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A fourth Order Pseudoparabolic Inverse Problem to Identify the Time Dependent Potential Term from Extra Condition

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Abstract:

In this work, the pseudoparabolic problem of the fourth order is investigated to identify the time -dependent potential term under periodic conditions, namely, the integral condition and overdetermination condition. The existence and uniqueness of the solution to the inverse problem are provided. The proposed method involves discretizing the pseudoparabolic equation by using a finite difference scheme, and an iterative optimization algorithm to resolve the inverse problem which views as a nonlinear least-square minimization. The optimization algorithm aims to minimize the difference between the numerical computing solution and the measured data. Tikhonov's regularization method is also applied to gain stable results. Two examples are introduced to explain the reliability of the proposed scheme. Finally, the results showed that the time dependent potential terms are successfully reconstructed, stable and accurate, even in inclusion of noise.

Keywords: Von Neumann stability analysis, Finite difference method, Tikhonov regularization method, Pseudoparabolic inverse problem, Inverse problem.

مسألة عكسية لمعادلة شبه قطع مكافئ من الرتبة الرابعة لتحديد الجهد المعتمد على الزمن من شرط اضافي

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الخلاصة

في هذا العمل، تم دراسة معادلة شبه القطع المكافئ من الرتبة الرابعة لإيجاد الجهد المعتمد على الزمن تحت شروط دورية، شرط التكامل، شروط إضافية. أيضا قد تم ذكر وجود الحل ووحدانية. الطريقة المقترحة تتضمن من خلال حل معادلة شبه القطع المكافى بآستخدام مخطط الفروقات المنتهية وخوارزمية الامثلية تكرارية لحل المسألة المعكوسة التي تم اعتبارها كمسألة اصغر التربيعات اللاخطية. تهدف هذة خوارزمية إلى تقليل الفرق بين الحل العددي المحسوب والبيانات المقاسة. تم تطبيق طريقة التنظيم تيخونوف أيضًا للحصول

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على نتائج مستقرة. تم تقديم مثالين لشرح موثوقية الطريقة المقترحة. في النهاية، تُظهر النتائج أن الجهد المعتمد على الزمن تم ايجاده بنجاح وأن الطريقة المستخدمة مستقرة ودقيقة، حتى في حالة وجود أخطاء عشوائية.

1. Introduction

For the inverse problems, the identification of the unknown coefficients of the parabolic problem has many applications in engineering and science. The identified unknown coefficients of the parabolic inverse problems are very interesting to many researchers recently. In [1], the authors presented two parabolic inverse problems for the identification of the space and time-dependent coefficients from the overdetermination conditions. Also, Marek et al. studied Penne's formulation of the bioheat transfer equation for the estimation parameters [2]. In [3], the authors presented the one-dimensional parabolic inverse problem for recovering the heat source and time-dependent thermal conductivity with the heat flux overdetermination condition, for the other related works, see [4-6].

The pseudoparabolic equations of a higher order play a vital role in the mathematical modelling of moisture transfer, fluid filtration and heat propagation [7]. The pseudoparabolic inverse problems have been utilized in modelling various phenomena such as the wave processes, chemical, engineering, diffusion, plasma physics and heat conduction [8]. In addition, they have many applications in real life phenomena such as the theory of small oscillation of a rotating fluid [9] and infiltration of homogeneous fluids in strata [10].

Moreover, in [11], the authors analyzed the uniqueness and existence of the solution of the third order pseudoparabolic inverse problem with periodic and integral conditions. Antotsev et al. [12] proved the unique solvability for the pseudoparabolic inverse problem with a P-Laplacian and under a nonlocal integral overdetermination condition by using the Galerkin method. A. I. Ismailov in [13] theoretically studied the two-dimensional pseudoparabolic inverse problem with the additional integral conditions. In [14], the authors analysed the existence and uniqueness of the solution of third order pseudoparabolic inverse problem with periodic and integral conditions. For the other related work of pseudoparabolic inverse problems see [15–18].

Many other researchers have examined the pseudoparabolic inverse problems to identify the unknown time-dependent coefficients. In studies [19], [20], the pseudoparabolic inverse problem was presented to determine the unknown coefficient of filtration and diffusion. An inverse problem of reformulation of an unknown potential element had been studied [21]. Irem and Timar in [22] solved the quasilinear pseudoparabolic equation under periodic boundary conditions and overdetermination data to determine the coefficient and source term. While in [23] the fractional multi-dimensional pseudoparabolic nonlinear source term problem is solved by the meshless radial basis function method.

Aysel and Yashar in 2020 established the existence and uniqueness of hyperbolic inverse problems for the fourth order to determine the lowest coefficient [24]. Whereas, in 2022 Huntul and Abbas presented higher order inverse problem to reconstruct the time–dependent potential coefficient numerically [25], [26]. The authors in [27] studied the pseudoparabolic inverse problem for the fourth order to identify the time–dependent potential term. The study in [28] discussed the pseudo hyperbolic inverse problem from higher order to reconstruct the potential term numerically. Yashar et al. in [29] presented a hyperbolic inverse problem from higher order to prove the exitances and uniqueness then they identified the unknown time-dependent coefficients.

In this study, the pseudoparabolic inverse problem was presented of the fourth order to investigate the retrieval of potential time- dependent coefficient numerically, for the first time, with periodic boundary conditions and non-local integral conditions. The integral type over specification data was utilized for recovering the unique potential term. The stability of the FDM proposed scheme is discussed. The uniqueness and existence for the consideration problem were proved in [30].

This study is organized as follows: The mathematical form of the inverse problem is given in Section 2, and in Section 3, the FDM is used to discretize the direct problem. Section 4 presents the numerical technique of functional minimization and the numerical results of the inverse problem. Finally, in Section 5, the conclusions are highlighted.

2. Mathematical formulation problem

Let $Q_T \coloneqq \{0 \le x \le 1, 0 \le t \le T\}$ be a rectangle domain and consider the following inverse problem of determining a pair of functions $(u(x,t),\, p(t))$, which satisfies the one-dimensional pseudoparabolic equation of the form

$$u_t = bu_{xxt} - a(t)u_{xxxx} + p(t)\,u + f(x,t), \tag{1}$$

with nonlocal initial condition

$$u(x,0) + \delta\, u(x,T) = \varphi(x), \qquad 0 \le x \le 1, \tag{2}$$

and periodic conditions

$$u(0,t) = u(1,t),\ \ u(0,t)_x = u(1,t)_x\ \ , u(0,t)_{xx} = u(1,t)_{xx} \quad 0 \le t \le T, \tag{3}$$

And the non-local integral condition

$$\int_0^1 u(x,t)dx = 0, \qquad 0 \le t \le T, \tag{4}$$

and the final overdetermination condition

$$u(0,\ t) - \int_0^t \lambda(\tau)u(1,\tau)d\tau = h(t), \qquad 0 \le t \le T, \tag{5}$$

where $b > 0$ and $\delta \ge 0$ are the given numbers.

The equations (1)-(5) are called the inverse problem where $a(t) > 0$ is the time-dependent function where $a(t)$ is a positive function that depends on t. If we assume $b = 0$ in Eq. (1), then we get a heat equation that has been investigated by many authors [27], [31]. The functions f, φ, λ and h are given functions. In this problem, $p(t)$ is the potential term, and $u(x,t)$ represents the temperature distribution of the rectangle at position x and time t. These functions are unknown. The unique solvability of the inverse problem has been established in [30] and the following their unique solvability theorems:

Definition 1. The classical solution to the inverse boundary value problem (1)-(5) means the pair $\{u(x,t), p(t)\}$ and functions $u(x,t) \in \bar{C}^{4,1}(Q_T), p(t) \in C\,[0,T]$ that satisfy equation (1) in Q_T, condition (2) in [0,1] and conditions (3)-(5) in [0,T], where $\bar{C}^{4,1}(Q_T) = \{u(x,t): u(x,t) \in C^{2,1}(Q_T), u_{txx}, u_{xxxx} \in C(Q_T)\}$.

Theorem 1. Let $b > 0, \delta \ge 0, \varphi(x) \in C\,[0,1], f(x,t) \in C(Q_T), \int_0^1 f(x,t)dx = 0\ , 0 < a(t) \in C\,[0,T], h(t) \in C^1[0,T], h(t) \ne 0\ (0 \le t \le T), \lambda(t) \in C\,[0,T], \delta\lambda(t) = 0\ (0 \le t \le T)$ and the following compatibility conditions:

$$\int_0^1 \varphi(x)dx = 0, \varphi(0) = h(0) + \delta h(T).$$

Then the inverse problem of finding a solution to the problem (1)-(5) is equivalent to the problem of determining the functions $u(x,t) \in \bar{C}^{4,1}(Q_T)$ and $p(t) \in C[0,T]$, from (1)-(3) and

$$u_{xxx}(0,t) = u_{xxx}(1,t) \quad (0 \le t \le T). \tag{6}$$

$$\lambda(t)u(1,t) + h'(t) - bu_{txx}(0,t) + a(t)u_{xxxx}(0,t) = p(t)\left(\int_0^1 \lambda(\tau)u(1,\tau)d\tau + h(t)\right) + f(0,t)(0 \le t \le T). \tag{7}$$

Lemma 1: Let us assume that the data of inverse problem (1)–(3), (6), and (7) satisfy the following conditions:

(1). $\varphi(x) \in W_2^{(5)}(0,1)$, $\varphi(0) = \varphi(1), \varphi'(0) = \varphi'(1), \varphi''(0) = \varphi''(1), \varphi'''(0) = \varphi'''(1)$, $\varphi^{(4)}(0) = \varphi^{(4)}(1)$;

(2). $f(x,t)$, $f_x(x,t)$, $f_{xx}(x,t) \in C(Q_T)$, $f_{xxx}(x,t) \in L_2(Q_T), f(0,t) = f(1,t)$, $f_x(0,t) = f_x(1,t), f_{xx}(0,t) = f_{xx}(1,t) \quad (0 \le t \le T)$;

(3). $b > 0, \delta \ge 0, \lambda(t)$ and $a(t) \in C[0,T], h(t) \in C^1[0,T]$, $h(t) \neq 0 \quad (0 \le t \le T)$.

Theorem 2. Let the conditions (1)-(3) be satisfied and

$$(A(T) + 2)^2 B(T) < 1.$$

Then problem (1)– (3), (6), (7) has a unique solution in $K = K_R\left(\|z\|_{E_T^5} \le R = A(T) + 2\right)$ in the space E_T^5 only, where

$$A(T) = A_1(T) + A_2(T), \qquad B(T) = B_1(T) + B_2(T).$$

$$A_1(T) = \|\varphi(x)\|_{L_2(0,1)} + (1+\delta)\sqrt{T}\|f(x,t)\|_{L_2(Q_T)} + 2\sqrt{3}\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)} + \frac{2\sqrt{3}}{b}(1+\delta)\sqrt{T}\|f_{xxx}(x,t)\|_{L_2(Q_T)},$$

$$B_1(T) = (1+\delta)\left(1+\frac{\sqrt{3}}{b}\right)T,$$

$$A_2(T) = \|[h(t)]^{-1}\|_{C[0,T]}\left\{\|h'(t) - f(0,t)\|_{C[0,T]} + \|\lambda(t)\|_{C[0,T]}\left(\|\varphi(x)\|_{L_2(0,1)} + (1+\delta)\sqrt{T}\|f(x,t)\|_{L_2(Q_T)}\right) + \left(\sum_{k=1}^{\infty}\zeta_k^{-2}\right)^{\frac{1}{2}}\left[\left\|\|f_x(x,t)\|_{C[0,T]}\right\|_{L_2(0,1)} + \left(\|\lambda(t)\|_{C[0,T]} + \frac{1}{b}\|a(t)\|_{C[0,T]}\right)\left(\left\|\varphi^{(3)}(x)\right\|_{L_2(0,1)} + \frac{\sqrt{T}(1+\delta)}{b}\|f_x(x,t)\|_{L_2(Q_T)}\right)\right]\right\},$$

$$B_2(T) = \|[h(t)]^{-1}\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\zeta_k^{-2}\right)^{\frac{1}{2}}\left[\left(\|\lambda(t)\|_{C[0,T]} + \frac{1}{b}\|a(t)\|_{C[0,T]}\right)\frac{T(2+\delta)}{b} + T\|\lambda(t)\|_{C[0,T]} + 1\right].$$

Theorem 3. Let all the conditions of Theorem 1 be satisfied, and

$$\int_0^1 f(x,t)dx = 0 \ \ (0 \le t \le T), \qquad \delta\lambda(t) = 0 \ (0 \le t \le T)$$

and the compatibility conditions are met:

$$\int_0^1 \varphi(x)dx = 0, \varphi(0) = h(0) + \delta h(T).$$

Then the inverse problem (1)– (5) has a classical solution in the ball $K = K_R\left(\|z\|_{E_T^5} \le R = A(T) + 2\right)$ from E_T^5 the only.

Proof: see [30].

3. Discretization of the direct solver

Consider the direct solver for inverse problem contains the equations (1)- (4) and requires the output data (5). In this direct problem, the only unknown quantity that should be determined is $u(x,t)$ that is all other components are given. Discretizing Eq. (1) by a form of the FDM as follows: Denote the $u(x_i, t_j) = u_{i,j}$, and $f(x_i, t_j) = f_{i,j}$ where the space node is $x_i = i\Delta x$, the time node is $t_j = j\Delta t$, the space step length is $\Delta x = \frac{1}{M}$ and the time step length is $\Delta t = \frac{T}{N}$ for $i = 0,1,\dots,M,\ j = 0,1,2,\dots,N$ where M, N are positive integers. Based on the FDM scheme (FTCS) forward time central space, Eq. (1) can be expressed as follows:

$$\frac{u_{i,j+1}-u_{i,j}}{\Delta t} = -a\left(\frac{u_{i+2,j+1}-4u_{i+1,j+1}+6u_{i,j+1}-4u_{i-1,j+1}+u_{i-2,j+1}}{2(\Delta x)^4} + \frac{u_{i+2,j}-4u_{i+1,j}+6u_{i,j}-4u_{i-1,j}+u_{i-2,j}}{2(\Delta x)^4}\right) + \frac{b}{\Delta t}\left(\frac{u_{i+1,j+1}-2u_{i,j+1}+u_{i-1,j+1}}{(\Delta x)^2}\right) - \frac{b}{\Delta t}\left(\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{(\Delta x)^2}\right)$$

$$+p_j u_{ij} + f_{i,j} \quad , i = 2,3,\dots,M,\ \ j = 0,1,\dots,N \qquad (8)$$

$$u(x_i,0) + \delta\, u(x_i,T) = \varphi(x_i), \qquad i = 2,3,\dots,M$$

$$u(0,t_j) = u(1,t_j), \quad u_x(0,t_j) = u_x(1,t_j), \qquad u_{xx}(0,t_j) = u_{xx}(1,t_j), \quad j = 0,1,\dots,N. \ \ (9)$$

The first periodic condition gives $u_{0,j} = u_{M,j}$, for all $j = 0,1,\dots,N$ and the second periodic condition gives,

$u_{-1,j} = u_{M-1,j}$, for all $j = 0,1,\dots,N$,

while the third periodic condition discretization gives

$u_{-2,j} = u_{M-2,j}$ for all $j = 0,1,2,\dots,N$.

Using the trapezoidal rule approximation to the integral in (4) to reach the following expression,

$$u_{Mj} + \sum_{i=1}^{M-1} u_{ij} = 0, \qquad j = 0,1,\dots,N. \qquad (10)$$

Also, the approximate formula for overdetermination condition Eq. (5) via trapezoidal rule is given as follows:

$$h\,(t_j) = u_{0j} - \frac{1}{2N}\{(\lambda_1 u_{M,1} + \lambda_N u_{M,N}) + \sum_{j=2}^{N-1} \lambda_j u_{M,j}\}. \tag{11}$$

Then Eq. (8) can be rearranged into the following difference equation

$$\frac{\gamma}{2}u_{i-2,j+1} - (\alpha+2\gamma)u_{i-1,j+1} + (1+2\alpha+3\gamma)u_{i,j+1} - (\alpha+2\gamma)u_{i+1,j+1} + \frac{\gamma}{2}u_{i+2,j+1} = -\frac{\gamma}{2}u_{i-2,j} - (\alpha-2\gamma)u_{i-1,j} + (1+2\alpha-3\gamma+\omega_j)u_{i,j} - (\alpha-2\gamma)u_{i+1,j} - \frac{\gamma}{2}u_{i+2,j} + \Delta t\, f_{i,j}, \quad i = 2,3,\dots,M, j = 0,1,\dots,N \tag{12}$$

$$\boldsymbol{D}v^{j+1} = \boldsymbol{E}v^j + Z$$

The last difference equation can be expressed in a more convenient way as the following linear algebraic system

$$D = \begin{pmatrix} 1+2\alpha+3\gamma & -(\alpha+2\gamma) & \frac{\gamma}{2} & 0 & \dots & \frac{\gamma}{2} & -(\alpha+2\gamma) \\ -(\alpha+2\gamma) & 1+2\alpha+3\gamma & -(\alpha+2\gamma) & \frac{\gamma}{2} & 0 & 0 & \frac{\gamma}{2} \\ \frac{\gamma}{2} & -(\alpha+2\gamma) & 1+2\alpha+3\gamma & -(\alpha+2\gamma) & \frac{\gamma}{2} & 0 & 0 \\ 0 & \frac{\gamma}{2} & -(\alpha+2\gamma) & 1+2\alpha+3\gamma & -(\alpha+2\gamma) & \frac{\gamma}{2} & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \frac{\gamma}{2} & -(\alpha+2\gamma) & 1+2\alpha+3\gamma & -(\alpha+2\gamma) & \frac{\gamma}{2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\gamma}{2} & 0 & 0 & \frac{\gamma}{2} & -(\alpha+2\gamma) & 1+2\alpha+3\gamma & -(\alpha+2\gamma) \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}_{M\times M}$$

$$E = \begin{pmatrix} 1+2\alpha-3\gamma+\omega_j & -(\alpha-2\gamma) & -\frac{\gamma}{2} & 0 & 0 & -\frac{\gamma}{2} & -(\alpha-2\gamma) \\ -(\alpha-2\gamma) & 1+2\alpha-3\gamma+\omega_j & -(\alpha-2\gamma) & -\frac{\gamma}{2} & 0 & 0 & -\frac{\gamma}{2} \\ -\frac{\gamma}{2} & -(\alpha-2\gamma) & 1+2\alpha-3\gamma+\omega_j & -(\alpha-2\gamma) & -\frac{\gamma}{2} & 0 & 0 \\ 0 & -\frac{\gamma}{2} & -(\alpha-2\gamma) & 1+2\alpha-3\gamma+\omega_j & -(\alpha-2\gamma) & -\frac{\gamma}{2} & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & -\frac{\gamma}{2} & -(\alpha-2\gamma) & 1+2\alpha-3\gamma+\omega_j & -(\alpha-2\gamma) & -\frac{\gamma}{2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\frac{\gamma}{2} & 0 & 0 & -\frac{\gamma}{2} & -(\alpha-2\gamma) & 1+2\alpha-3\gamma+\omega_j & -(\alpha-2\gamma) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{M\times M}$$

$$Z = \begin{pmatrix} \Delta t f_{0,j} \\ \Delta t f_{1,j} \\ \vdots \\ \Delta t f_{M-3,j} \\ \Delta t f_{M-2,j} \\ 0 \end{pmatrix}$$

where $v^j = (u_{0,j}, u_{1,j}, \dots, u_{M-1,j})$.

3.1 The stability analysis for the proposed scheme

In this subsection, we apply the Von Neumann stability analysis for the direct problem [32], [33]. We take $f(x,t) = 0$, for simplicity, and assuming the local constant $p_j = \hat{g}$ for known level in Eq. (12) where $\hat{g} = \max_{t=[0,T]}|p(t)|, \hat{a} = \max_{t=[0,T]}|a(t)|$, then the difference equation becomes:

$$\frac{\gamma}{2}u_{i-2,j+1} - (\alpha + 2\gamma)u_{i-1,j+1} + (1 + 2\alpha + 3\gamma)u_{i,j+1} - (\alpha + 2\gamma)u_{i+1,j+1} + \frac{\gamma}{2}u_{i+2,j+1}$$
$$= -\frac{\gamma}{2}u_{i-2,j} - (\alpha - 2\gamma)u_{i-1,j} + (1 + 2\alpha - 3\gamma + \omega)u_{i,j} - (\alpha - 2\gamma)u_{i+1,j}$$
$$-\frac{\gamma}{2}u_{i+2,j}, \quad (13)$$

where $\alpha = \frac{b}{(\Delta x)^2}$, $\gamma = \frac{\hat{a}\Delta t}{(\Delta x)^4}$, $\omega = \Delta t\hat{g}$,

The decomposition method of the numerical solution into the Fourier sum is applied as follows:

$$u_{i,j} = S^j e^{wi\theta}, \quad (14)$$

where S is the amplification factor, the phase angle $\theta = \emptyset\Delta x$, where $\emptyset = \frac{2\pi}{N}$ and $w = \sqrt{-1}$ and Δx is the space length. If |S|< 1, then we said S to be satisfying the von Neumann condition. To find S, substitute the above data into Eq. (13) as follows:

$$\frac{\gamma}{2} S^{j+1}e^{w\theta(i-2)} - (\alpha + 2\gamma) S^{j+1}e^{w\theta(i-1)} + (1 + 2\alpha + 3\gamma) S^{j+1}e^{w\theta i}$$
$$- (\alpha + 2\gamma) S^{j+1}e^{w\theta(i+1)} + \frac{\gamma}{2} S^{j+1}e^{w\theta(i+2)}$$
$$= -\frac{\gamma}{2} S^j e^{w\theta(i-2)} - (\alpha - 2\gamma) S^j e^{w\theta(i-1)} + (1 + 2\alpha - 3\gamma + g) S^j e^{w\theta i}$$
$$- (\alpha - 2\gamma) S^j e^{w\theta(i+1)} - \frac{\gamma}{2} S^j e^{w\theta(i+2)}$$

simplifying the above equation, we get:

$$\big(\gamma\cos 2\theta - 2(\alpha + 2\gamma)\cos\theta + (1 + 2\alpha + 3\gamma)\big)S$$
$$= -\gamma\cos 2\theta - 2(\alpha - 2\gamma)\cos\theta + (1 + 2\alpha - 3\gamma + g) \quad (15)$$

Eq. (15) can be written as follows:

$S = \frac{-\gamma\cos 2\theta - 2(\alpha-2\gamma)\cos\theta + (1+2\alpha-3\gamma+g)}{\gamma\cos 2\theta - 2(\alpha+2\gamma)\cos\theta + (1+2\alpha+3\gamma)}$.

Now, taking the absolute value, then

$$|S| = \left|\frac{-\gamma\cos 2\theta - 2(\alpha - 2\gamma)\cos\theta + (1 + 2\alpha - 3\gamma + g)}{\gamma\cos 2\theta - 2(\alpha + 2\gamma)\cos\theta + (1 + 2\alpha + 3\gamma)}\right|$$

$$|\gamma\cos 2\theta - 2(\alpha + 2\gamma)\cos\theta + (1 + 2\alpha + 3\gamma)|$$
$$\leq \gamma|\cos 2\theta| + 2|\alpha + 2\gamma||\cos\theta| + |1 + 2\alpha + 3\gamma|$$

$$\leq \gamma + 2|\alpha + 2\gamma| + |1 + 2\alpha + 3\gamma|, \quad (16)$$

since $M, N > 0, \alpha = \frac{b}{(\Delta x)^2} = bM^2, \gamma = \frac{a\Delta t}{(\Delta x)^4} = \frac{aM^4}{N}$, substituting in Eq. (16), the right-hand side will become

$$\leq \frac{aM^4}{N} + 2\left|bM^2 + 2\frac{aM^4}{N}\right| + \left|1 + 2bM^2 + 3\frac{aM^4}{N}\right|$$
$$\leq \frac{aM^4}{N} + 2bM^2 + 4\frac{aM^4}{N} + 1 + 2bM^2 + 3\frac{aM^4}{N} = 1 + 4bM^2 + 7\frac{aM^4}{N} > 1.$$

Since b> 0, thus we get |S|< 1, then the method is unconditionally stable.

3.2 Example for the direct problem

We consider the direct problem (1)-(4) with T=1, $a = b = 0.0001$ and the following input data:

$$u(x,0) = \frac{\cos(2\pi x)}{e^1}, \qquad x \in [0,1]$$

$$p(t) = \cos(2\,\pi\,t)\,, \qquad t \in [0,T]$$

$$f(x,t) = e^{-t}(-0.311996 - 0.367879\cos(2\pi\,t))\cos(2\pi x), \quad (x,t) \in Q_T$$

the analytic solution

$$u(x,t) = e^{-1-t}\cos(2\pi x), \quad (x,t) \in Q_T$$

and overdetermination condition

$$h\,(t) = e^{-1-t} + 2.32544 * 10^{-8}, \qquad t \in [0,T].$$

This solution can be verified by the direct substitution into governing equation. The numerical and analytical results for the temperature distribution $u(x,t)$ at coarse mesh size $M = N = 40$, is depicted in Figure 1 and a very good accuracy is obtained as illustrated in the absolute error graph which is about 10^{-3} magnitude, see the right plot. Figure 2 displays the computational required data in comparison with the analytical one for $h(t)$ for $\delta = 0$, and $\lambda = -0.0000001$ and excellent agreement is also obtained.

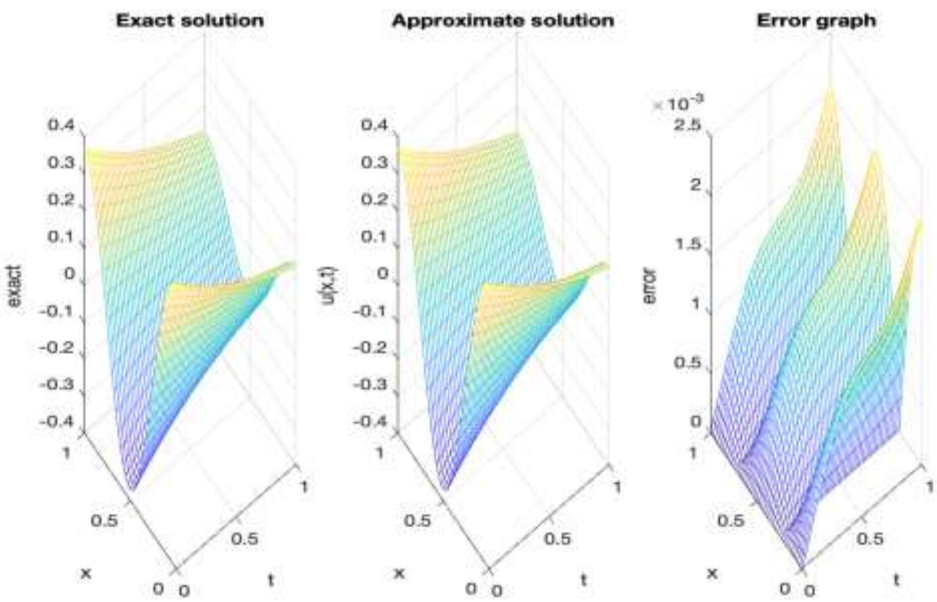


Figure 1: Analytical and computational temperature distributions for $u(x,t)$ and the absolute error of Example 1

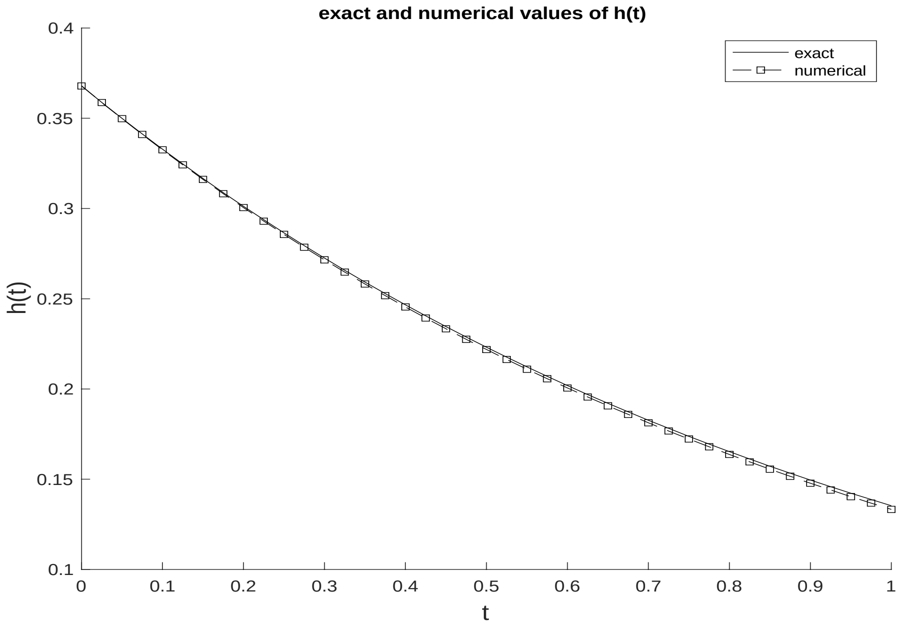


Figure 2: The analytical and computational curve for $h(t)$ with $\delta = 0$ for the forward problem of Example 1.

4. The Computational approach for the inverse problem

Our goal in this section is devoted to solve the inverse problem. To find the stable reconstructions for unknown coefficient $p(t)$, in addition to the heat distribution $u(x,t)$ that satisfies Eqs. (1)- (5). This problem is numerically solved by minimizing the gap between extra measurement data (5) and computed solutions. To gain suitable results, we apply Tikhonov's regularization method due to ill-posedness of the problem. The cost functional can be constructed from (5) for more details, see [34]–[38];

$$K(p) = \left\| u(0,t) - \int_0^1 \lambda(t) u(1,t) dt - h(t) \right\|^2 + \beta \|p(t)\|^2, \tag{17}$$

and the approximate formula is given by

$$K\left(\underline{p}\right) = \sum_{j=1}^{N} \left(u(0,t_j) - \int_0^1 \lambda(t_j) u(1,t_j) dt - h(t_j) \right)^2 + \beta \sum_{j=1}^{N} p_j^2, \tag{18}$$

where $\beta \geq 0$ is the regularization parameter, and the norm is the usual norm over [0,T]:

The objective function (17) is minimized by subroutine *lsqnonlin* from MATLAB optimization toolbox. This routine tries to solve the nonlinear least squares curve fitting problem that starts from the initial guess. The upper and lower bounds on the variable p are specified as $10^{-2} \leq p \leq 10^2$. Also, in this routine, it is not required that the gradient which is supplied by the user is computed inside the routine via some FDM formulas.

The following parameters are essential to start the optimization processes of (18), the minimization will terminate when the following prescribed parameters are achieved:

- Allowed number of iterations = $10^{2} *$ (No. of variables).
- Specified solution and objective function Tolerance = 10^{-20}.

The inverse problem is solved with respect to noisy/ exact measurement data in (5). The additive noise is presented in :

$$h^{\epsilon}(t_j) = h(t_j) + \epsilon_j, \quad j = 1,2,\ldots,N, \tag{19}$$

where ϵ is a normal Gaussian random vector and standard deviation μ is:

$$\mu = q \times \max_{t\in[0,T]} |h(t)|, \tag{20}$$

where q represents the percentage of noise. Here we use the *normrnd* built-in function to generate the random variables $\epsilon = (\epsilon_j)$ $j = 1,2,\ldots,N$ as follows:

$$\epsilon = normrnd(0, \mu, N).$$

4.1 Results and discussion

We introduce a couple of test examples for the inverse problem. To explain and validate the stability and accuracy of the computational procedure which is based on the finite difference method combined with the minimization of functional (18).

To assess the reconstruction accuracy of the potential term, we use the root mean squares error *rmse* which is given by the next expression:

$$rmse(p) = \sqrt{\frac{1}{N}\sum_{j=1}^{N}\left(p_{j-numerical} - p_{exact}(t_j)\right)^2}, \tag{21}$$

Example 1:

Consider the inverse problem (1)-(5), with input data as in the example of the direct problem. The initial guess was taken $p(0) = 1$. It is easy to check that the input data verifies the conditions of Theorem 1- 3, hence the inverse pseudoparabolic problem (1)-(5) has a unique solution. The noise free numerical solution retrieval is discussed when, $q = 0$ i.e., no noise included in (20). The associated results for this case are plotted in Figure 3 and clear agreement is obtained. The objective function (18) is represented in Figure 5, and a speed declining convergence is seen for achieving a shorter stationary value of order O(10^{-8}) in 10 iterations only.

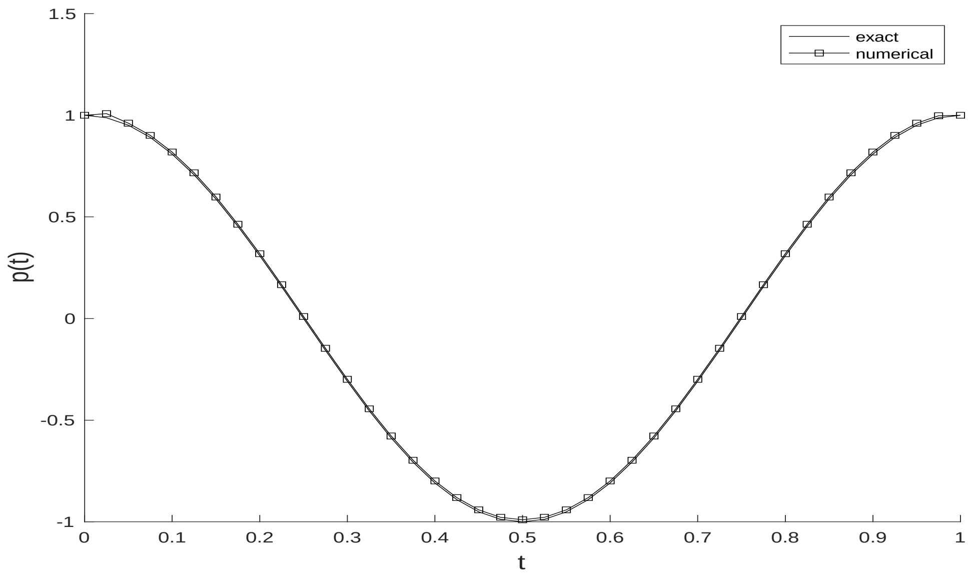


Figure 3: Numerical and exact solution for potential term $p(t)$ when $M = N = 40$.

Now, to examine the stability of the numerical solution with respect to noise in the data (5), as it is defined in (19). We add $q \in \{1, 5\}\%$ noise, Figure 4 shows the exact $p(t)$ in different cases of noise. Figure 5 presents the objective function (18) of $p(t)$ with various cases of noise.

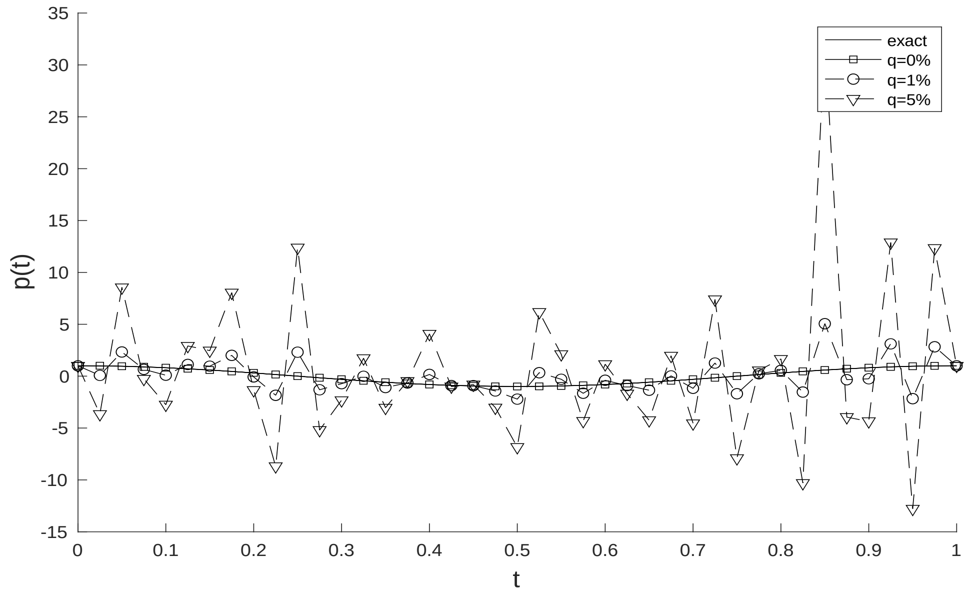


Figure 4: Numerical reconstructions and exact solution for $p(t)$, with noise level $q = \{0, 1\%, 5\%\}$, without regularization applied for Example 1.

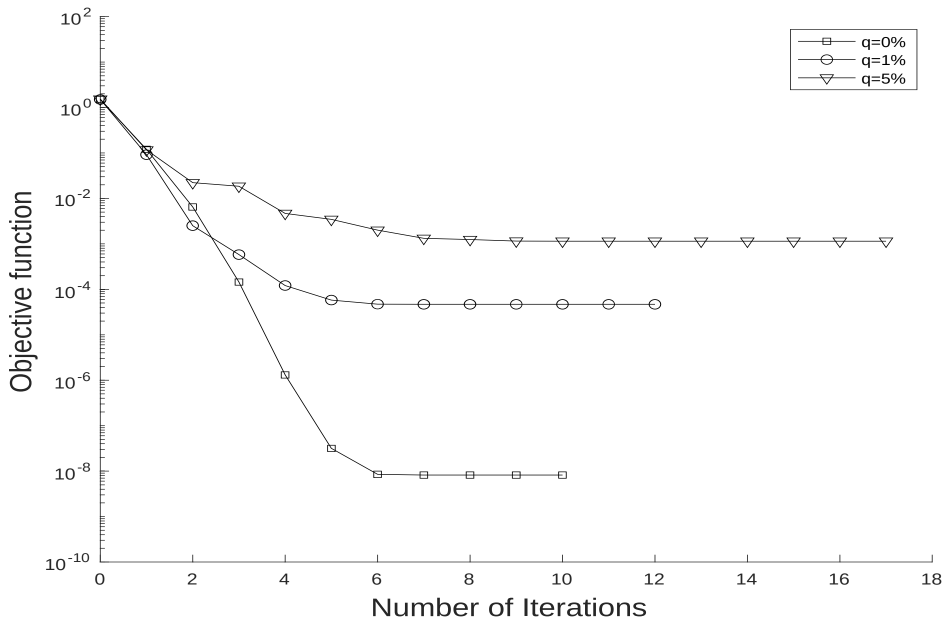


Figure 5: The unregularized objective function (18), with q $= \{0, 1\%, 5\%\}$noise data.

In this stage, we employ the Tikhonov regularization method to obtain stable reconstruction for $p(t)$. Where regularization parameters $\beta = \{10^{-4}, 10^{-3}\}$ was chosen for both noise data $q = 1\%, 5\%$ by trial and error. Figures 7 and 9 show the objective function (18) decreases rapidly in a relatively small number of iterations. The Tikhonov approach with selected parameters gives a reasonable and stable approximate solution of potential term $p(t)$ for both cases (see Figures 6 and 8), one can observe that these choices of β give the stable and accurate approximate solution for $p(t)$ in no more than 385 s. Table 1 presents the associated values including $rmse(p)$ which show that a reasonable range of values can be seen, with the best retrieval occurring at the smallest $rmse(p)$. The numerical and exact temperatures $u(x,t)$, with $q = 1\%$ noise, $\beta = 10^{-4}$, $q = 5\%$, noise, $\beta = 10^{-3}$, as well as the absolute error between them are plotted in Figure 10.

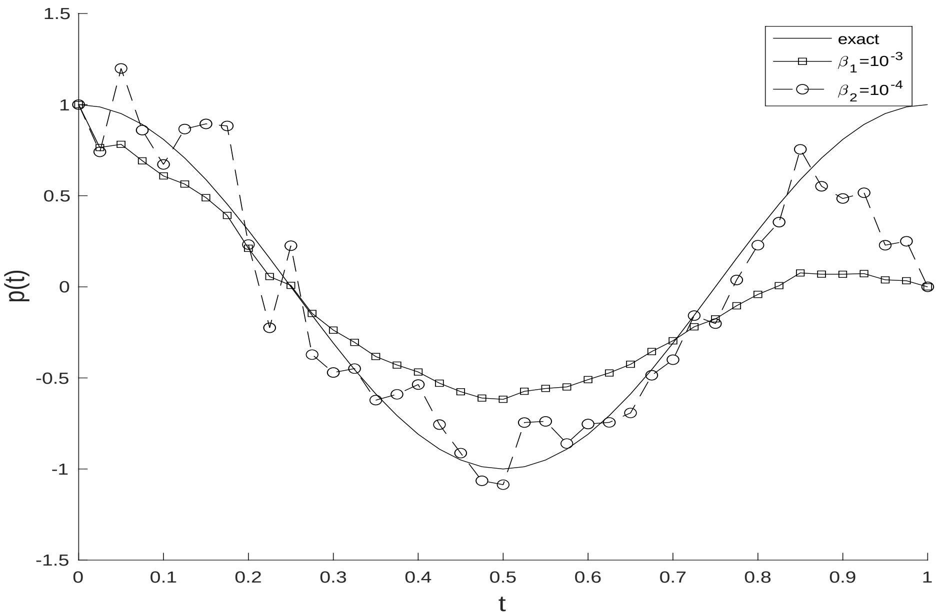


Figure 6: Numerical reconstructions and exact solution for $p(t)$, with regularization parameter $\beta = \{10^{-4}, 10^{-3}\}$ and $q = 1\%$ noise.

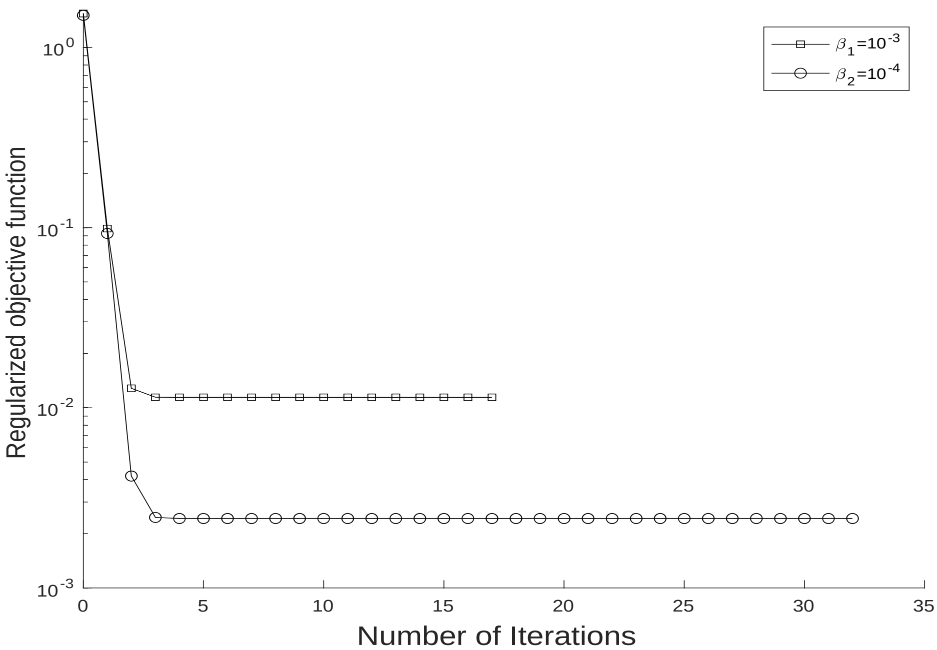


Figure 7: The regularized objective function (18), with regularization parameter $\beta = \{10^{-4}, 10^{-3}\}$ and $q = 1\%$ noise.

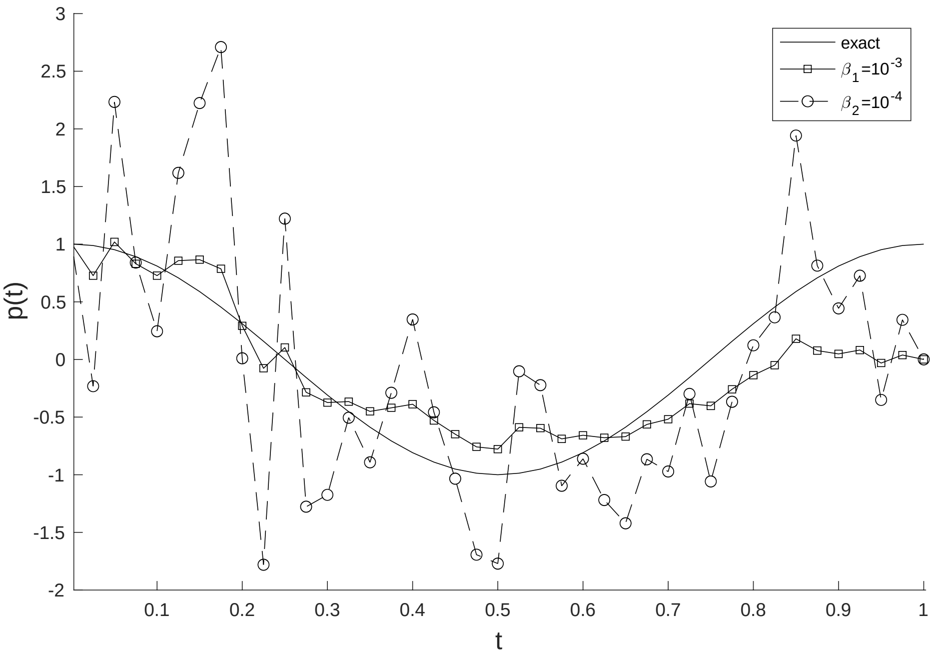


Figure 8: Numerical reconstructions and exact solution for $p(t)$, with regularization parameter $\beta = \{10^{-4}, 10^{-3}\}$ and $q = 5\%$ noise.

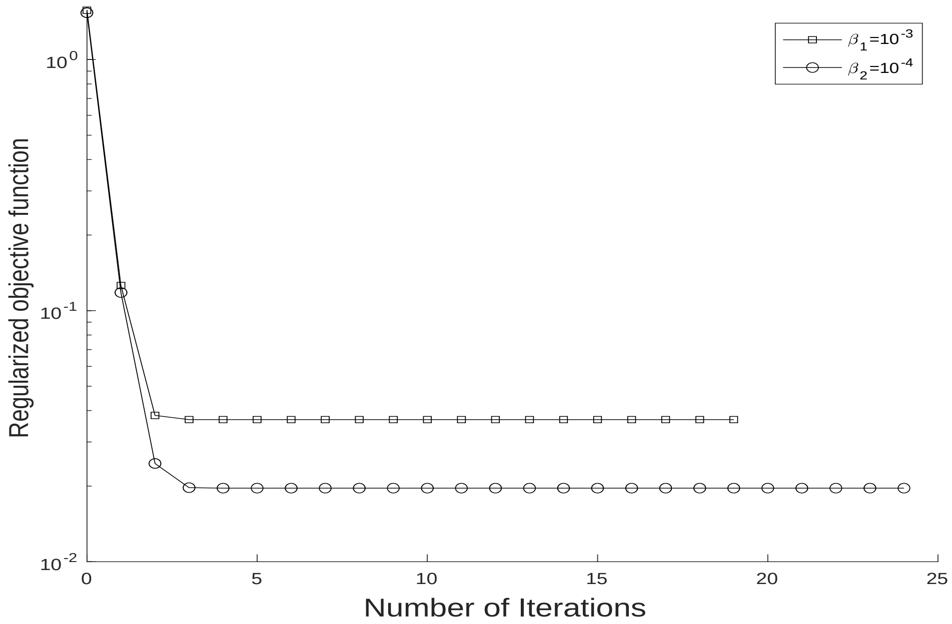
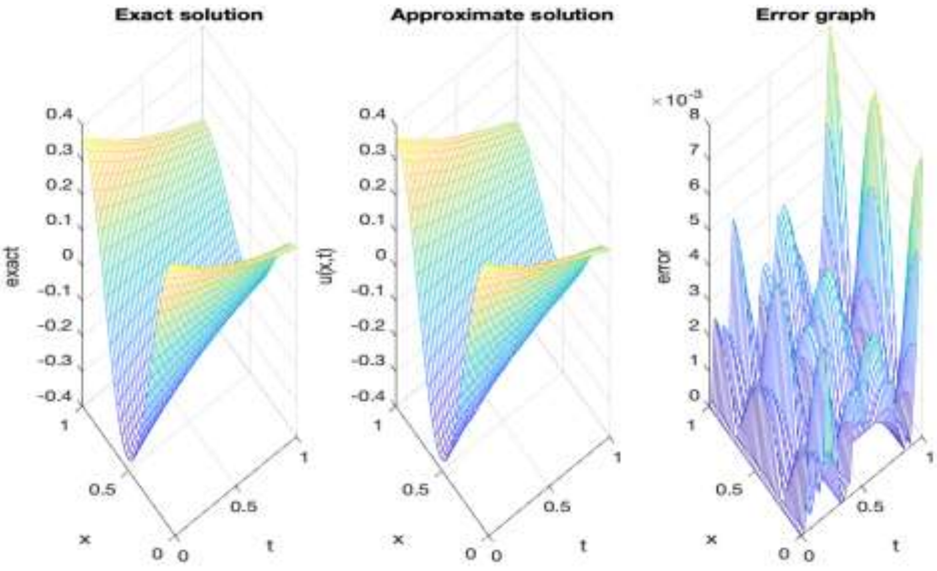
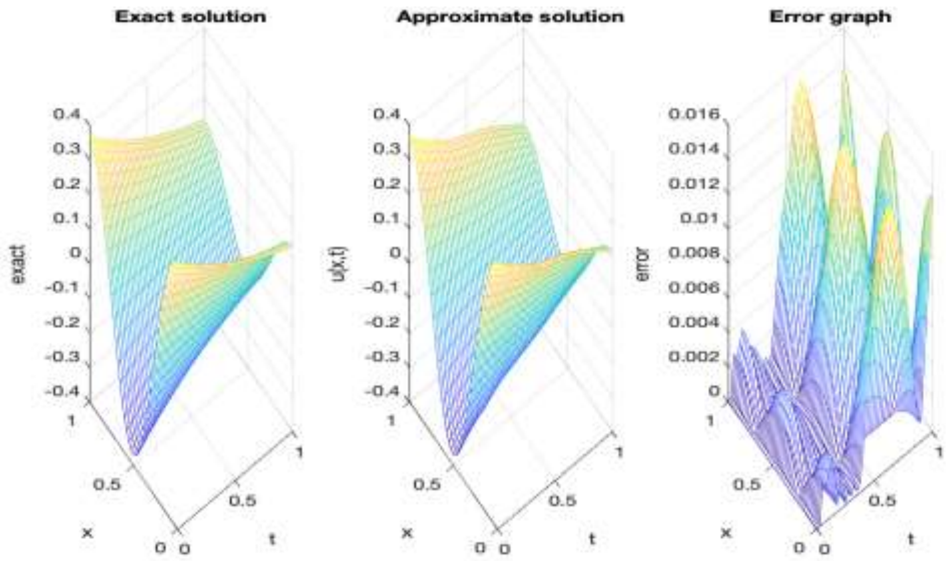


Figure 9: The regularized objective function (18), with regularization parameter $\beta = \{10^{-4}, 10^{-3}\}$ and $q = 5\%$ noise.



(a)



(b)

Figure 10: Numerical and exact temperature $u(x,t)$ with (a) $q = 1\%$ and $\beta = 10^{-4}$, (b) $q = 5\%$ noise and $\beta = 10^{-3}$.

Table 1: Number of iterations that required to achieve the minimization, value of the minimised functional (18) at final iteration, $rmse$ value and consumed computational time in seconds, for Example 1 with $q \in \{1\%, 5\%\}$ noise.

$q = 1\%$	$\beta = 10^{-3}$	$\beta = 10^{-4}$	$\beta = 10^{-5}$
No. of iterations	18	33	39
Objective function (18) at final iteration	0.0114	0.0024	5.5129E-04
$rmse(p)$	0.4097	0.2919	0.6385
computational time (seconds)	183	337	385
q = 5%	$\beta = 10^{-3}$	$\beta = 10^{-4}$	$\beta = 10^{-5}$
No. of iterations	20	25	36
Objective function (18) at final iteration	0.0368	0.0196	0.0091
$rmse(p)$	0.4156	0.8913	3.0201
computational time (seconds)	202	279	374

Example 2:

Consider the inverse problem (1)-(5) with T=1 with $a = b = 0.001$ and $\lambda = -10^{-6}$ and the following input data:

$$u(x, 0) = \cos(2\pi x), \quad x \in [0,1]$$

$$p(t) = 0.5 + |t - 0.5|, \quad t \in [0, T]$$

$$f(x, t) = e^{-2t}(-1.02041 - |t - 0.5|)\cos(2\pi x), \quad (x, t) \in Q_T$$

the analytic solution is given by

$$u(x, t) = e^{-2t}\cos(2\pi x), \quad (x, t) \in Q_T$$

and overdetermination condition

$$h\,(t) = e^{-2t} + 4.32332 * 10^{-8}, \quad t \in [0, T]$$

Now, in the beginning, we attempt to retrieve the unknown potential term $p(t)$ and the temperature $u(x, t)$ for exact input data, i.e. $q = 0$, as well as, for $q \in \{0.1\%, 1\%\}$ noisy data without regularization. The objective functional (18) is depicted in Figure 13 and it observed the speed convergence in early iterations i.e. from iteration 0 to 8, then steadily for the rest of iterations and the minimization processes terminated when allowed tolerance for solution or objective function is reached which is set to 10^{-8}. The related numerical solution of $p(t)$ obtained inversions is presented in Figure 11 with no noise. Whilst, the noise amount increases from 0.1% to 1% as illustrated in Table 2 which indicates the impact of noise inclusion.

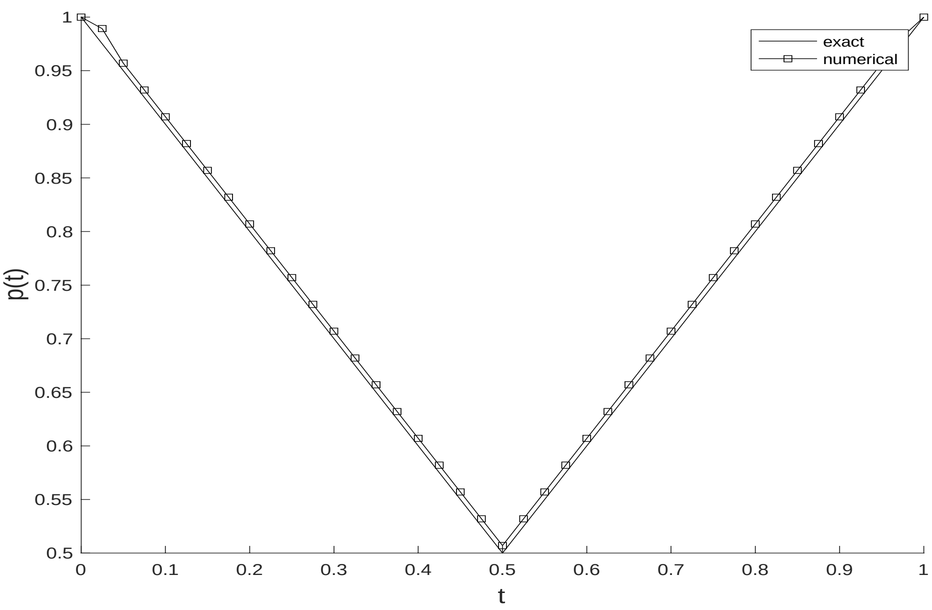


Figure 11: Numerical and exact solution for potential term $p(t)$ when $M = N = 40$.

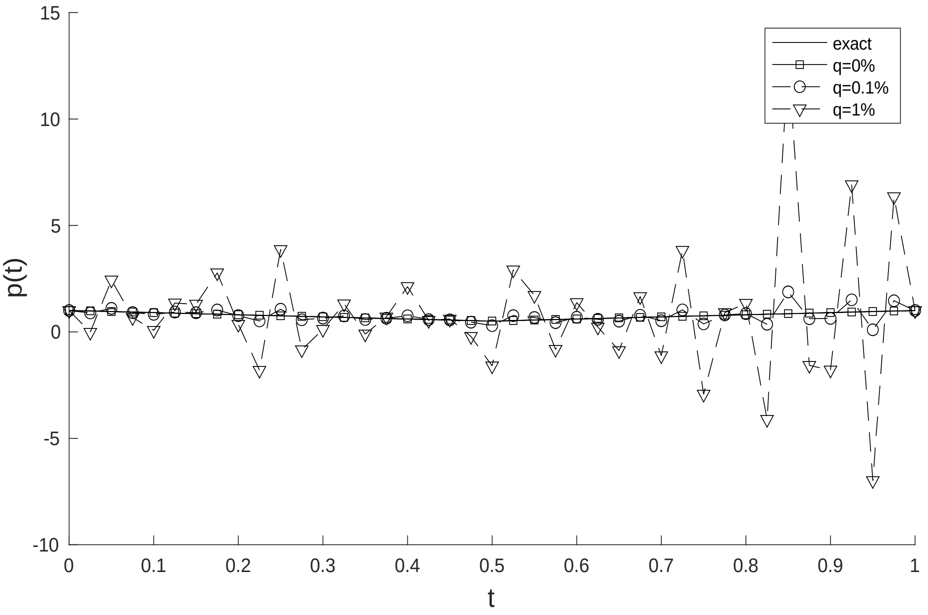


Figure 12: Numerical reconstructions in comparison with exact solution for $p(t)$, with noise level $q = \{0, 0.1\%, 1\%\}$, without regularization applied Example 2.

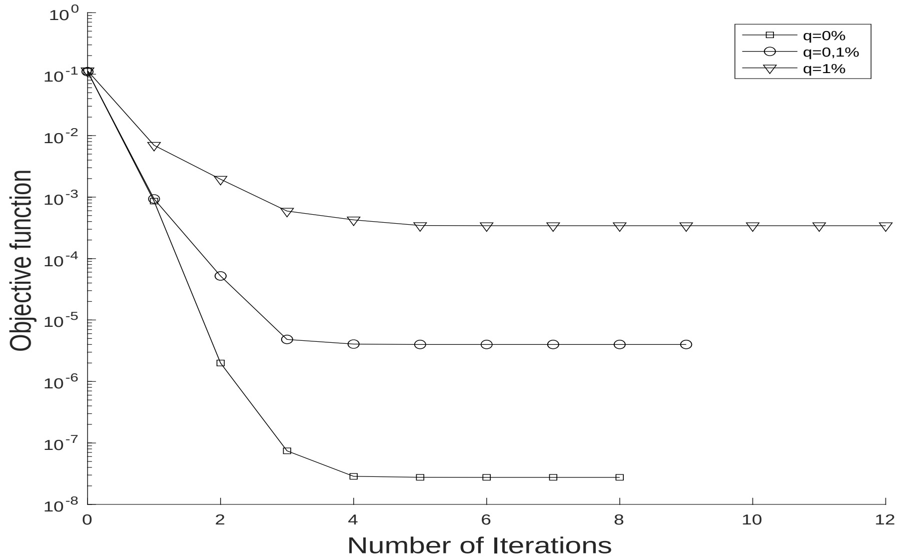


Figure 13: The unregularized objective function (18), with $\mathrm{q} = 0, 0.1\%, 1\%$ noise data, Example 2.

To restore stability, some regularization should be applied. To replicate real input data, noise of $q \in \{0.1, 1\}\%$ is included with regularization $\beta = \{10^{-5}, 10^{-4}\}$ for case $q = 0.1\%$ and $\beta = \{10^{-4}, 10^{-3}\}$ for case $q = 1\%$. Figures 15 and 17 reveal the objective function minimization (18). For both cases, no more than 26 iterations are taken to achieve a minimum value after speed convergence in early iterations followed by steady and slow convergence to reach a stationary value. Figures 14 and 16 show the reconstruct potential unknown coefficient. Before instabilities, it begins to show up when the noise levels increase from 0.1% to 1%. A very excellent agreement which established when there is $\beta = 10^{-3}$, and $\beta = 10^{-4}$, respectively. Moreover, Table 2 informs that the associated values show a reasonable range of values that can be seen with the best retrieval occurring at the smallest $rmse(p)$.

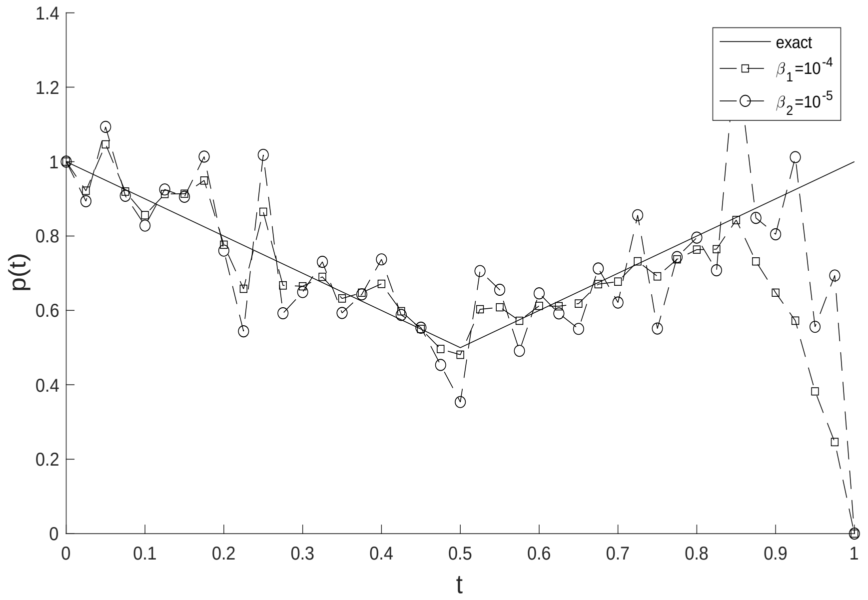


Figure 14: Numerical reconstructions and exact solution for $p(t)$, with regularization parameter $\beta = \{10^{-5}, 10^{-4}\}$ and q $= 0.1\%$ noise.

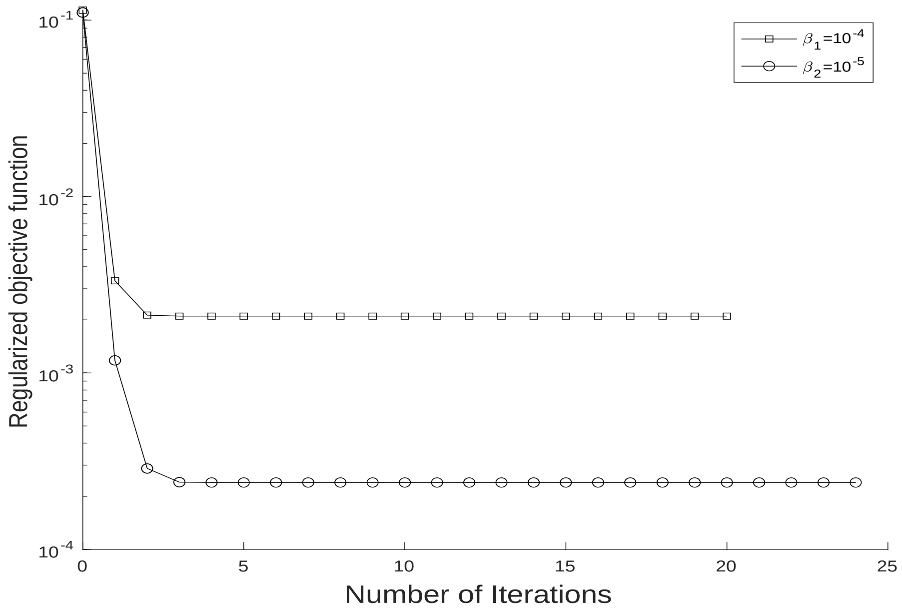


Figure 15: The regularized objective function (18), with regularization parameter $\beta = \{10^{-5}, 10^{-4}\}$ and $q = 0.1\%$ noise.

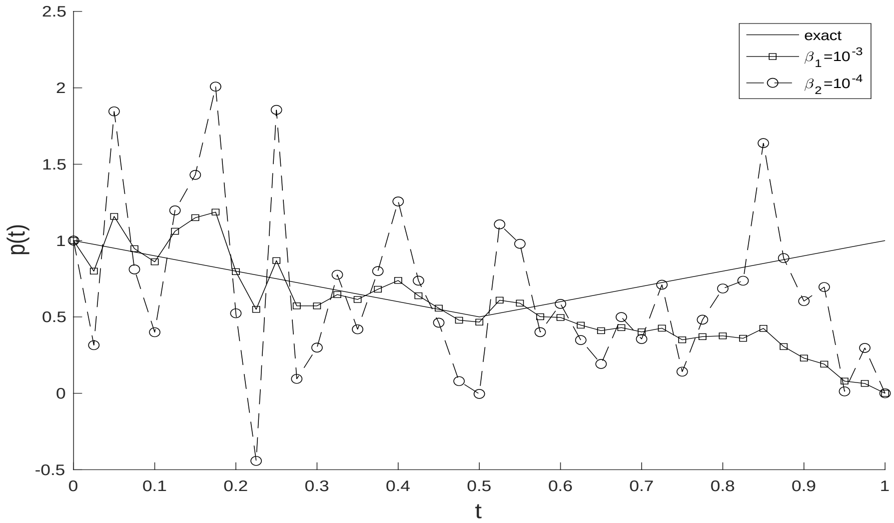


Figure 16: Numerical reconstructions and exact solution for $p(t)$, with regularization parameter $\beta = \{10^{-4}, 10^{-3}\}$ and $q = 1\%$ noise.

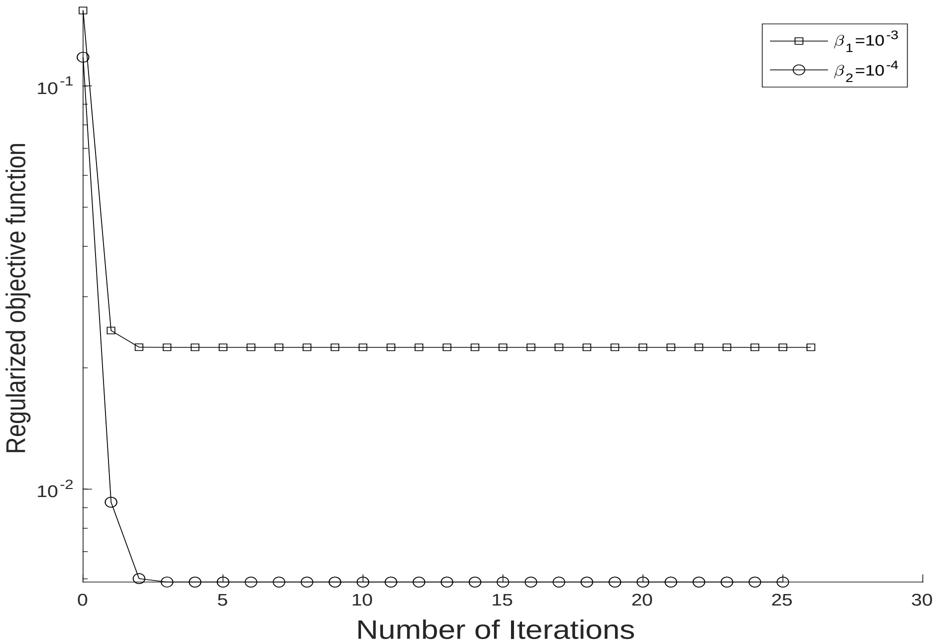
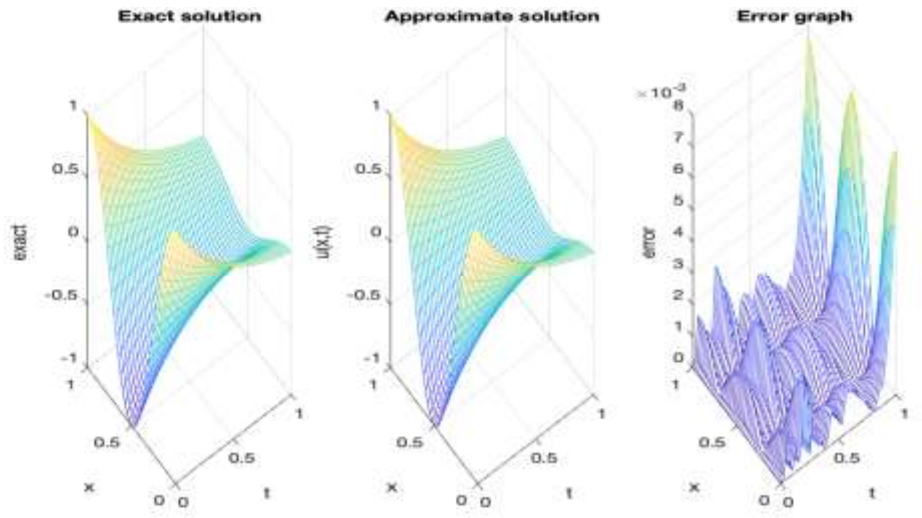
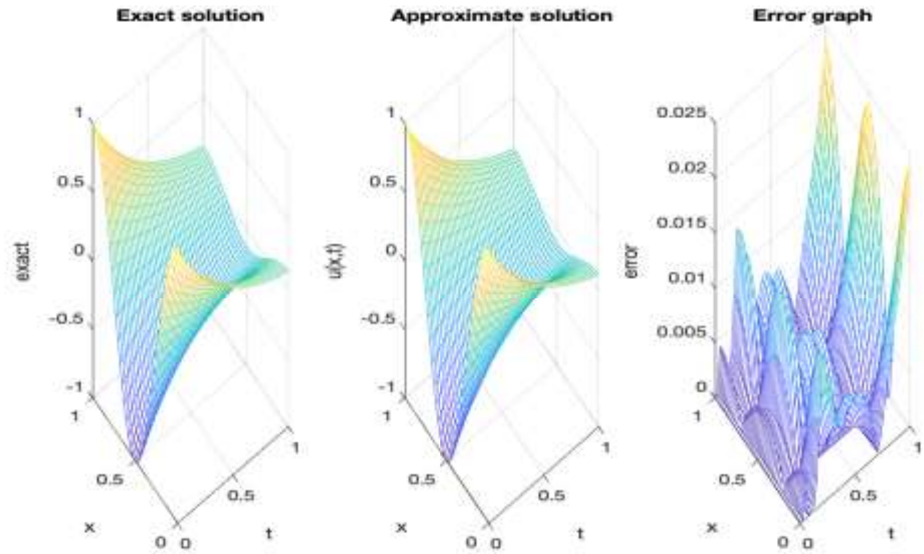


Figure 17: The regularized objective function (18), with regularization parameter $\beta = \{10^{-4}, 10^{-3}\}$ and $q = 1\%$ noise.



(a)



(b)

Figure 18: Numerical and exact temperature $u(x,t)$ with (a) $q = 0.1\%$ and $\beta = 10^{-4}$, (b) $q = 1\%$ noise and $\beta = 10^{-3}$.

Table 2: Number of iterations required to achieve the minimization, value of the minimised functional (18) at final iteration, $rmse$ value and consumed computational time, for Example 2 with $q \in \{0.1\%, 1\%\}$ noise.

$q = 0.1\%$	$\beta = 10^{-3}$	$\beta = 10^{-4}$	$\beta = 10^{-5}$
No. of iterations	31	21	25
Objective function (18) at final iteration	0.0175	0.0021	2.3940E-04
$rmse(p)$	0.3420	0.2323	0.2162
computational time (seconds)	327	224	259
$q = 1\%$	$\beta = 10^{-3}$	$\beta = 10^{-4}$	$\beta = 10^{-5}$
No. of iterations	27	26	26
Objective function (18) at final iteration	0.0225	0.0059	0.0020
$rmse(p)$	0.3747	0.5535	1.3412
computational time(seconds)	286	271	269

5. Conclusions

The fourth order pseudoparabolic inverse problem to identify numerically the potential coefficient has been investigated under periodic and nonlocal boundary conditions and overdetermination data. The finite difference scheme in cooperation with the trapezoidal rule has been used for direct problems. The Von Neumann technique was employed to study the stability of the proposed numerical direct method. Therefore, to reconstruct the stability, Tikhonov's regularization was employed. Stable results are obtained under various noise levels.

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