Nilpotency of Centralizers in Prime Rings

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Abstract
E. C. Posner proved that if λ and δ are derivations of a prime ring R with characteristic not equal 2 , then $\lambda\delta = 0$ implies that either $\lambda = 0$ or $\delta = 0$. David W.Jensen extend this result by showing that , without any characteristic restriction , $\lambda\delta^m = 0$ implies either $\lambda = 0$ or $\delta^{4m-1} = 0$, also he proved that $\lambda^n\delta = 0$ implies either $\delta^2 = 0$ or $\lambda^{12n-9} = 0$, and finally , in general when $\lambda^n\delta^m = 0$, he showed that if λ and δ are commute , then at least one of the derivations must be nilpotent .Here we ask the possibility if the same results of David can be satisfied on R with replacing the derivations λ and δ with centralizers T and G.

Keywords: semiprime ring, prime ring, derivation, left (right) centralizer, centralizer, Jordan centralizer , nilpotent centralizer .

المتمركزات عديمة القوى في الحلقات الأولية

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الخلاصة :
اثبت بوسنر انه إذا كانت λ و δ مشتقات لحلقة اولية R بحيث إن مميز إل$R\neq 2$, إذا كانت $\lambda\delta = 0$ فأنه اما $\lambda = 0$ أو $\delta = 0$. وسع دافيد هذه النتيجة بحيث بين انه بدون إي قيد على مميز ألR, إن $\lambda\delta^m = 0$ يحقق اما $\lambda = 0$ او $\delta^{4m-1} = 0$, واثبت ايضا ان $\lambda^n\delta = 0$ يحقق اما $\delta^2 = 0$ او $\lambda^{12n-9} = 0$ و اخيرا اثبت بصورة عامة انه اذا كان $\lambda^n\delta^m = 0$, λ و δ تتبادل مع بعضها فأنه على الاقل واحدة منهم تكون عديمة القوى . خلال هذا البحث سنقوم بتطبيق نتائج دافيد الثلاثة على المتمركزات .

Introduction:
Throughout this paper R will represent an associative ring. Recall that R is a prime ring if $aRb = 0$ implies that $a = 0$ or $b = 0$ (where $a, b \in R$) , and R is semiprime ring if $aRa = 0$ implies that $a = 0$ (Where $a \in R$) . A ring R is n-torsion free if $nx = 0$ implies that $x = 0$ (where $x \in R$ and n is a positive integer) . An additive mapping $\lambda: R \to R$ is called a derivation if $\lambda(xy) = \lambda(x)y + x\lambda(y)$ holds for all $x, y \in R$. An additive mapping $T: R \to R$ is called left (right) centralizer if $T(xy) = T(x)y$ $(T(xy) = x\,T(y))$ holds for all $x , y \in R$. T *is* called centralizer if it is both left and right centralizer. A centralizer is said to be nilpotent if $T^n(x) = 0$ for some fixed integer n. An additive mapping $T: R \to R$ is called left (right) Jordan centralizer in case $T(x^2) = T(x)x$ $\left(T(x^2) = xT(x)\right)$ holds for all $x \in R$. Following

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ideas from [1] ,Zalar has proved in [2] that any left (right) Jordan centralizer on a 2-torsion free semiprime ring is a left (right) centralizer. J. Vukman [3] shows that for a semiprime ring R with extended centroid C if $3T(xyx) = T(x)yx + xT(y)x + xyT(x)$ holds for all $x, y \in R$ then there exists $\alpha \in C$ such that $T(x) = \alpha x$, for all $x \in R$ .Other results concerning centralizer in prime and semiprime ring can be found in [4 -9] . David W. Jensen [10] showed that ,if $\lambda\delta^m = 0$ implies either $\lambda = 0$ or $\delta^{4m-1} = 0$, also he proved that $\lambda^n\delta = 0$ implies either $\delta^2 = 0$ or $\lambda^{12n-9} = 0$, and finally , in general when $\lambda^n\delta^m = 0$, he showed that if λ and δ are commute , then at least one of the derivations must be nilpotent .Here we ask the possibility if the same result can be satisfied on R with replacing the derivations λ and δ with centralizers T and G.

First we shall prove the following two Lemmas which shall be used throughout the proof of our results :

Lemma 1: Assume T is a centralizer of a prime ring R and there is a nonzero element $a \in R$, such that $a(T^nR) = 0$ or $(T^nR)a = 0$, then $T^{2n-1} = 0$.

Proof: Assume first $a(T^nR) = 0$. Then for all $x, y \in R$ we have :

$$aT^n(xy) = axT^n(y) , \quad \text{for all } x, y \in R \qquad (1)$$

Replacing y in (1) by $T^{n-1}(y)$ yields that $axT^{2n-1}(y) = 0$, for all , $y \in R$. Therefore , $aRT^{2n-1}(y) = 0$, for all $y \in R$. Hence since R is a prime ring and $a \neq 0$ we have that $T^{2n-1}(y) = 0,$ for all $y \in R$, so $T^{2n-1} = 0$.

Similarly , if we begin with $(T^nR)a = 0$ we can get that $T^{2n-1} = 0$.

Lemma 2 : If T and G are centralizers of a prime ring R and $TG = 0$, then either $T = 0$ or $G^2 = 0$.

Proof: For all $x, y \in R$, we have

$0 = TG(xy) = T(G(x)y) = G(x)\,T(y)$, for all $x, y \in R$ (2)

Replacing x by $G(x)$ in (2) we get that $G^2(x)\,T(y) = 0$, for all $x, y \in R$.

Assume $G^2(x) \neq 0$, then by Lemma 1 we have that $T(y) = 0,$ for all $y \in R$, hence $T = 0$.

Next we will prove the main results in this research which will show that for a prime ring R and centralizers T, G of R , if $T^nG^m = 0$, then at least one of the centralizers will be nilpotent if $n = 1$, $m = 1$ or T and G are commute .

Theorem 1 : Let T and G be centralizers of a prime ring , and let $TG^m = 0$ where m is a positive integer . Then either $T = 0$ or $G^r = 0$, where $r \leq 4m - 1$.

Proof : We proceed by induction , when $m = 1$, Lemma 2 implies that the result is true . Assume the theorem is true for $m = 1,2, \ldots, k - 1$.

Now assume $TG^k = 0$, then for all x and y in ,

$TG^k(xy) = T\big(xG^k(y)\big) = 0$ (3)

Replacing x by $G^{k-1}(x)$ and y by $G^k(y)$ in (3) yields

$0 = T\big(G^{k-1}(x)\,G^{2k}(y)\big) = T(G^{k-1}(x))G^{2k}(y) = TG^{k-1}(x)G^{2k}(y)$, for all , $y \in R$.

If $TG^{k-1}(x) \neq 0$, then by Lemma 1 we get that $G^{4k-1} = 0$. On the other hand if $TG^{k-1}(x) = 0$ for all $\in R$, then by induction hypothesis we get that the theorem is true .

Theorem 2 : Let T and G be centralizers of a prime ring , let $T^nG = 0$, where n is a positive integer . Then either $G^2 = 0$ or $T^t = 0$, where $t \leq 12n - 9$.

Proof : Let S be the set of all centralizers of R .

Claim : S is a Lie ring .

Let $T, G \in S$,

$(T - G)(xy) = T(xy) - G(xy) = T(x)y - G(x)y$

$= \big(T(x) - G(x)\big)y = (T - G)(x)\,y$.

Hence $T - G \in S$.

Now we want to prove $TG \in S$.

$TG(xy) = T(G(x)y) = TG(x)y$.

So $TG \in S$. Hence S is a ring .

Now ,

$(TG - GT)(xy) = TG(xy) - GT(xy)$

$$= TG(x)y - GT(x)y = (TG - GT)(x)\, y$$

Thus $TG - GT \in S$. Hence S is a Lie ring of R .

Therefore $[G,T] = GT - TG$ is a centralizer of R , $[GT - TG, T] = GT^2 - 2TGT + T^2G$ is a centralizer , also $[GT^2 - 2TGT + T^2, T] = GT^3 - 3TGT^2 + 3T^2GT - T^3G$ is a centralizer . Continue in this way we may conclude that

$\sum_{i=0}^{2n-1} \binom{2n-1}{i} (-1)^i\, T^i G T^{2n-1-i}$ is a centralizer (4)

The coefficients are not germane to the rest of the proof , so we suppress them from here on out .Using the assumption that $T^nG = 0$ and (4) we get that $GT^{2n-1} + TGT^{2n-2} + \ldots + T^{n-1}GT^n$ is a centralizer. Since

$(GT^{2n-1} + TGT^{2n-2} + \ldots + T^{n-1}GT^n)G = 0$ (5)

Then by applying Lemma2 on (5) gives us that $G^2 = 0$ or

$GT^{2n-1} + TGT^{2n-2} + \ldots + T^{n-1}GT^n = 0$ (6)

If $G^2 \neq 0$, then premultiplying (6) by T^{n-1} and using $T^nG = 0$ to obtain $T^{n-1}GT^{2n-1} = 0$. Premultiplying (5) by T^{n-2} it follows that

$$T^{n-2}GT^{2n-1} + T^{n-1}GT^{2n-2} = 0$$

$$\Rightarrow (T^{n-2}GT^{2n-1} + T^{n-1}GT^{2n-2})T = 0$$

$$\Rightarrow T^{n-2}GT^{2n-1} = 0$$

Premultiplying (6) by T^{n-3} it follows that

$$T^{n-3}GT^{2n-1} + T^{n-2}GT^{2n-2} + T^{n-1}GT^{2n-3} = 0$$

$$\Rightarrow (T^{n-3}GT^{2n-1} + T^{n-2}GT^{2n-2} + T^{n-1}GT^{2n-3})T^2 = 0$$

$$\Rightarrow T^{n-3}GT^{2n+1} = 0$$

Continuing this algorithm we arrive at $GT^{3n-2} = 0$.

Applying Theorem 1 completes the proof .

Theorem 3 : Assume T and G are centralizers of a prime ring R , and assume $T^nG^m = 0$, where n and m are positive integers . If T and G are commute then at least one of them is nilpotent .

Proof : First by our hypothesis we have for all $x, y \in R$

$$0 = T^nG^m\left(T^{mn}(x)G^{(m-1)m}T^{n-1}(y)\right) = T^n\left(T^{mn}(x)G^{m^2}T^{n-1}(y)\right)$$

$$= T^{(m+1)n}(x)G^{m^2}T^{n-1}(y)$$

If $T^{n-1}G^{m^2} \neq 0$, then by using Lemma 1 we have that $T^{2(m+1)n-1} = 0$.

If $T^{n-1}G^{m^2} = 0$. Hence , for $x, y \in R$

$$0 = T^{n-1}G^{m^2}\left(T^{m^2(n-1)}(x)T^{n-2}G^{(m^2-1)m^2}(y)\right)$$

$$= T^{n-1}\left(T^{m^2(n-1)}(x)T^{n-2}G^{m^4}(y)\right)$$

$= T^{(m^2-1)(n-1)}(x)T^{n-2}G^{m^4}(y)$, for all $x, y \in R$.

If $T^{n-2}G^{m^4} \neq 0$, by Lemma 1 we have that $T^{2(m^2-1)(n-1)-1} = 0$.

If $T^{n-2}G^{m^4} = 0$ continue by applying the same way above we arrive at $G^{m^{2^n}} = 0$. Then G is a nilpotent centralizer .

Remark 1: The assumption that R is prime is essential, as the following example shows :

Example : Let F be a field and $D_2(F)$ the ring of 2 by 2 diagonal matrices over the field F , and let T, G be centralizers of $D_2(F)$ defined by

$T\left(\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}\right) = \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}$ and $G\left(\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & y \end{bmatrix}$, for all $x, y \in F$

Easily one can show that the centralizers commute with each other , and $TG = 0$, but none one of the centralizers are nilpotent .

Remark 2: In this research we applied the same results in [11] by using centralizers instead of derivations, so we shall give examples to illustrate that there is no relation between derivation and centralizer.

Example 1: Let F be a field and $D_2(F)$ the ring of 2 by 2 diagonal matrices over the field F, and let T be a mapping of $D_2(F)$ defined by

$T\left(\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}\right) = \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}$, for all $x, y \in F$

Easily one can show that T is a centralizer but it is not a derivation .

Example 2: Let R be the ring of 2 by 2 upper triangle matrices over a field, and let d be a mapping of R defined by

$d\left(\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} - \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, for all $x, y, z \in F$

Then d *is* a derivation but not centralizer.

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