x) = 0$, $i = 0,1,2,\ldots,\lceil\alpha\rceil - 1, \alpha > 0$.

Proof. The lemma can be demonstrated by plugging attributes (ii) and (iii) from equation (11) into equation (13) □

The following theorem applies the fractional calculus to the operational matrix of SCP derivatives that is given in equation (15).

Theorem [28] **4.2.** Let $\psi(x)$ be a shifted Chebyshev vector defined in $\psi(x) = \left[T_{N_1,0}(x), T_{N_1,1}(x), \ldots, T_{N_1,M_1}(x)\right]'$ and $\alpha > 0$. Then

$$D^{\alpha}\psi(x) \cong \mathcal{D}^{(\alpha)}\psi(x), \qquad (18)$$

$\mathcal{D}^{(\alpha)}$ is the $(M_1+1)\times(M_1+1)$ SCOM of order α derivatives in the Caputo concept which is defined as follows:

$$\mathcal{D}^{(\alpha)} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ & \vdots\ \vdots\ \vdots & & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ \mathcal{B}^{\alpha}(\lceil\alpha\rceil,0) & \mathcal{B}^{\alpha}(\lceil\alpha\rceil,1) & \mathcal{B}^{\alpha}(\lceil\alpha\rceil,2) & \cdots & \mathcal{B}^{\alpha}(\lceil\alpha\rceil,M_1) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \mathcal{B}^{\alpha}(i,0) & \mathcal{B}^{\alpha}(i,1) & \mathcal{B}^{\alpha}(i,2) & \cdots & \mathcal{B}^{\alpha}(i,M_1) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \mathcal{B}^{\alpha}(M_1,0) & \mathcal{B}^{\alpha}(M_1,1) & \mathcal{B}^{\alpha}(M_1,2) & \cdots & \mathcal{B}^{\alpha}(M_1,M_1) \end{bmatrix}$$

where

$$\mathcal{B}^{\alpha}(i,j) = \sum_{k=\lceil\alpha\rceil}^{i} \frac{(-1)^{i-k}2\mathrm{i}(i+k-1)!\,\Gamma(k-\alpha+\frac{1}{2})}{\mathcal{C}_j(N_1)^{\alpha}\Gamma(k+\frac{1}{2})\Gamma(k-\alpha-j+1)\Gamma(k+j-\alpha+1)(i-k)!}. \qquad (19)$$

5. Fractional Differentiation in Shifted Chebyshev Operational Matrix

In terms of SCPs, a function $u(x)$ defined for $0 \le x \le N_1$ can be written as

$$u(x) = \sum_{i=0}^{\infty} c_i\, T_{N_1,i}(x), \qquad (20)$$

where c_i denote the coefficients which are given by

$$c_i = \frac{1}{h_i}\int_0^{N_1} u(x)\, T_{N_1,i}(x) w_{N_1}(x)dx\,, i = 0,1,2,\ldots. \qquad (21)$$

we study the (M_1+1)-term SCPs in practice, so that

$$u_{M_1}(x) \approx \sum_{i=0}^{M_1} c_i\, T_{N_1,i}(x) = C'\psi(x), \tag{22}$$

where C and $\psi(x)$ are the shifted Chebyshev coefficient and vector, respectively. They are given by
$C' = [c_0, c_1, \dots, c_N]$, $\psi(x) = \left[T_{N_1,0}(x), T_{N_1,1}(x), \dots, T_{N_1,M_1}(x)\right]'$,respectively.

We may simulate a function $u(x,t)$ defined for $0 \le x \le N_1$ and $0 \le t \le N_3$ which relies on double SCPs by expanding the above property of the two variable functions as

$$u(x,t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij}\, T_{N_1,i}(x) T_{N_3,j}(t), \tag{23}$$

where

$$a_{ij} = \frac{1}{h_i h_j}\int_0^{N_1}\int_0^{N_3} u(x,t)\, T_{N_1,i}(x) T_{N_3,j}(t) w_{N_1,N_3}(x,t)\,dt\,dx, \tag{24}$$

such that $w_{N_1,N_3}(x,t) = w_{N_1,i}(x) w_{N_3,j}(t)$.

In practice, we consider the (M_1+1) and (M_3+1)-terms double SCPs with regard to x, t such that

$$u_{M_1,M_3}(x,t) \approx \sum_{i=0}^{M_1}\sum_{j=0}^{M_3} a_{ij}\, T_{N_1,i}(x) T_{N_3,j}(t) = \psi(x)' A\, \psi(t), \tag{25}$$

where the shifted Chebyshev coefficient matrix and vector A and $\psi(t)$ are given by
$A = \{a_{ij}\}_{i,j=0}^{M_1,M_3}$, $\psi(t) = \left[T_{N_3,0}(t), T_{N_3,1}(t), \dots, T_{N_3,M_3}(t)\right]'$, respectively.
Therefore, to approximate a three-variable function $u(x,y,t)$ defined for $0 \le x \le N_1, 0 \le y \le N_2$ and $0 \le t \le N_3$ that depends on the triple Chebyshev series as follows

$$u(x,y,t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \tilde{u}_{kij} T_{N_1,i}(x) T_{N_2,j}(y) T_{N_3,j}(t), \tag{26}$$

where

$$\tilde{u}_{kij} = \frac{1}{h_i h_j h_k}\int_0^{N_1}\int_0^{N_2}\int_0^{N_3} u(x,y,t)\, T_{N_1,i}(x) T_{N_2,j}(y) T_{N_3,k}(t) w_{N_1,N_2,N_3}(x,y,t)\,dt\,dy\,dx, \tag{27}$$

such that $w_{N_1,N_2,N_3}(x,y,t) = w_{N_1,i}(x) w_{N_2,j}(y) w_{N_3,k}(t)$.
In reality, we evaluate the triple SCPs $(M_1+1), (M_2+1)$ and (M_3+1)-terms with respect to x, y, t so that

$$u_{M_1,M_2,M_3}(x,y,t) \approx \sum_{i=0}^{M_1}\sum_{j=0}^{M_2}\sum_{k=0}^{M_3} \tilde{u}_{kij} T_{N_1,i}(x) T_{N_2,j}(y) T_{N_3,k}(t) = \psi(t)'\, \ddot{U}\, \psi(x) \otimes \psi(y) \tag{28}$$

where the Kronecker tensor product is $\otimes$, the shifted Chebyshev vectors $\psi(x)$, $\psi(y)$ and $\psi(t)$ are given by

$$\left.\begin{aligned} \psi(x) &= \left[T_{N_1,0}(x), T_{N_1,1}(x), \dots, T_{N_1,M_1}(x)\right]' \\ \psi(y) &= \left[T_{N_2,0}(y), T_{N_2,1}(y), \dots, T_{N_2,M_2}(y)\right]' \\ \psi(t) &= \left[T_{N_3,0}(t), T_{N_3,1}(t), \dots, T_{N_3,M_3}(t)\right]' \end{aligned}\right\}. \tag{29}$$

The block structure of the shifted Chebyshev coefficient matrix $\ddot{U}$ is:

$$\ddot{U} = \begin{bmatrix} \tilde{u}_{000} & \tilde{u}_{001} & \cdots & \tilde{u}_{00M_2} & \tilde{u}_{010} & \tilde{u}_{011} & \cdots & \tilde{u}_{0M_1M_2} \\ \tilde{u}_{100} & \tilde{u}_{101} & \cdots & \tilde{u}_{10M_2} & \tilde{u}_{110} & \tilde{u}_{111} & \cdots & \tilde{u}_{1M_1M_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \tilde{u}_{M_300} & \tilde{u}_{M_301} & \cdots & \tilde{u}_{M_30M_2} & \tilde{u}_{M_310} & \tilde{u}_{M_311} & \cdots & \tilde{u}_{M_3M_1M_2} \end{bmatrix} \tag{30}$$

6- Methodology Description

The choice of collocation points has a major effect on the spectral solution's efficiency and convergence. One of the most significant elements that is used in approximation is the equally spaced mesh points $x_i = \frac{iN_1}{M_1}$, $i = 0,1,\dots,M_1$. It should be noted that one cannot utilize the collocation strategy with equally spaced nodes for a differential equation with the discontinuity at $x = 0$ and $x = N_1$ in the area $[0, N_1]$, because the associations abscissas 0 and N_1 must be utilized as 2 points from the collocation nodes. We utilize the collocation approach with $x_i = x_1, x_2, \dots, x_{N-1}$ nodes to treat the 2D nonlinear TFZKEs; i.e., just by gathering, this equation only at $M_3 \times (M_1 - 1) \times (M_2 - 1)$ is equally spaced grid points$(0, N_3), (0, N_1)$ and $(0, N_2)$, respectively. These equations together with ICs and BCs can be solved using one of the iteration methods to produce $(M_1 + 1) \times (M_2 + 1) \times (M_3 + 1)$ nonlinear system of equations.

Put, $\boldsymbol{P}_{M_1}(0, N_1) = span\{T_{N_1,0}(x), T_{N_1,1}(x), \dots, T_{N_1,M_1}(x)\}$.We recall the equally spaced grid depends on the Chebyshev generators, $\boldsymbol{P}_{M_1}(0, N_1)$ represents the group of all algebraic polynomials of order M_1 which is any positive number.

We will use equation, $x_i = \frac{iN_1}{M_1}$, $i = 0,1,\dots,M_1$ to design the numerical solution procedure for (1) relying on SCPs, according to the specified conditions, in a series or matrix form into the shifted Chebyshev vectors $\psi(x)$, $\psi(y)$ and $\psi(t)$ define by equation (29). Similarly, the shifted Chebyshev coefficient matrix $\ddot{U}$ is given by equation (30).

The space - time fractional derivatives of linear and nonlinear functions can be approximated as follows:

$$\left.\begin{aligned}
\frac{\partial^\alpha u(x,y,t)}{\partial t^\alpha} &= \left[D_{N_3}^{(\alpha)}\psi(t)\right]' \ddot{U}\psi(x)\otimes\psi(y),\\
\frac{\partial u^{\beta_1}(x,y,t)}{\partial x} &= \psi'(t)\left(\ddot{U}\right)^{\beta_1}\left[D_{N_1}^{(1)}\psi(x)\right]\otimes\psi(y),\\
\frac{\partial^3 u^{\beta_2}(x,y,t)}{\partial x^3} &= \psi'(t)\left(\ddot{U}\right)^{\beta_2}\left[D_{N_1}^{(3)}\psi(x)\right]\otimes\psi(y),\\
\frac{\partial^3 u^{\beta_3}(x,y,t)}{\partial x\partial y^2} &= \psi'(t)\left(\ddot{U}\right)^{\beta_3}\left[D_{N_1}^{(1)}\psi(x)\right]\otimes\left[D_{N_2}^{(2)}\psi(y)\right].
\end{aligned}\right\} \tag{31}$$

By applying the proposed method for 2D nonlinear TFZKEs based on equally spaced grid points in the matrix form that is given in equation (1), we have

$$\psi'(t)\left[D_{N_3}^{(\alpha)}\right]' \ddot{U}\psi(x)\otimes\psi(y) + \gamma_1\psi'(t)\left(\ddot{U}\right)^{\beta_1}\left[D_{N_1}^{(1)}\psi(x)\right]\otimes\psi(y) + \gamma_2\psi'(t)\left(\ddot{U}\right)^{\beta_2}\left[D_{N_1}^{(3)}\psi(x)\right]\otimes\psi(y) + \gamma_3\psi'(t)\left(\ddot{U}\right)^{\beta_3}\left[D_{N_1}^{(1)}\psi(x)\right]\otimes\left[D_{N_2}^{(2)}\psi(y)\right] = 0. \tag{32}$$

We collocate (32) at $M_3 \times (M_1 - 1) \times (M_2 - 1)$ points, as

$$\psi'(t_k)\left[D_{N_3}^{(\alpha)}\right]' \ddot{U}\psi(x_i)\otimes\psi(y_j) + \gamma_1\psi'(t_k)\left(\ddot{U}\right)^{\beta_1}\left[D_{N_1}^{(1)}\psi(x_i)\right]\otimes\psi(y_j) + \gamma_2\psi'(t_k)\left(\ddot{U}\right)^{\beta_2}\left[D_{N_1}^{(3)}\psi(x_i)\right]\otimes\psi(y_j) + \gamma_3\psi'(t_k)\left(\ddot{U}\right)^{\beta_3}\left[D_{N_1}^{(1)}\psi(x_i)\right]\otimes\left[D_{N_2}^{(2)}\psi(y_j)\right] = 0 \tag{33}$$

For $i = 1,2,\dots,M_1 - 1$, $j = 1,2,\dots,M_2 - 1$ and $k = 1,2,\dots,M_3$.

where x_i $(0 \le i \le M_1)$ and y_j $(0 \le j \le M_2)$ are roots of SCP for the space of $T_{N_1,i}(x)$ and $T_{N_2,j}(y)$, respectively, while t_k $(0 \le k \le N_3)$ are the roots of $T_{N_3,k}(t)$, that construct a system of $M_3 \times (M_1 - 1) \times (M_2 - 1)$, the nonlinear system of equations with unknown extension coefficients $\tilde{u}_{kij}$ that are derived from ICs and BCs by utilize equations ((2)-(6)), as follows

$$\left.\begin{array}{ll} \psi'(0)\ddot{U}\psi(x_i)\otimes\psi(y_j) = f_0(x_i, y_j), & 0 \le i \le N_1, 0 \le j \le N_2 \\ \psi'(t_k)\ddot{U}\psi(0)\otimes\psi(y_j) = f_1(y_j, t_k), & 0 \le j \le N_2, 0 \le k \le N_3 \\ \psi'(t_k)\ddot{U}\psi(N_1)\otimes\psi(y_j) = f_2(y_j, t_k), & 0 \le j \le N_2, 0 \le k \le N_3 \\ \psi'(t_k)\ddot{U}\psi(x_i)\otimes\psi(0) = f_3(x_i, t_k), & 0 \le i \le N_1, 0 \le k \le N_3 \\ \psi'(t_k)\ddot{U}\psi(x_i)\otimes\psi(N_2) = f_4(x_i, t_k), & 0 \le i \le N_1, 0 \le k \le N_3 \end{array}\right\} \quad (34)$$

This results in $(M_3 + 1) \times ((N_1 + 1) \times (N_2 + 1))$ a nonlinear mathematical model, which may be solved using the Levenberg-Marquardt process with $\ddot{U}$ as the variable and an initial estimate of all zeros to minimize formulas (33)-(34). Consequently, the approximate solution $u_{N_3,N_1,N_2}(x, y, t)$ at the point (x_i, y_j, t_k) given in equation (28) can be calculated.

The same approach as in [22] can be used to estimate an upper bound of the maximum absolute errors obtained for the approximation. Therefore, the error bound is an important aspect of the method's convergence.

7- Illustrative Examples

In this section, we apply the approach which has been presented in section 6 for solving the 2D nonlinear TFZKEs in the two cases based on the first kind SCPs. TFZKEs were initially converted into non-linear algebraic equations (33) and (34). By applying the Levenberg-Marquardt approach to minimize those equations as a collection of least squares solutions, using $\ddot{U}$ as the variable. The estimated surface of u is then obtained using this $\ddot{U}$ in (38) $u(x, y, t)$.

In these two cases, we take $\gamma_1 = 1, \gamma_2 = \gamma_3 = \frac{1}{8}$ and 2, $N_1 = N_2 = N_3 = 1$, $\beta_1 = \beta_2 = \beta_3 = 2$ and 3, and using equally spaced mesh points. Tables 1 and 3 show that the maximum errors are evaluated by solving (2,2,2) and (3,3,3) of TFKZE under SCPs study on $x \in [0, N_1], y \in [0, N_2]$ and $t \in [0, N_3]$ at $\alpha = 0.67$ and $\alpha = 0.75$ when $M_1 = M_2 = M_3 = 2 - 10$, but Tables 2 and 4 show that the numerical solutions of TFKZE (2,2,2) and TFKZE (3,3,3) under SCPs study on $x \in [0, N_1], y \in [0, N_2]$ and $t \in [0, N_3]$ $\alpha = 0.67$ and $\alpha = 0.75$ when $M_1 = M_2 = M_3 = 10$ and the results compare with the optimal homotopy asymptotic method (OHAM) [29], perturbation iteration algorithm (PIA) [30] and variational iteration method (VIM) [31].

Case 1:

Consider the 2D nonlinear TFKZE (2,2,2) [29]:

$$D_t^{\alpha} u(x, y, t) + \left(u^2(x, y, t)\right)_x + \frac{1}{8}\left(u^2(x, y, t)\right)_{xxx} + \frac{1}{8}\left(u^2(x, y, t)\right)_{xyy} = 0, \quad 0 < \alpha \le 1 \quad (35)$$

So, the exact solution is

$$u(x, y, t) = \frac{4}{3}\lambda sinh^2(x + y - \lambda t) \quad (36)$$

under the ICs and BCs:

$$u(x, y, t) = \frac{4}{3}\lambda sinh^2(x + y) \quad (37)$$

$$u(0, y, t) = \frac{4}{3}\lambda sinh^2(y - \lambda t) \quad (38)$$

$$u(N_1, y, t) = \frac{4}{3}\lambda sinh^2(N_1 + y - \lambda t) \quad (39)$$

$$u(x, 0, t) = \frac{4}{3}\lambda sinh^2(x - \lambda t) \quad (40)$$

$$u(x, N_2, t) = \frac{4}{3}\lambda sinh^2(x + N_2 - \lambda t) \quad (41)$$

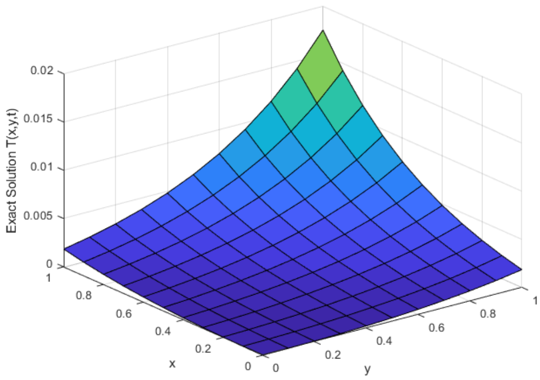
Figure 1 compares the numerical as well as exact solutions of Case 1 at $\alpha = 0.67,\ 0.75$ whereas Figure 2 shows the maximum error values for all data sets, with the best quality for $M_1 = M_2 = M_3 = 10$ at almost 3.7×10^{-5} at $\alpha = 0.67$ and 3.6×10^{-5}at $\alpha = 0.75$, and the worst quality for $M_1 = M_2 = M_3 = 2$ at well under 2.3×10^{-5}.

Table 1: Maximum errors obtained for Case 1 (TFKZE(2,2,2)-SCPs) at $\lambda = 0.001$.

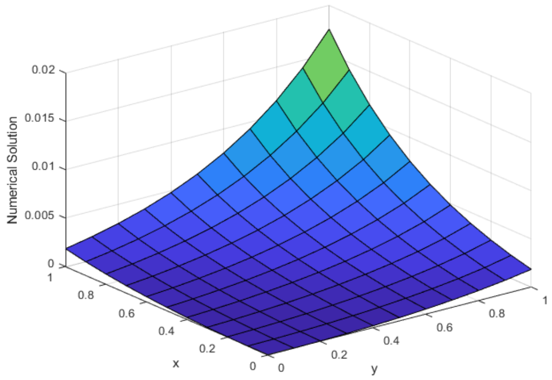
$M_1 = M_2 = M_3$	**Maximum Error** $\alpha = 0.67$	**Maximum Error** $\alpha = 0.75$
2	2.344918451282476e-05	2.297312760942793e-05
3	2.749039955281599e-05	2.699780172533799e-05
4	3.097769048145214e-05	3.028499685707410e-05
5	3.257643191689145e-05	3.185477927975053e-05
6	3.413182884638623e-05	3.333157156296118e-05
7	3.485303128310073e-05	3.403341670914391e-05
8	3.573894703296030e-05	3.488457255431884e-05
9	3.612858609417912e-05	3.526198003008744e-05
10	3.671049747537743e-05	3.582855564614271e-05

Table 2: Numerical solutions obtained for Case 1 (TFKZE(2,2,2)-SCPs) at λ=0.001.

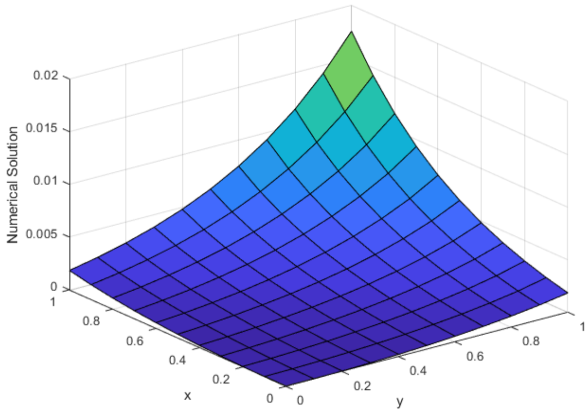
x	**y**	**T**	**PIA** [30]	**OHAM** [29]	**TFKZE-SCPs**	**Error TFKZE-SCPs** $\alpha = 0.67$	**PIA** [30]	**OHAM** [29]	**TFKZE-SCPs**	**Error TFKZE-SCPs** $\alpha = 0.75$
			$\alpha = 0.67$				$\alpha = 0.75$			
0.1	0.1	0.2	5.31854e-05	5.39424e-05	5.14440e-05	1.40235e-06	5.32747e-05	5.39530e-05	5.18111e-05	2.12764e-06
		0.3	5.28631e-05	5.39094e-05	5.05438e-05	2.49475e-06	5.29757e-05	5.39191e-05	5.09304e-05	2.95369e-06
		0.4	5.25777e-05	5.38798e-05	4.97274e-05	3.34028e-06	5.27039e-05	5.38881e-05	5.01132e-05	3.71622e-06
0.6	0.6	0.2	2.95493e-03	3.02730e-03	3.02718e-03	5.30583e-06	2.96356e-03	3.02837e-03	3.02870e-03	7.80483e-06
		0.3	2.92662e-03	3.02397e-03	3.02345e-03	9.32833e-06	2.93717e-03	3.02496e-03	3.02506e-03	1.07232e-05
		0.4	2.90307e-03	3.02099e-03	3.02007e-03	1.23286e-05	2.91448e-03	3.02182e-03	3.02167e-03	1.33788e-05
0.9	0.9	0.2	1.06822e-02	1.14179e-02	1.15237e-02	7.74379e-06	1.07716e-02	1.14303e-02	1.15262e-02	1.07509e-05
		0.3	1.04487e-02	1.13792e-02	1.15174e-02	1.33227e-05	1.05488e-02	1.13907e-02	1.15201e-02	1.44666e-05
		0.4	1.02777e-02	1.13447e-02	1.15117e-02	1.71773e-05	1.03736e-02	1.13543e-02	1.15144e-02	1.77393e-05



$\alpha = 0.67$



$\alpha = 0.67$



$\alpha = 0.75$

Figure 1: Numerical solution and exact solution for Case 1 at $N_3 = N_1 = N_2 = 1, M_1 = M_2 = M_3 = 10$.

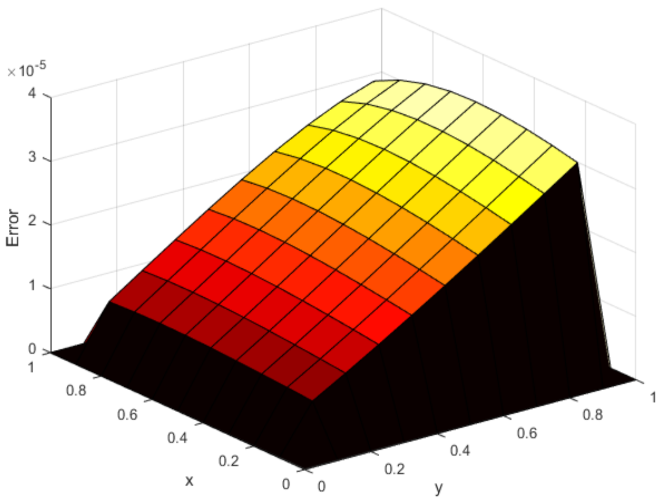


Figure 2: Error for Case 1 at $\alpha = 0.75,\ N_3 = N_1 = N_2 = 1,\ M_1 = M_2 = M_3 = 10$.

Case 2:
Consider 2D nonlinear TFKZE (3,3,3) [19]:

$$D_t^{\alpha} u(x,y,t) + \left(u^3(x,y,t)\right)_x + 2\left(u^3(x,y,t)\right)_{xxx} + 2\left(u^3(x,y,t)\right)_{xyy} = 0, \quad 0 < \alpha \leq 1 \qquad (42)$$

So, the exact solution

$$u(x,y,t)=\frac{3}{2}\lambda sinh\left(\frac{1}{6}(x+y-\lambda t)\right) \tag{43}$$

under ICs and BCs :

$$u(x,y,0)=\frac{3}{2}\lambda sinh\left(\frac{1}{6}(x+y)\right) \tag{44}$$

$$u(0,y,t)=\frac{3}{2}\lambda sinh\left(\frac{1}{6}(y-\lambda t)\right) \tag{45}$$

$$u(N_1,y,t)=\frac{3}{2}\lambda sinh\left(\frac{1}{6}(N_1+y-\lambda t)\right) \tag{46}$$

$$u(x,0,t)=\frac{3}{2}\lambda sinh\left(\frac{1}{6}(x-\lambda t)\right) \tag{47}$$

$$u(x,N_2,t)=\frac{3}{2}\lambda sinh\left(\frac{1}{6}(x+N_2-\lambda t)\right) \tag{48}$$

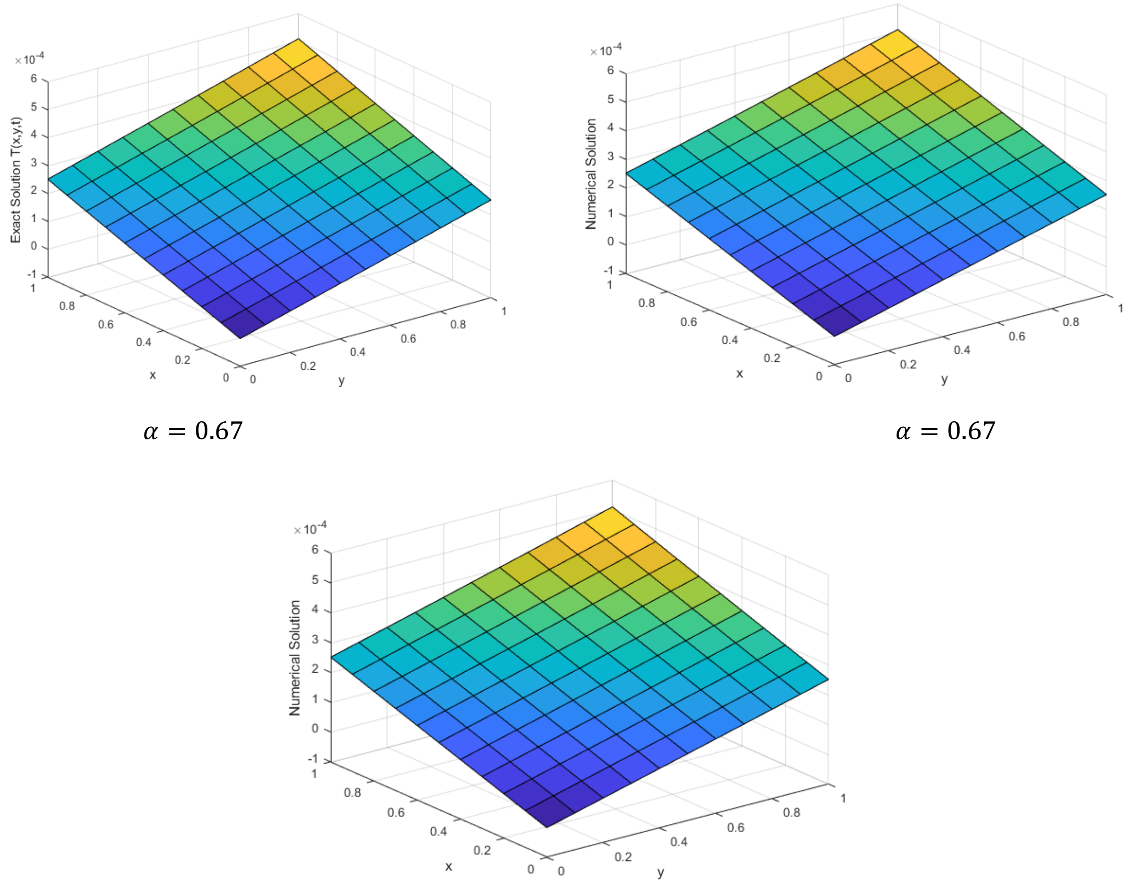
Figure 3 shows a comparison of the numerical and exact solutions of Case 2 at $\alpha=0.67$, 0.75 and while Figure 4 shows the maximum error values for all data sets, with the greatest result for $M_1=M_2=M_3=10$ at well under 2.6×10^{-7} at $\alpha=0.67$ and $\alpha=0.75$, and the worst achievement for $M_1=M_2=M_3=2$ at almost 2.5×10^{-7}.

Table 3: Maximum errors obtained for Case 1 (TFKZE(3,3,3)-SCPs) at $\lambda=0.001$.

$M_1=M_2=M_3$	Maximum Error $\alpha=0.67$	Maximum Error $\alpha=0.75$
2	2.534725168110781e-07	2.534725885772931e-07
3	2.561893359584975e-07	2.561893928360146e-07
4	2.578436190190780e-07	2.578436970204854e-07
5	2.589316778409925e-07	2.589317506534084e-07
6	2.596969405678235e-07	2.596970219645185e-07
7	2.602632199962289e-07	2.602632983050076e-07
8	2.606986545373546e-07	2.606987372997648e-07
9	2.610437364801931e-07	2.610438169214349e-07
10	2.613238086275247e-07	2.613238916645633e-07

Table 4: Numerical solutions obtained for Case 2 (TFKZE(3,3,3)-SCPs) at λ=0.001.

x	y	t	VIM [31]	OHAM [29]	TFKZE-SCPs	Error TFKZE-SCPs	VIM [31]	OHAM [29]	TFKZE-SCPs	Error TFKZE-SCPs
			$\alpha=0.67$			$\alpha=0.67$	$\alpha=0.75$			$\alpha=0.75$
0.1	0.1	0.2	5.000912783e-05	5.00091e-05	5.000925835e-05	5.002632799e-08	5.000914105e-05	5.00091e-05	5.000925855e-05	5.002652923e-08
		0.3	5.000907777e-05	5.00090e-05	5.000925786e-05	7.503969096e-08	5.000909430e-05	5.00091e-05	5.000925807e-05	7.503990303e-08
		0.4	5.000903250e-05	5.00090e-05	5.000925741e-05	1.000530859e-07	5.000905153e-05	5.00091e-05	5.000925762e-05	1.000532977e-07
0.6	0.6	0.2	3.020038194e-04	3.02003e-04	3.020040024e-04	5.100174542e-08	3.020038425e-04	302004e-04	3.020040026e-04	5.100194666e-08
		0.3	3.0200375160-04	3.02003e-04	3.020040019e-04	7.650271211e-08	3.020037779e-04	302004e-04	3.020040021e-04	7.650292418e-08
		0.4	3.020036895e-04	3.02003e-04	3.020040014e-04	1.020036408e-07	3.020037195e-04	302004e-04	3.020040017e-04	1.020038526e-07
0.9	0.9	0.2	4.567801885e-04	4.56780e-04	4.567804387e-04	5.226524736e-08	4.567802187e-04	45678e-04	4.567804389e-04	5.226544860e-08
		0.3	4.567800915e-04	4.56780e-04	4.567804383e-04	7.839790053e-08	4.567801293e-04	45678e-04	4.567804385e-04	7.839811260e-08
		0.4	4.567800089e-04	4.56780e-04	4.567804378e-04	1.045304727e-07	4.567800482e-04	45678e-04	4.567804380e-04	1.045306845e-07



$\alpha = 0.67$ $\alpha = 0.67$

$\alpha = 0.75$

Figure 3: Numerical solution and exact solution for Case 2 at $N_3 = N_1 = N_2 = 1, M_1 = M_2 = M_3 = 10$.

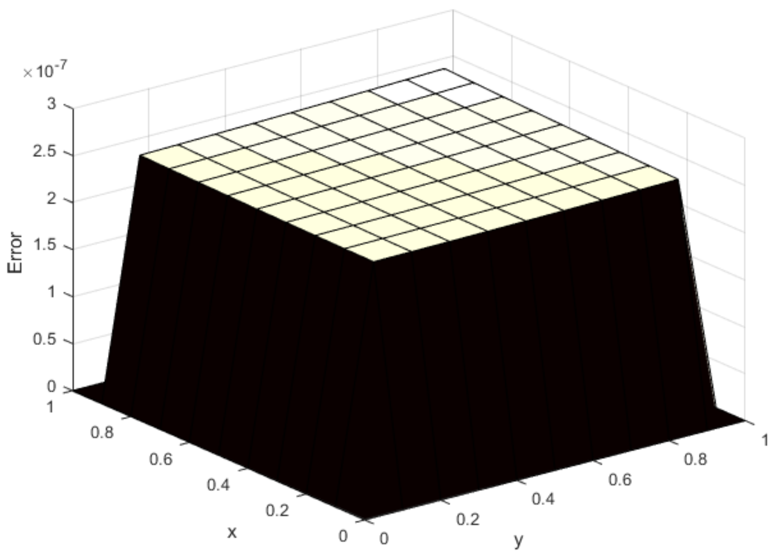


Figure 4: Error for Case 2 at $\alpha = 0.75,\ N_3 = N_1 = N_2 = 1,\ M_1 = M_2 = M_3 = 10$.

8. Conclusions

An approximate strategy for successfully solving 2D nonlinear TFZKEs is proposed in this paper. The Caputo formula describes the fractional derivatives. The proposed operational matrix formulation technique is based on the collocation method of SCPs. The numerical results for (2,2,2) and (3,3,3) of TFKZE -SCPs show that the present approach has accurate results and good convergence relying on Figures 1-4 by utilizing fewer grid points than other analytic techniques. The provided numerical technique for solving linear and nonlinear fractional order models is shown to be very efficient and convenient.

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