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Mann Iteration Processes on Uniform Convex n-Banach Space

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Abstract
Let Y be a uniformly convex n-Banach space, M be a nonempty closed convex subset of Y, and S:M→M be a nonexpansive mapping. The purpose of this paper is to study some properties of uniform convex set that help us to develop iteration techniques for approximation of fixed point of nonlinear mapping by using the Mann iteration processes in n-Banach space.

Keywords: Mann iteration process; Fixed point; Nonexpansive mapping; Uniformly convex, n-Banach space

تكرار مان على فضاء n- بناخ المحدب والمنتظم

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الخلاصة
لتكن Y فضاء بناخ من النمط n المحدب والمنتظم, وM مجموعه جزئية محدبة غير خالية ومغلقه من Y, و S:M→M دالة غير توسعية. الغرض من البحث هو دراسة الخواص الجديدة للمساحة المحدبة الموحدة التي تساعدنا في تطوير تقنيات التكرار لتقريب النقطة الثابتة لدالة غير خطية بواسطة تكرار مان لفضاء بناخ من النمط n.

Introduction

Let Y be a real linear space of dimension greater than 1 and ‖.,...,.‖ : Y × Y→ ℝ satisfying the following conditions:

a) $\|x_1,\dots,x_n\| = 0$ if and only if $x_1,x_2,\dots,x_n$ are linearly dependent vectors;
b) $\|x_1,\dots,x_n\|$ is invariant under permutations of $x_1,\dots,x_n$;
c) $\|\lambda x_1,x_2,\dots,x_n\| = |\lambda|\|x_1,x_2,\dots,x_n\|$ for all $\lambda \in \mathbb{R}$ and $x_1,\dots,x_n \in Y$
d) $\|x_0+x_1,x_2,\dots,x_n\| \le \|x_0,x_2,\dots,x_n\|+\|x_1,x_2,\dots,x_n\|$ for all $x_0,x_1,\dots,x_n \in Y$

then ‖.,…,.‖ is called an n-norm on Y and (Y,‖.,…,.‖) is called a linear n-normed space [1],
In the following, we need the concept of n-Banach space. A treatment of 2-Banach space can be found in [2]. The notion of n-Banach space and related concepts such as Cauchy sequence and convergence sequencesas given below are discussed briefly in [1].
A sequence $\{x_n\}$ in n-normed space (Y,‖.,…,.‖) is said to be a converge to x ∈ X, if for all $z_1,z_2,\dots,z_{n-1} \in Y$,

$$\lim_{n\to\infty} \| x_n - x,\ z_1,z_2,\dots,z_{n-1} \| = 0$$

Also, a sequence $\{x_n\}$ in n-normed space (Y,‖.,…,.‖) is said to be a Cauchy sequence if for all $z_1,z_2,\dots,z_{n-1} \in Y$,

$$\lim_{m,n\to\infty} \| x_n - x_m, z_1,z_2,\dots,z_{n-1}\| = 0.$$

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Note that, every convergent sequence is Cauchy sequence. If the converse is true, then the n-normed space is called n–Banach space.

In 1965, Browder [3] and Göhde [4] independently proved that every nonexpansive self-mapping of a closed convex and bounded subset of a uniformly convex Banach space has a fixed point.

In [5], It is shown that a technique of Mann is fruitful in finding a fixed point on Banach space of monotone nonexpansive mapping**.**

In this paper, we generalized the concept of the uniformly convex set on n-Banach space and discuss some its properties that help us to study approximate fixed point by Mann iteration process, under nonexpansive mapping:

Let M be a nonempty closed convex subset of a real uniformly convex n-Banach space Y, A self-mapping S:M→M is said to be nonexpansive If

$$\|S(x) - S(y), z_1, z_2, \ldots, z_{n-1}\| \leq \|x - y, z_1, z_2, \ldots, z_{n-1}\| \text{ for all } x, y \in M$$

And $z_1, z_2, \ldots, z_{n-1} \in Y$, and S is said to be quasi-nonexpansive provided that if S(p) = p then

$$\|S(x) - p, z_1, z_2, \ldots, z_{n-1}\| \leq \|x - p, z_1, z_2, \ldots, z_{n-1}\|$$

This paper consists of two sections, In section 1 we study the uniform convex in n-normed space, basic properties and some theorem on uniformly convex n-Banach space. In section 2, we give some result on Mann iteration on uniform convex n-Banach space.

$1 UNIFORM CONVEXITY

In this section, we introduce the definition of uniformly convex on n-normed space, and discuss some its properties,

Definition 1.1

A linear n–normed space $(Y, \|., \ldots, .\|)$ is said to be uniformly convex if for every ϵ in (0,2] and a nonzero z in Y, there exists a $\delta > 0$ such that $\|x, z_1, z_2, \ldots, z_{n-1}\| \leq 1$, $\|y, z_1, z_2, \ldots, z_{n-1}\| \leq 1$ and $\|x-y, z_1, z_2, \ldots, z_{n-1}\| \geq \epsilon$ imply that

$$\|\tfrac{1}{2}(x+y), z_1, z_2, \ldots, z_{n-1}\| \leq 1-\delta$$

The following theorem is the generalization of the theory found in [6, p.54] of uniformly convex Banach space to uniformly convex n-Banach space.

Theorem 1.2

Let Y be a uniformly convex n–Banach space, then we have:

a) The elements x, y, $z_1, z_2, \ldots, z_{n-1} \in Y$ with $\|x, z_1, z_2, \ldots, z_{n-1}\| \leq r$, $\|y, z_1, z_2, \ldots, z_{n-1}\| \leq r$, $\|x-y, z_1, z_2, \ldots, z_{n-1}\| \geq \epsilon$ for any r and ϵ with $r \geq \epsilon > 0$ there exist a $\delta = \delta(\epsilon/r, z_1, z_2, \ldots, z_{n-1})$ such that

$$\|\tfrac{1}{2}(x+y), z_1, z_2, \ldots, z_{n-1}\| \leq r[1-\delta].$$

b) The elements x, y, $z_1, z_2, \ldots, z_{n-1} \in Y$ with $\|x, z_1, z_2, \ldots, z_{n-1}\| \leq r$, $\|y, z_1, z_2, \ldots, z_{n-1}\| \leq r$, $\|x-y, z_1, z_2, \ldots, z_{n-1}\| \geq \epsilon$ for any r and ϵ with $r \geq \epsilon > 0$ there exist a δ such that

$$\|\alpha x+(1-\alpha)y, z_1, z_2, \ldots, z_{n-1}\| \leq r\,[1-2\min\{\alpha, 1-\alpha\}\,\delta(\epsilon/r, z_1, z_2, \ldots, z_{n-1})] \text{ for all } \alpha \in (0,1)$$

Proof.

a) suppose that $\|x, z_1, z_2, \ldots, z_{n-1}\| \leq r$, $\|y, z_1, z_2, \ldots, z_{n-1}\| \leq r$ and $\|x-y, z_1, z_2, \ldots, z_{n-1}\| \geq \epsilon > 0$, thus

$\|\frac{x}{r}, z_1, z_2, \ldots, z_{n-1}\| \leq 1$, $\|\frac{y}{r}, z_1, z_2, \ldots, z_{n-1}\| \leq 1$ and $\|\frac{x-y}{r}, z_1, z_2, \ldots, z_{n-1}\| \geq \frac{\epsilon}{r} > 0$

By the definition (1.1), there exists $\delta = \delta(\epsilon/r, z_1, z_2, \ldots, z_{n-1}) > 0$ such that

$$\|\frac{x+y}{2r}, z_1, z_2, \ldots, z_{n-1}\| \leq 1-\delta.$$

Therefore,

$$\|\frac{x+y}{2}, z_1, z_2, \ldots, z_{n-1}\| \leq r[1-\delta]$$

b) when $\alpha = \frac{1}{2}$, we are done by part (a). If $\alpha \in (0, \frac{1}{2}]$, so by part (a) and $\|y, z_1, z_2, \ldots, z_{n-1}\| \leq r$ we have

$$\begin{aligned}\|\alpha x+(1-\alpha)y, z_1, z_2, \ldots, z_{n-1}\| &= \|\alpha(x+y)+(1-2\alpha)y, z_1, z_2, \ldots, z_{n-1}\| \\ &\leq 2\alpha\,\|\frac{x+y}{2}, z_1, z_2, \ldots, z_{n-1}\| + (1-2\alpha)\,\|y, z_1, z_2, \ldots, z_{n-1}\|\end{aligned}$$

$\leq 2\alpha r\,[1-\delta]+(1-2\alpha)r$
$\leq r\,[1-2\alpha\delta]$.

Now by the choice of $\alpha \in [\frac{1}{2},1)$, so by part (a) and $\|x, z_1,z_2,\ldots,z_{n-1}\| \leq r$ we have

$\|\alpha x+(1-\alpha)y, z_1,z_2,\ldots,z_{n-1}\| = \|(2\alpha-1)x + (1-\alpha)(x+y), z_1,z_2,\ldots,z_{n-1}\|$
$\leq (2\alpha-1)\|x, z_1,z_2,\ldots,z_{n-1}\| + 2(1-\alpha)\|\frac{x+y}{2}, z_1,z_2,\ldots,z_{n-1}\|$
$\leq r\,[1-2(1-\alpha)\,\delta]$

then,

$$\|\alpha x+(1-\alpha)y, z_1,z_2,\ldots,z_{n-1}\| \leq r\,[1-2\min\{\alpha,1-\alpha\}\,\delta] \quad ■$$

Using theorem (1.2), we obtain the following, which has important applications in approximation of fixed point of nonlinear mapping in n-Banach space.

Proposition 1.3

Let Y be a uniformly convex n–Banach space and let $\{\alpha_n\}$ be a sequence of real numbers in (0,1) bounded away from 0 and 1. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in Y such that

$$\lim_{n\to\infty} sup\|x_n, z_1,z_2,\ldots,z_{n-1}\| \leq a, \lim_{n\to\infty} sup\|y_n, z_1,z_2,\ldots,z_{n-1}\| \leq a$$

and $\lim_{n\to\infty} sup\|\alpha_n x_n + (1-\alpha_n) y_n, z_1,z_2,\ldots,z_{n-1}\| = a$, for some $a \geq 0$. Then

$$\lim_{n\to\infty} \|x_n - y_n, z_1, z_2, \ldots, z_{n-1}\| = 0$$

Proof.

If a = 0 is trivial. So, Let $a > 0$. Assume for contradiction, that $\{x_n - y_n\}$ does not converge to 0. Then there exists a subsequence $\{x_{n_j} - y_{n_j}\}$ of $\{x_n - y_n\}$ such that $inf_j\|x_{n_j} - y_{n_j}, z_1,z_2,\ldots,z_{n-1}\| > 0$. Note $\{\alpha_n\}$ is bounded away from 0 and 1, and there exists two positive number γ and β such that $0 < \gamma \leq \alpha_n \leq \beta < 1$ for all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} sup\|x_n, z_1,z_2,\ldots,z_{n-1}\| \leq a$ and $\lim_{n\to\infty} sup\|y_n, z_1,z_2,\ldots,z_{n-1}\| \leq a$ we may suppose an $r \in (a,a+1)$ for a subsequence $\{n_j\}$ suchthat $\|x_{n_j}, z_1,z_2,\ldots,z_{n-1}\| \leq r$, $\|y_{n_j}, z_1,z_2,\ldots,z_{n-1}\| \leq r$, $a < r$. Choose $r \geq \epsilon > 0$ such that

$$2\gamma(1-\beta)\delta(\epsilon/r, z_1,z_2,\ldots,z_{n-1}) < 1 \text{ and } \|x_{n_j} - y_{n_j}, z_1,z_2,\ldots,z_{n-1}\| \geq \epsilon > 0$$

From Theorem (1.2) part (b), we have

$\|\alpha_{n_j} x_{n_j} + (1-\alpha_{n_j}) y_{n_j}, z_1,z_2,\ldots,z_{n-1}\| \leq r[1-2\alpha_{n_j}(1-\alpha_{n_j})\delta(\epsilon/r, z_1,z_2,\ldots,z_{n-1})]$
$\leq r[1-2\gamma(1-\beta)\delta(\epsilon/r, z_1,z_2,\ldots,z_{n-1})] < a$

for all $j \in \mathbb{N}$ which contradicts the hypothesis. ■

Next, we give some characterizations of uniform convexity in n–Banach space

Proposition 1.4

Let Y be a n–Banach space and $r \geq \epsilon > 0$, Then the following are equivalent:

a) Y is uniformly convex

b) For two sequences $\{x_n\}$ and $\{y_n\}$ in Y for all $n \in \mathbb{N}$, if

$\|x_n, z_1, z_2,\ldots,z_{n-1}\| \leq r$, $\|y_n, z_1,z_2,\ldots,z_{n-1}\| \leq r$ and

$\lim_{n\to\infty} \|x_n + y_n, z_1,z_2,\ldots,z_{n-1}\| = 2$, then $\lim_{n\to\infty} \|x_n - y_n, z_1, z_2, \ldots, z_{n-1}\| = 0$ (1)

Proof

(a) ⇒ (b). assume Y is uniformly convex. Let $\{x_n\}$ and $\{y_n\}$ be two sequenece Y and $\{\alpha_n\} = \{\frac{1}{2}\} \subseteq (0,1)$ Since

$\lim_{n\to\infty} sup\|\frac{1}{2}x_n + \frac{1}{2}y_n, z_1,z_2,\ldots,z_{n-1}\| = \frac{1}{2}\lim_{n\to\infty} sup\|x_n + y_n, z_1,z_2,\ldots,z_{n-1}\| = 1$ and

$$\lim_{n\to\infty} sup\|x_n, z_1,z_2,\ldots,z_{n-1}\| = 1, \lim_{n\to\infty} sup\|y_n, z_1,z_2,\ldots,z_{n-1}\| = 1$$

Thus by proposition (1.3) we have

$$\lim_{n\to\infty} \|x_n - y_n, z_1, z_2, \ldots, z_{n-1}\| = 0$$

(b) ⇒ (a). assume condition (1) is satisfied. If Y is not uniformly convex for $\epsilon > 0$, and the two sequences $\{x_n\}$ and $\{y_n\}$ in Y such that

(i) $\|x_n, z_1,z_2,\ldots,z_{n-1}\| \le 1$, $\|y_n, z_1,z_2,\ldots,z_{n-1}\| \le 1$
(ii) $\|x_n$-$y_n, z_1,z_2,\ldots,z_{n-1}\| \ge \epsilon$
iii) $\|x_n + y_n , z_1,z_2,\ldots,z_{n-1}\| > 2$ (1- $\delta(\epsilon, z_1,z_2,\ldots,z_{n-1})$)
Let $\delta(\epsilon, z_1,z_2,\ldots,z_{n-1}) = \frac{1}{n}$, then

$$\|x_n + y_n , z_1,z_2,\ldots,z_{n-1}\| > 2\ (1 - \frac{1}{n})$$

Therefore

$$\lim_{n\to\infty} \|x_n + y_n , z_1, z_2, \ldots , z_{n-1}\| > 2$$

But, $\lim_{n\to\infty} \| x_n + y_n, z_1,z_2,\ldots,z_{n-1}\| = 2$
This is a contradiction, then, must be Y uniformly convex ■

$2 Mann Iteration Processes on n-Banach Space.

In this section, we introduce the Mann iteration on n-Banach space, basic properties and some theorems on Mann iteration on uniformly convex n-Banach space

First, we introduce the Mann iteration in n-normed space.

Definition 2.1

Let M be a nonempty convex subset of a convex n-normed space Y and S:M→M a mapping. Let $\{\alpha_n\}$ be a sequence satisfying $0 \le \alpha_n \le 1$ for all n∈ ℕ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Define a sequence $\{x_n\}$ in M by

$$x_1 \in M,$$
$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S x_n,\ n \in \mathbb{N}$$

Then $\{x_n\}$ is called the Mann iteration.

The following results are very useful for approximation of the fixed point of nonexpansive type mapping.

proposition .2.2

Let M be a closed convex subset of n-Banach space Y and S:M→M be aNonexpansive mapping, p be a fixed point of S, where $\{x_{n+1}\}$ is a Mann iteration define by (2.1) then,
a) $\{\| x_{n+1} - p,\ z_1,z_2,\ldots,z_{n-1} \|\}$ is a decreasing sequence.
Moreover, $\lim_{n\to\infty} \|x_n - p ,\ z_1, z_2, \ldots , z_{n-1}\|$ exists
b) $\{\|x_{n+1} - Sx_{n+1} ,\ z_1,z_2,\ldots,z_{n-1}\|\}$ is a decreasing sequence.
Moreover, $\lim_{n\to\infty} \|\ x_{n+1} - Sx_{n+1} ,\ z_1, z_2, \ldots , z_{n-1}\|$ exists

Proof

(a) Let p is a fixed point of S, and by using definition (2.1) we have for all n∈ ℕ.

$$\begin{aligned}\|x_{n+1} - p,\ z_1,z_2,\ldots,z_{n-1}\| &= \|(1-\alpha_n)(x_n - p) + \alpha_n (Sx_n - p),\ z_1,z_2,\ldots,z_{n-1}\| \\ &\le (1-\alpha_n)\|x_n - p ,\ z_1,z_2,\ldots,z_{n-1}\| + \alpha_n\|Sx_n - p ,z_1,z_2,\ldots,z_{n-1}\| \\ &\le \|x_n - p,\ z_1,z_2,\ldots,z_{n-1}\|\end{aligned}$$

Therefore, the sequence $\{\|x_n - p,\ z_1,z_2,\ldots,z_{n-1}\|\}$ is a decreasing and hence it follows that $\lim_{n\to\infty} \|x_n - p ,\ z_1, z_2, \ldots , z_{n-1}\|$ exists

(b). By using (2.1) we have.

$$\begin{aligned}\|x_{n+1} - Sx_{n+1} ,\ z_1,z_2,\ldots,z_{n-1}\| &= \|(1-\alpha_n)(x_n - Sx_n) + (Sx_n - Sx_{n+1}),\ z_1,z_2,\ldots,z_{n-1}\| \\ &\le (1-\alpha_n)\|x_n - Sx_n ,\ z_1,z_2,\ldots,z_{n-1}\| \\ &\quad + \|x_n - x_{n+1} ,\ z_1,z_2,\ldots,z_{n-1}\| \\ &= (1-\alpha_n)\|x_n - Sx_n,\ z_1,z_2,\ldots,z_{n-1}\| \\ &\quad + \|x_n - ((1-\alpha_n)x_n + \alpha_n Sx_n,\ z_1,z_2,\ldots,z_{n-1}\| \\ &= \|x_n - Sx_n ,\ z_1,z_2,\ldots,z_{n-1}\|\end{aligned}$$

Therefore, the sequence $\{\|x_n - Sx_n ,\ z_1,z_2,\ldots,z_{n-1}\|\}$ is a decreasing and hence it follows that $\lim_{n\to\infty} \|x_{n+1} - Sx_{n+1} ,\ z_1, z_2, \ldots , z_{n-1}\|$ exists ■

The next results will be used in proof proposition (2.4)

proposition 2.3

Let $\{x_n\}$ and $\{y_n\}$ be two sequences in a uniformly convex n–Banach space Y such that,

$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n y_n$ and $\|y_n, z_1,z_2,\ldots,z_{n-1}\| \le \|x_n, z_1,z_2,\ldots,z_{n-1}\|$, for all n∈ ℕ

Where $\{\alpha_n\}$ is a sequence of nonnegative number in [0,1] with $\sum_{n=1}^{\infty} min\{\alpha_n, 1-\alpha_n\} = \infty$, then $0 \in \overline{\{x_n - y_n\}}$.

Proof

Assume for contradiction that, $\|x_n-y_n, z_1,z_2,\ldots,z_{n-1}\| \geq \epsilon >0$ for all n∈ ℕ, by using Mann iteration then,

$$\|x_{n+1}, z_1,z_2,\ldots,z_{n-1}\| \leq \|x_n, z_1,z_2,\ldots,z_{n-1}\| \leq \ldots \leq \|x_1, z_1,z_2,\ldots,z_{n-1}\| \text{ for all } n \in \mathbb{N}.$$

By using theorem (1.2) part (b), then there exist a $\delta = \delta(\epsilon/r, z_1,z_2,\ldots,z_{n-1})$ such that.

$$\|x_{n+1}, z_1,z_2,\ldots,z_{n-1}\| = \|(1-\alpha_n)x_n+\alpha_n y_n, z_1,z_2,\ldots,z_{n-1}\| \leq \|x_n, z_1, z_2,\ldots,z_{n-1}\| \left[1-2\min\{\alpha_n, 1-\alpha_n\}\delta(\epsilon/\|x_1,z_1,z_2,\ldots,z_{n-1}\|, z_1,z_2,\ldots,z_{n-1})\right]$$

Therefore,

$$\|x_n, z_1,z_2,\ldots,z_{n-1}\| \leq \prod_{i=1}^{n-1} \|x_1, z_1, z_2, \ldots, z_{n-1}\| \left[1-2\min\{d_i, 1-d_i\}\delta\right]$$

for all n>1, Since

$$\sum_{n=1}^{\infty} min\{\alpha_n,1-\alpha_n\} = \infty \text{ and } \prod_{i=k}^{n}(1-\alpha_i) \leq \exp(-\sum_{i=k}^{n}\alpha_i) \to 0$$

Thus

$\|x_{n+1}, z_1,z_2,\ldots,z_{n-1}\| \leq \|x_1, z_1,z_2,\ldots,z_{n-1}\| \exp(-2\,\delta\sum_{i=1}^{n-1}\min\{\alpha_i, 1-\alpha_i\}) \to 0$ as n→ ∞,
and hence $\lim_{n\to\infty}\|x_n, z_1, z_2, \ldots, z_{n-1}\| = \lim_{n\to\infty}\|y_n, z_1, z_2, \ldots, z_{n-1}\| = 0$
this is a contradiction ■

Proposition 2.4

Let M a nonempty convex subset in a uniformly convex n-Banach space Y and S:M→M a mapping, with satisfies the condition,

$$\|S(x)-p, z_1,z_2,\ldots,z_{n-1}\| \leq \|x-p, z_1,z_2,\ldots,z_{n-1}\| \text{ for all } x, z_1,z_2,\ldots,z_{n-1} \in M$$

and p is a fixed point of S, Define a sequence $\{x_n\}$ in M by using (2.1) and $\sum_{n=1}^{\infty} min\{\alpha_n,1-\alpha_n\} = \infty$. Then lim inf $\|x_n - Sx_n, z_1,z_2,\ldots,z_{n-1}\| = 0$

Proof.

By using Proposition (2.2) we obtain That $\lim_{n\to\infty}\|x_n - p, z_1, z_2, \ldots, z_{n-1}\|$ exists, for p is a fixed point of S. Therefore,

$$\|Sx_n - p, z_1,z_2,\ldots,z_{n-1}\| \leq \|x_n - p, z_1,z_2,\ldots,z_{n-1}\| \text{ for all } n \in \mathbb{N}$$

And

$$x_{n+1} - p = (1-\alpha_n)\, x_n + \alpha_n(Sx_n - p) \text{ for all } n \in \mathbb{N}$$

From proposition (2.3), we obtain that lim inf $\|x_n - Sx_n, z_1,z_2,\ldots,z_{n-1}\| = 0$ ■

The following proposition we will demonstrate using the characteristic (2.2) and (1.3)

Proposition 2.5

Let M be a nonempty closed convex subset of a uniformly convex n-Banah space Y and S:M→M be a quasi–nonexpansive mapping that has at least one fixed point p. Let $\{x_n\}$ be the mann iteration defined by (2.1), where $\{\alpha_n\}$ is a sequence of nonnegative numbers that is bounded away from 0 and 1. Then

$$\lim_{n\to\infty}\|x_n - Sx_n, z_1,z_2,\ldots,z_{n-1}\| = 0.$$

Proof

By using Proposition (2.2), we obtain That $\lim_{n\to\infty}\|x_n - p, z_1, z_2, \ldots, z_{n-1}\|$ exists, Suppose that $\lim_{n\to\infty}\|x_n - p, z_1, z_2, \ldots, z_{n-1}\| = r$. Since $x_{n+1} - p = (1-\alpha_n)(x_n - p) + \alpha_n(Sx_n - p)$ and $\|Sx_n - p, z_1,z_2,\ldots,z_{n-1}\| \leq \|x_n - p, z_1,z_2,\ldots,z_{n-1}\|$, It follows from Proposition (1.3) that $\lim_{n\to\infty}\|x_n - Sx_n, z_1,z_2,\ldots,z_{n-1}\| = 0$. ■

From the theorem found in paper [7, p.370] which states (Let S be a nonexpansive of a convex subset of uniform convex Banach space Y into itself with at least one fixed point, assume $\sum \alpha_n(1-\alpha_n)$ diverges, then the sequence$\{(I-S)\, x_n\}$ converge to 0 for each initial point $x_1 \in Y$), we generalized this theorem to n-Banach space

Theorem 2.6

Let S be a nonexpansive of a convex subset of uniform convex n-Banach space Y Into itself with at least one fixed point, assume $\sum \alpha_n(1-\alpha_n)$ diverges, then the sequence$\{(I-S)\, x_n\}$ converge to 0 for each initial point $x_1 \in Y$, where $\{x_{n+1}\}$ is defined by (2.1)

Proof

If p is a fixed point of S , by using proposition (2.2) part (a),

$$\|x_{n+1} - p, z_1,z_2,\ldots,z_{n-1}\| \leq \|x_n - p, z_1,z_2,\ldots,z_{n-1}\| \text{ for all } n \in \mathbb{N} \quad (1)$$

Since S is nonexpansive,

$$\|x_n - Sx_n, z_1,z_2,\ldots,z_{n-1}\| \leq \|(x_n - p) - (Sx_n - p), z_1,z_2,\ldots,z_{n-1}\|$$

$$\leq 2\|x_n\text{- p}, z_1,z_2,\ldots,z_{n-1}\|$$

We may assume there is an $a > 0$ such that $\|x_n$- p, $z_1,z_2,\ldots,z_{n-1}\| \geq$ a for all n. and by using proposition (2.2) part (b),

$$\|x_{n+1} - Sx_{n+1}, z_1,z_2,\ldots,z_{n-1}\| \leq \|x_n - Sx_n, z_1,z_2,\ldots,z_{n-1}\|$$

If {(I-S) x_n} dose not converge to 0 , then there is an $\epsilon > 0$ such that

$$\|x_n - Sx_n, z_1,z_2,\ldots,z_{n-1}\| \geq \epsilon \text{ for all n.}$$

let d =2 δ($\epsilon / \|x_1 - p, z_1, z_2, \ldots, z_{n-1}\|$, $z_1,z_2,\ldots,z_{n-1}$). By using theorem (1.2) part (b) and (1) we get

$$\begin{aligned}\|x_{n+1} - p,z_1,z_2,\ldots,z_{n-1}\| &\leq \|x_n\text{- p}, z_1,z_2,\ldots,z_{n-1}\|\text{-}\propto_n(1\text{-} \propto_n)\, d\, \|x_n\text{- p}, z_1,z_2,\ldots,z_{n-1}\| \\ &\leq \|x_{n-1}\text{- p}, z_1,z_2,\ldots,z_{n-1}\| - \propto_{n-1}(1\text{-} \propto_{n-1}) \\ &\quad d\|x_{n-1}\text{- p},z_1,z_2,\ldots,z_{n-1}\|\text{-}\propto_n (1 - \propto_n)d\|x_n\text{- p},z_1,z_2,\ldots,z_{n-1}\| \\ &\leq \|x_{n-1}\text{- p} ,z_1,z_2,\ldots,z_{n-1}\| - \|x_n\text{- p}, z_1,z_2,\ldots,z_{n-1}\| \\ &\quad d(\propto_{n-1}(1\text{-} \propto_{n-1})+\propto_n(1\text{-} \propto_n))\end{aligned}$$

By induction we have,

$$\begin{aligned}\|x_{n+1} - p,z_1,z_2,\ldots,z_{n-1}\| &\leq \|x_{n-1}\text{- p}, z_1,z_2,\ldots,z_{n-1}\|\text{-}\|x_1\text{- p}, z_1,z_2,\ldots,z_{n-1}\| \\ &\quad \text{-}\|x_n\text{- p}, z_1,z_2,\ldots,z_{n-1}\|\, d \textstyle\sum_{k=1}^{n} \propto_k (1 - \propto_k).\end{aligned}$$

Therefore,

$$a\,(1+d \textstyle\sum_{k=1}^{n} \propto_k (1 - \propto_k)) \leq \|x_1\text{- p}, z_1,z_2,\ldots,z_{n-1}\|$$

This gives a contradiction since the series on the left diverges. ■

Definition 2.7

Let Y be an n-Banach space and let M a subset of Y. A mapping S:M→Y is called demicompact if it has the property that whenever $\{x_n\}$is a bounded sequence in Y and $\{ Sx_n - x_n\}$ is convergent, then there exists a subsequence $\{x_{n_k}\}$of $\{x_n\}$ which is convergent.

Corollary 2.8

Suppose M is a closed bounded convex subset of Y, S and$\{\alpha_n\}$satisfy the hypothesis of theorem (2.6). Suppose I-S maps closed bounded subset of M into closed subset of M (in particular, if S is demi compact (2.7)). Then for each $x_1 \in$M, the iteration process defined by (2.1) converges to a fixed point of S.

Proof

Let V denote the closure of the set of iteration x_n. By Theorem (1.14), 0 is a cluster point of (I-S) V. But (I-S) V is closed so 0 ∈ (I-S) V. Then there exists a subsequence x_{n_i} converging to a fixed point p. By equation (1) in theorem (2.6) and $x_{n_i} \to$ p imply that $x_n \to$ p

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