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Admissible Classes of Seven-Parameter Mittag-Leffler Operator with Third-Order Differential Subordination Properties

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Abstract
The main purpose of this paper, is to characterize new admissible classes of linear operator in terms of seven-parameter Mittag-Leffler function, and discuss sufficient conditions in order to achieve certain third-order differential subordination and superordination results. In addition, some linked sandwich theorems involving these classes had been obtained.

Keywords: Admissible function, Differential subordination, Differential superordination, Mittag-Leffler Function, Sandwich lemmas.

الفئات المقبولة لعامل ميتاج–لفلر ذي المعلمات السبعة مع خصائص التبعية التفاضلية من الدرجة الثالثة

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الخلاصة
الغرض الرئيسي من هذا البحث هو توصيف فئات جديدة من العامل الخطي ذا الصلة بدالة– ميتاج–لفلر ذات المعلمات السبعة، ومناقشة الشروط الكافية من أجل تحقيق بعض نتائج التبعية التفاضلية من الدرجة الثالثة ومتعلقاتها. بالإضافة إلى ذلك، تم الحصول على بعض نظريات الساندويتش المرتبطة التي تنطوي على تلك الفئة.

1. Introduction

The history of differential subordination theory was related to differential inequalities in real variable concept that involve real-valued technique, due to the requirements to modify these differential inequalities to its complex analogue. Thence, the notion of differential subordination for holomorphic functions raised in the monographs of Lindelöf [1], Littlewood [2], and Rogosinski [3]. Since then, hundreds of articles interested to study that concept, especially second-order differential subordination with their various applications, see [4-10].

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In 1982, Goldstein et al. [11] assumed a third-order differential subordination involving geometric aspects and some Miller problems. After that, Punnusamy and Juneja [12] deals with third-order differential subordination with new specific properties and simple classes. Thereafter, Antonio and Miller [13] could address a general class linked to third-order differential subordination that had significant applications for univalent functions and operators.

Let H be the class of holomorphic functions in the open unit disk U, also let
$\mathbb{H}[\alpha, n] = \{f \in \mathbb{H}: f(z) = \alpha + \alpha_n z^n + \alpha_{n+1} z^{n+1} + \cdots\}$; $\alpha, \alpha_{n+i} \in \mathbb{C}\,(i = 0,1, \dots ;\, n \in \mathbb{N})$. Additionally, the class A denotes the subclass of $\mathbb{H}$ consisting of the normalized functions by the form, [14]:

$$f(z) = z + \sum_{n=2}^{\infty} \alpha_n z^n\,,\ \alpha_i \in \mathbb{C}\,(i = 2, \dots, n). \tag{1.1}$$

For two functions f_1 and f_2 belong to $\mathbb{H}$, we say that the function f_1 subordinate to f_2 , written $f_1 \prec f_2$ if there exists a schwarz function w, such that $f_1(z) = f_2\big(w(z)\big)$. If f_2 univalent, then $f_1 \prec f_2$ if and only if $f_1(0) = f_2(0)$ and $f_1(\mathbb{U}) \subset f_2(\mathbb{U})$. Note that, if f_1 subordinate to f_2, then f_2 superordinate to f_1, [15].

Let $\psi: \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ and h be univalent in $\mathbb{U}$ and ρ be analytic in $\mathbb{U}$ such that ρ satisfies the third-order differential subordination

$$\psi\left(\rho(z), z\acute{\rho}(z), z^2\acute{\acute{\rho}}(z), z^3\acute{\acute{\acute{\rho}}}(z); z\right) \prec h(z) \tag{1.2}$$

then ρ is called a solution of (1.2). The univalent function μ is called dominant of the solutions (1.2) if $\rho \prec \mu$ for all ρ satisfying (1.2). A dominant $\tilde{\mu}$ that satisfies $\tilde{\mu} \prec \mu$ for all dominants μ of (1.2) is said to be the best dominant of (1.2), [13].

Analogously, let $\psi: \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ and h be analytic in $\mathbb{U}$. If ρ and $\psi\left(\rho(z), z\acute{\rho}(z), z^2\acute{\acute{\rho}}(z), z^3\acute{\acute{\acute{\rho}}}(z); z\right)$ are univalent in $\mathbb{U}$ such that ρ satisfies the third-order differential superordination

$$h(z) \prec \psi\left(\rho(z), z\acute{\rho}(z), z^2\acute{\acute{\rho}}(z), z^3\acute{\acute{\acute{\rho}}}(z); z\right) \tag{1.3}$$

then ρ is called a solution of (1.3). The univalent function μ is called subordinant of the solutions of the differential subordination if $\mu \prec \rho$ for all ρ satisfying (1.3). A subordinant $\tilde{\mu}$ that satisfies $\mu \prec \tilde{\mu}$ for all subordinants μ of (1.3) is said to be the best subordinant of (1.3), [13].

In order to apply the idea of this work, we recall the seven-parameter Mittag-Leffler linear operator $M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z): A \to A$ which introduced by Rasheed and Majeed [16], as

$$\begin{aligned} M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z) &= T^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(z) * f(z) \\ &= z + \sum_{n=0}^{\infty} \frac{(a)_n}{a} \frac{(b)_n}{b} \frac{c}{(c)_n} \frac{\Gamma(\tau_1 + \lambda_1)\Gamma(\tau_2 + \lambda_2)}{\Gamma(\tau_1 n + \lambda_1)\Gamma(\tau_2 n + \lambda_2) n!} \alpha_n z^n. \end{aligned} \tag{1.4}$$

where $f \in A$ given in (1.1) and $T^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(z)$ is the normalization of the seven-parameter Mittag-Leffler function $E^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(z)$ proposed by Rasheed and Majeed

$$T^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(z) = \frac{c\Gamma(\tau_1+\lambda_1)\Gamma(\tau_2+\lambda_2)}{ab}\left(E^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(z) - \frac{1}{\Gamma(\lambda_1)\Gamma(\lambda_2)}\right). \quad (1.5)$$

Moreover, some computations of certain results were essentially connected to those relations:

$$z\left(M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z)\right)' = (a+1)M^{a+1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z) - aM^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z), \quad (1.6)$$

and

$$\tau_1 z\left(M^{a,b,c}_{\tau_1,\lambda_{1+1},\tau_2,\lambda_2}f(z)\right)' = (\tau_1+\lambda_1)M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z) - \lambda_1 M^{a,b,c}_{\tau_1,\lambda_1+1,\tau_2,\lambda_2}f(z). \quad (1.7)$$

The central method of this paper, is to assume a sufficient conditions to suggest admissible classes considering the linear operator $M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z)$ that mentioned above, then investigate certain third-order differential subordination properties associated with these new classes, and its dual superordination properties. Further, some sandwich type theorems had been confirmed due to the collection of subordination and superordination results.

2. Preliminaries

In this portion, we recall the fundamental definitions and results that will be used throughout this paper:

Definition 2.1: [14] Let Λ be the set of all functions $\mu(z)$ that are holomorphic and univalent on $\overline{U}/E(\mu)$, where $\overline{\mathbb{U}} = \mathbb{U} \cup \partial\mathbb{U}$, with

$$E(\mu) = \{s \in \partial U: \lim_{z\to s}\mu(z)\} = \infty,$$

such that $\acute{\mu}(z) \neq 0$ for $s \in \partial\mathbb{U}/E(\mu)$. Note that, Λ_β denote the subclass of Λ where $\mu(0) = \beta$ for $\beta \in \mathbb{C}$.

Definition 2.2: [13] Let $\Omega \subseteq \mathbb{C}, \mu \in$, and n be a positive integer. The class of admissible functions $\Psi_n[\Omega,\mu]$ containing those functions $\psi: \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ that satisfy the following admissibility condition:

$$\psi(u,v,w,r;z) \notin \Omega$$

whenever

$$u = \mu(s), \qquad v = ks\acute{\mu}(s), \qquad Re\left(\frac{w}{v}+1\right) \geq kRe\left(\frac{s\acute{\acute{\mu}}(s)}{\acute{\mu}(s)}+1\right),$$

and

$$Re\frac{r}{v} \geq k^2 Re\left(\frac{s^2\acute{\acute{\mu}}(s)}{\acute{\mu}(s)}+1\right),$$

where $z \in \mathbb{U}, s \in \partial\mathbb{U}/E(\mu), k \geq n$ and $n \in \mathbb{N}/\{1\}$.

Definition 2.3: [17] Let $\Omega \subseteq \mathbb{C}$, $\mu \in \mathbb{H}[a,n]$ with $n \in \mathbb{N}$ and $\acute{\mu}(z) \neq 0$. The class of admissible functions $\acute{\Psi}[\Omega,\mu(z)]$ containing those functions $\psi: \mathbb{C}^4 \times \overline{\mathbb{U}} \to \mathbb{C}$ that satisfy the following admissibility condition:

$$\Psi(u,v,w,r;s) \in \Omega$$

whenever

$$u = \mu(s), \qquad v = \frac{z\acute{\mu}(z)}{\iota}, \qquad Re\left(\frac{w}{v}+1\right) \leq \frac{1}{\iota}Re\left(\frac{z\acute{\acute{\mu}}(z)}{\acute{\mu}(z)}+1\right),$$

and

$$Re\left(\frac{r}{v}+1\right) \leq \frac{1}{\iota^2} Re\left(\frac{z^2 \dddot{\mu}(z)}{\acute{\mu}(z)}\right),$$

where $z \in \mathbb{U}, s \in \partial\mathbb{U}, \iota \geq n$ and $n \in \mathbb{N}/\{1\}$.

Theorem 2.4: [13] Let $\rho \in \mathbb{H}[\alpha, n]$ with $n \geq 2$, and let $\mu \in \Lambda_\beta$, such that

$$Re\left(\frac{z^2 \ddot{\mu}(z)}{\acute{\mu}(z)}\right) \geq 0, \quad \left|\frac{z\acute{\rho}(z)}{\acute{\mu}(s)}\right| \leq k,$$

Where $z \in \mathbb{U}, s \in \partial\mathbb{U}/E(\mu)$ and $k \geq n \geq 2$.

If $\Omega \subseteq \mathbb{C}, \psi \in \Psi[\Omega, \mu]$, and $\psi\left(\rho(z), z\acute{\rho}(z), z^2\ddot{\rho}(z), z^3\dddot{\rho}(z); z\right) \in \Omega$, then $\rho(z) \prec \mu(z)$.

Theorem 2.5: [17] Let $\psi \in \acute{\Psi}_n[\Omega, \mu]$, $\rho \in \Lambda_\beta$ and $\mu \in \mathbb{H}[\alpha, n]$. Suppose that $\psi\left(\rho(z), z\acute{\rho}(z), z^2\ddot{\rho}(z), z^3\dddot{\rho}(z); z\right)$ is univalent in U, such that

$$Re\left(\frac{z^2 \ddot{\mu}(z)}{\acute{\mu}(z)}\right) \geq 0, \quad \left|\frac{z\acute{\rho}(z)}{\acute{\mu}(s)}\right| \leq \iota,$$

where $z \in \mathbb{U}, s \in \partial\mathbb{U}/E(\mu)$ and $\iota \geq n \geq 2$.

Then $\Omega \subset \left\{\psi\left(\rho(z), z\acute{\rho}(z), z^2\ddot{\rho}(z), z^3\dddot{\rho}(z); z\right) : z \in \mathbb{U}\right\}$ which implies $\mu(z) \prec \rho(z)$.

3. Third-order differential subordination results

This section, aim to construct new admissible classes in terms of the seven-parameter Mittag-Leffler linear operator $M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z)$ recalled in (1.4), then investigates certain third-order differential subordination significant properties.

Definition 3.1: Let $\Omega \subseteq \mathbb{C}$, $\mu \in$. The class of admissible functions $\Phi[\Omega, \mu]$ containing those functions $\phi: \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ that satisfy the following admissibility condition:

$$\phi(\gamma_1, \gamma_2, \gamma_3, \gamma_4; z) \notin \Omega$$

whenever

$$\gamma_1 = \mu(s), \qquad \gamma_2 = \frac{ks\acute{\mu}(s) + a\mu(s)}{(a+1)},$$

$$Re\left(\frac{(a+1)(a+2)\gamma_3 - (2a+1)(a+1)\gamma_2 + a^2\gamma_1}{(a+1)\gamma_2 - a\gamma_1}\right) \geq kRe\left(\frac{s\ddot{\mu}(s)}{\acute{\mu}(s)} + 1\right),$$

and

$$Re\left(\frac{(a+1)(a+2)[(a+3)\gamma_4 - 3(a+2)\gamma_3 + 3(a+1)\gamma_2 - a\gamma_1]}{(a+1)\gamma_2 - a\gamma_1}\right) \geq k^2 Re\left(\frac{s^2\dddot{\mu}(s)}{\acute{\mu}(s)} + 1,\right)$$

where $z \in \mathbb{U}, s \in \partial\mathbb{U}/E(\mu)$ and $k \in \mathbb{N}/\{1\}$.

Consider a member of the class $\Phi[\Omega, \mu]$ that concurring the linear operator $M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z)$, the next theorem establishes certain conditions to get a dominant for that operator.

Theorem 3.2: Let $\phi \in \Phi[\Omega, \mu]$ and $\mu \in \Lambda_1$. If $f \in A$ such that

$$Re\left(\frac{s\ddot{\mu}(s)}{\acute{\mu}(s)}\right) \geq 0, \quad \left|\frac{M^{a+1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z) - M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z)}{\acute{\mu}(z)}\right| \leq k\left|\frac{1}{a+1}\right|, \qquad (3.1)$$

and

$$\left\{\phi\left(M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z), M^{a+1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z), M^{a+2,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z), M^{a+3,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z); z\right) : z \in U\right\} \subset \Omega, \quad (3.2)$$

then

$$M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z) \prec \mu(z). \tag{3.4}$$

Proof. Suppose that

$$M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z) = \rho(z), \tag{3.5}$$

by virtue of relation (1.6), yields that

$$M^{a+1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z) = \frac{1}{a+1}[z\acute{\rho}(z) + a\rho(z)], \tag{3.6}$$

$$M^{a+2,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z) = \frac{1}{a+2}\left[\frac{1}{a+1} z^2\acute{\acute{\rho}}(z) + 2z\acute{\rho}(z) + a\rho(z)\right], \tag{3.7}$$

and

$$M^{a+3,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z) = \frac{1}{a+3}\left[\frac{1}{(a+1)(a+2)} z^3\acute{\acute{\acute{\rho}}}(z) + \frac{3}{a+1} z^2\acute{\acute{\rho}}(z) + 3z\acute{\rho}(z) + a\rho(z)\right]. \tag{3.8}$$

Now, we define

$$\gamma_1 = u, \qquad \gamma_2 = \frac{1}{a+1}(v + au), \qquad \gamma_3 = \frac{1}{a+2}\left[\frac{1}{(a+1)} w + 2v + au\right],$$

and

$$\gamma_4 = \frac{1}{(a+3)}\left[\frac{1}{(a+1)(a+2)} r + \frac{3}{a+1} w + 3v + au\right].$$

So, the transformation $\psi: \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ can be define as

$$\psi(u, v, w, r; z) = \phi(\gamma_1, \gamma_2, \gamma_3, \gamma_4; z), \tag{3.9}$$

from relations (3.5) to (3.8), we conclude

$$\psi\left(\rho(z), z\acute{\rho}(z), z^2\acute{\acute{\rho}}(z), z^3\acute{\acute{\acute{\rho}}}(z); z\right)$$
$$= \phi\left(M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z), M^{a+1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z), M^{a+2,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z), M^{a+3,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z); z\right), \tag{3.10}$$

by expression (3.9) we obtain

$$\psi\left(\rho(z), z\acute{\rho}(z), z^2\acute{\acute{\rho}}(z), z^3\acute{\acute{\acute{\rho}}}(z); z\right) \in \Omega. \tag{3.10}$$

Note that,

$$\frac{w}{v} + 1 = \frac{(a+1)(a+2)\gamma_3 - (2a+1)(a+1)\gamma_2 + a^2\gamma_1}{(a+1)\gamma_2 - a\gamma_1}, \tag{3.11}$$

and

$$\frac{r}{v} = \frac{(a+1)(a+2)[(a+3)\gamma_4 - 3(a+2)\gamma_3 + 3(a+1)\gamma_2 - a\gamma_1]}{(a+1)\gamma_2 - a\gamma_1}. \tag{3.12}$$

Furthermore,

$$\left|\frac{z\acute{\rho}(z)}{\acute{\mu}(s)}\right| = \left|\frac{(a+1)M^{a+1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z) - aM^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z)}{\acute{\mu}(s)}\right| \le k. \tag{3.13}$$

Hence, the admissibility condition of $\phi \in \Phi[\Omega, \mu]$ is the same for ψ that defined in (1.2). Therefore, and by Theorem 2.4, we obtain $\rho(z) \prec \mu(z)$, that is $M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z) \prec \mu(z)$.■

Assume that Ω is a proper subset of $\mathbb{C}$, that means there exist a conformal mapping of $h(z)$ maps $\mathbb{U}$ onto Ω such that $h(\mathbb{U}) =$. As a direct conclusion of Theorem 3.2, we get the following theorem.

Theorem 3.3: Let $h(z)$ be analytic function in $\mathbb{U}$, $\phi \in \Phi[h,\mu]$ and $\mu \in \Lambda_1$. If $f \in A$ satisfies the condition (3.1). If

$$\phi\left(M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z), M^{a+1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z), M^{a+2,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z), M^{a+3,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z); z\right) \prec h(z), \quad (3.14)$$

then

$$M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z) \prec \mu(z). \quad (3.15)$$

The above results can be extended for the case that the behavior of the function $\mu(z)$ on the boundary of $\mathbb{U}$ is unknown, see the following corollaries.

Corollary 3.4: Let $\Omega \subseteq \mathbb{C}, \mu(z)$ be univalent in $\mathbb{U}$ with $\mu(0)=1$. Let $\phi \in \Phi[\Omega,\mu_t]$ for some $t \in (0,1)$, where $\mu_t(z)=\mu(tz)$. If

$$Re\left(\frac{s\acute{\mu}_t(s)}{\acute{\mu}_t(s)}\right) \geq 0, \quad \left|\frac{M^{a+1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z) - M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z)}{\acute{\mu_t}(z)}\right| \leq k\left|\frac{1}{a+1}\right|, \quad (3.16)$$

and

$$\phi\left(M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z), M^{a+1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z), M^{a+2,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z), M^{a+3,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z); z\right) \in \Omega, \quad (3.17)$$

then for $z \in \mathbb{U}$ and $s \in \partial\mathbb{U}\backslash E(\mu_t)$, we have

$$M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z) \prec \mu(z). \quad (3.18)$$

Proof. From Theorem 3.2, we obtain $M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z) \prec \mu_t(z)$. Hence, we conclude that $\mu_t(z) \prec \mu(z)$.■

Corollary 3.5: Let $\Omega \subseteq \mathbb{C}, \mu(z)$ be univalent in $\mathbb{U}$ with $\mu(0)=1$. Let $\phi \in \Phi[\Omega,\mu_t]$ for some $t \in (0,1)$, where $\mu_t(z)=\mu(tz)$. If the condition (3.16) satisfied such that

$$\phi\left(M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z), M^{a+1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z), M^{a+2,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z), M^{a+3,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z); z\right) \prec h(z), \quad (3.19)$$

then for $z \in \mathbb{U}$ and $s \in \partial\mathbb{U}\backslash E(\mu_t)$, we have $M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z) \prec \mu(z)$.

The following part considering the relation (1.7) as an essential tool to achieve another class of admissible linear operator that concurring $M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z)$.

Definition 3.6: Let $\Omega \subseteq \mathbb{C}, \mu \in$. The class of admissible functions $\Phi[\Omega,\mu]$ containing those functions $\phi: \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ that satisfy the following admissibility condition:

$$\phi(\gamma_1,\gamma_2,\gamma_3,\gamma_4;z) \notin \Omega$$

whenever

$$\gamma_1 = \mu(s), \qquad \gamma_2 = \frac{\tau_1 ks\acute{\mu}(s) + \lambda_1\mu(s)}{\sigma},$$

$$Re\left(\frac{\sigma(\gamma_3 - 2\gamma_2)(\sigma-1) + \lambda_1^2\gamma_1}{\tau_1\lambda_1(\gamma_2-\gamma_1) + \tau_1^2\gamma_2}\right) \geq kRe\left(\frac{s\acute{\acute{\mu}}(s)}{\acute{\mu}(s)} + 1\right),$$

and

$$Re\left(\frac{1}{\tau_1^3}\ [\sigma(\sigma-1)(\sigma-2)\gamma_4-\sigma(\sigma-1)^2\gamma_3 + [\sigma(\sigma-1)(\sigma+\lambda_1-1)-3\sigma\lambda_1(\sigma-2)+3\sigma\tau_1-2\sigma]\gamma_2 - [\sigma(\sigma-1)+3\lambda_1^2(\sigma-2)+3\tau_1\lambda_1-2\lambda_1]\ \gamma_1\right)\geq k^2Re\left(\frac{s^2\acute{\acute{\mu}}(s)}{\acute{\mu}(s)}+1,\right)$$

where $z\in\mathbb{U}, s\in\partial\mathbb{U}\backslash E(\mu), k\in\mathbb{N}/\{1\}$ and $\sigma=\tau_1+\lambda_1$.

Theorem 3.7: Let $\phi\in\Phi[\Omega,\mu]$ and $\mu\in\Lambda_1$. If $f\ \in A$ such that

$$Re\left(\frac{s\acute{\acute{\mu}}(s)}{\acute{\mu}(s)}\right)\geq 0,\quad \left|\frac{M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z)-M^{a,b,c}_{\tau_1,\lambda_1+1,\tau_2,\lambda_2}f(z)}{\acute{\mu}(s)}\right|\leq k\left|\frac{\tau_1}{\sigma}\right|, \tag{3.20}$$

and

$$\left\{\phi\left(M^{a,b,c}_{\tau_1,\lambda_1+1,\tau_2,\lambda_2}f(z),M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z),M^{a,b,c}_{\tau_1,\lambda_1-1,\tau_2,\lambda_2}f(z),M^{a,b,c}_{\tau_1,\lambda_1-2,\tau_2,\lambda_2}f(z);z\right):z\in U\right\}\subset\Omega, \tag{3.21}$$

then

$$M^{a,b,c}_{\tau_1,\lambda_1+1,\tau_2,\lambda_2}f(z)\prec\mu(z). \tag{3.22}$$

Proof. Suppose that

$$M^{a,b,c}_{\tau_1,\lambda_1+1,\tau_2,\lambda_2}f(z)=\rho(z), \tag{3.23}$$

by virtue of relation (1.7), yields that

$$M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z)=\frac{1}{\sigma}[\tau_1z\acute{\rho}(z)+\lambda_1\rho(z)], \tag{3.24}$$

hence,

$$M^{a,b,c}_{\tau_1,\lambda_1-1,\tau_2,\lambda_2}f(z)=\frac{1}{\sigma(\sigma-1)}\left[\tau_1^2z^2\acute{\acute{\rho}}(z)+\tau_1(\sigma+\lambda_1-1)z\acute{\rho}(z)+\lambda_1(\lambda_1-1)\rho(z)\right], \tag{3.25}$$

also,

$$M^{a,b,c}_{\tau_1,\lambda_1-2,\tau_2,\lambda_2}f(z) =\frac{1}{\sigma(\sigma-1)(\sigma-2)}\ \left[\tau_1^3z^3\acute{\acute{\acute{\rho}}}(z)+3\tau_1^2(\sigma-1)z^2\acute{\acute{\rho}}(z)+3\tau_1\lambda_1(\sigma_2) -\tau_1(3\tau_1-2)z\acute{\rho}(z)+\lambda_1(\lambda_1-1)(\lambda_1-2)\rho(z)\right]. \tag{3.26}$$

Now, define the transformation

$$\gamma_1=u,\quad \gamma_2=\frac{1}{\sigma}(\lambda_1u+\tau_1v),$$

$$\gamma_3=\frac{1}{\sigma(\sigma-1)}[\ \tau_1^2w+\tau_1(\sigma+\lambda_1-1)v+\lambda_1(\lambda_1-1)\ u],$$

and

$$\gamma_4=\frac{1}{\sigma(\sigma-1)(\sigma-2)}[\ \tau_1^3r+3\tau_1^2(\sigma-1)w+[3\tau_1\lambda_1(\sigma-2)-\tau_1(3\tau_1-2)]v +\lambda_1(\lambda_1-1)(\lambda_1-2)u].$$

So, the transformation $\psi:\mathbb{C}^4\times\mathbb{U}\to\mathbb{C}$, can be define as

$$\psi(u,v,w,r;z)=\phi(\gamma_1,\gamma_2,\gamma_3,\gamma_4;z), \tag{3.27}$$

From relations (3.23) to (3.26), we conclude

$$\psi\left(\rho(z),z\acute{\rho}(z),z^2\acute{\acute{\rho}}(z),z^3\acute{\acute{\acute{\rho}}}(z);z\right) =\phi\left(M^{a,b,c}_{\tau_1,\lambda_1+1,\tau_2,\lambda_2}f(z),M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z),M^{a,b,c}_{\tau_1,\lambda_1-1,\tau_2,\lambda_2}f(z),M^{a,b,c}_{\tau_1,\lambda_1-2,\tau_2,\lambda_2}f(z);z\right), \tag{3.28}$$

by expression (3.27), we have

$$\psi\left(\rho(z), z\acute{\rho}(z), z^2\acute{\acute{\rho}}(z), z^3\acute{\acute{\acute{\rho}}}(z)z\right) \in \Omega. \quad (3.29)$$

Note that,

$$\frac{w}{v} + 1 = \frac{\sigma(\gamma_3 - 2\gamma_2)(\sigma - 1) + \lambda_1^2\gamma_1}{\tau_1\lambda_1(\gamma_2 - \gamma_1) + \tau_1^2\gamma_2}, \quad (3.30)$$

and

$$\frac{r}{v} = \frac{1}{\tau_1^3}\, \sigma(\sigma-1)(\sigma-2)\gamma_4 - \sigma(\sigma-1)^2\gamma_3 + [\sigma(\sigma-1)(\sigma+\lambda_1-1) - 3\sigma\lambda_1(\sigma-2) + 3\sigma\tau_1 - 2\sigma]\gamma_2 - [\sigma(\sigma-1) + 3\lambda_1^2(\sigma-2) + 3\tau_1\lambda_1 - 2\lambda_1]\gamma_1. \quad (3.31)$$

Furthermore,

$$\left|\frac{z\acute{p}(z)}{\acute{\mu}(s)}\right| = \left|\frac{\sigma M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z) - \lambda_1 M^{a,b,c}_{\tau_1,\lambda_1+1,\tau_2,\lambda_2}f(z)}{\tau_1\acute{\mu}(s)}\right| \le k. \quad (3.32)$$

Hence, the admissibility condition of $\phi \in \Phi[\Omega, \mu]$ is the same for ψ that defined in (1.2). Therefore, and by Theorem 2.4, we obtain $\rho(z) \prec \mu(z)$, that is $M^{a,b,c}_{\tau_1,\lambda_1+1,\tau_2,\lambda_2}f(z) \prec \mu(z)$.■

Let Ω is a proper subset of $\mathbb{C}$, then there exists a conformal mapping $h(z)$ which take $\mathbb{U}$ onto Ω, i.e., $h(\mathbb{U}) = \Omega$. The next theorem is a direct conclusion of Theorem 5.

Theorem 3.8: Let $h(z)$ be analytic function in $\mathbb{U}, \phi \in \Phi[h, \mu]$ and $\mu \in \Lambda_1$. If $f \in A$ satisfies the condition (3.20). If

$$\phi\left(M^{a,b,c}_{\tau_1,\lambda_1+1,\tau_2,\lambda_2}f(z), M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z), M^{a,b,c}_{\tau_1,\lambda_1-1,\tau_2,\lambda_2}f(z), M^{a,b,c}_{\tau_1,\lambda_1-2,\tau_2,\lambda_2}f(z); z\right) \prec h(z), \quad (3.33)$$

then

$$M^{a,b,c}_{\tau_1,\lambda_1+1,\tau_2,\lambda_2}f(z) \prec \mu(z). \quad (3.34)$$

The next corollaries extend Theorem 3.7 and Theorem 3.8, for the case that the behavior of the function $\mu(z)$ on the boundary of $\mathbb{U}$ is unknown.

Corollary 3.9: Let $\Omega \subseteq \mathbb{C}, \mu(z)$ be univalent in $\mathbb{U}$ with $\mu(0) = 1$. Let $\phi \in \Phi[\Omega, \mu_t]$ for some $t \in (0,1)$, where $\mu_t(z) = \mu(tz)$. If

$$Re\left(\frac{s\acute{\acute{\mu_t}}(s)}{\acute{\mu_t}(s)}\right) \ge 0, \quad \left|\frac{M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z) - M^{a,b,c}_{\tau_1,\lambda_1+1,\tau_2,\lambda_2}f(z)}{\acute{\mu_t}(z)}\right| \le k\left|\frac{\tau_1}{\sigma}\right|, \quad (3.35)$$

and

$$\phi\left(M^{a,b,c}_{\tau_1,\lambda_1+1,\tau_2,\lambda_2}f(z), M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z), M^{a,b,c}_{\tau_1,\lambda_1-1,\tau_2,\lambda_2}f(z), M^{a,b,c}_{\tau_1,\lambda_1-2,\tau_2,\lambda_2}f(z); z\right) \in \Omega, \quad (3.36)$$

then for $z \in \mathbb{U}$ and $s \in \partial\mathbb{U}\backslash E(\mu_t)$, we have $M^{a,b,c}_{\tau_1,\lambda_1+1,\tau_2,\lambda_2}f(z) \prec \mu(z)$.

Proof. From Theorem 3.7, we obtain $M^{a,b,c}_{\tau_1,\lambda_1+1,\tau_2,\lambda_2}f(z) \prec \mu_t(z)$. Hence, we conclude that $\mu_t(z) \prec \mu(z)$. ■

Corollary 3.10: Let $\Omega \subseteq \mathbb{C}, \mu(z)$ be univalent in $\mathbb{U}$ with $\mu(0) = 1$. Let $\phi \in \Phi[\Omega, \mu_t]$ for some $\mu_t(z) = \mu(tz)$. If the condition (3.35) is satisfied such that

$$\phi\left(M^{a,b,c}_{\tau_1,\lambda_1+1,\tau_2,\lambda_2}f(z), M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z), M^{a,b,c}_{\tau_1,\lambda_1-1,\tau_2,\lambda_2}f(z), M^{a,b,c}_{\tau_1,\lambda_1-2,\tau_2,\lambda_2}f(z); z\right) \prec h(z), \quad (3.37)$$

then for $z \in \mathbb{U}$ and $s \in \partial\mathbb{U}/E(\mu_t)$, we have $M^{a,b,c}_{\tau_1,\lambda_1+1,\tau_2,\lambda_2}f(z) \prec \mu(z)$.

Remark 3.11: If we consider the special case of $\mu(z) = Bz, (B > 0)$ in Definition 3.1 and 3.6 respectively, we could achieve the following interesting results.

Definition 3.12: Let $\Omega \subseteq \mathbb{C}$ and $B \geq 0$. The class of admissible functions $[\Omega, B]$ containing those functions $\phi: \mathbb{C}^4 \times U \to \mathbb{C}$ that satisfy the following admissibility conditions:

$$\phi\left(Bz, \frac{1}{a+1}(k+a), \frac{1}{a+2}\left[\frac{X}{a+1} + 2k + a\right], \frac{1}{a+3}\left[\frac{1}{(a+2)(a+3)}Y + \frac{3}{a+1}X + 3k + a\right]; z\right) \notin \Omega$$

where $z \in \mathbb{U}, Re(X), Re(Y)$ and $k \in \mathbb{N}\backslash\{1\}$.

Corollary 3.13: Let $\phi \in \Phi[\Omega, B]$. If $f \in A$ such that

$$\left|M^{a+1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z) - M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z)\right| \leq k\left|\frac{1}{a+1}\right|B, \quad (3.38)$$

and

$$\phi\left(M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z), M^{a+1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z), M^{a+2,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z), M^{a+3,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z); z\right) \in \Omega, \quad (3.39)$$

then

$$\left|M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z) - 1\right| < B. \quad (3.40)$$

Corollary 3.13: Let $\phi \in \Phi[\Omega, B]$. If the condition (3.38) holds, and

$$\left|\phi\left(M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z), M^{a+1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z), M^{a+2,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z), M^{a+3,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z); z\right) - 1\right| < B, \quad (3.41)$$

then

$$\left|M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z) - 1\right| < B. \quad (3.42)$$

Definition 3.14: Let $\Omega \subseteq \mathbb{C}$ and $B \geq 0$. The class of admissible functions $[\Omega, B]$ containing those functions $\phi: \mathbb{C}^4 \times U \to \mathbb{C}$ that satisfy the following admissibility conditions:

$$\phi\left(Bz, \frac{1}{\sigma}(\lambda_1 + \tau_1 k), \frac{1}{\sigma(\sigma-1)}[\tau_1^2 X + \tau_1(\lambda_1 + \sigma - 1)k + \lambda_1(\lambda_1 - 1)], \frac{1}{\sigma(\sigma-1)(\sigma-2)}[\tau_1^3 Y + 3\tau_1^3(\sigma-1)X + (3\tau_1\lambda_1(\sigma-2) - \tau_1[3\tau_1 - 2])k]; z\right) \notin \Omega$$

where $z \in \mathbb{U}$, $Re(X), Re(Y)$, $k \in \mathbb{N}\backslash\{1\}$ and $\sigma = \tau_1 + \lambda_1$.

Corollary 3.15: Let $\phi \in \Phi[\Omega, B]$. If $f \in A$ such that

$$\left|M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z) - M^{a,b,c}_{\tau_1,\lambda_1+1,\tau_2,\lambda_2}f(z)\right| \leq k\left|\frac{\tau_1}{\sigma}\right|B, \quad (3.43)$$

and

$$\phi\left(M^{a,b,c}_{\tau_1,\lambda_1+1,\tau_2,\lambda_2}f(z), M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z), M^{a,b,c}_{\tau_1,\lambda_1-1,\tau_2,\lambda_2}f(z), M^{a,b,c}_{\tau_1,\lambda_1-2,\tau_2,\lambda_2}f(z); z\right) \in \Omega, \quad (3.44)$$

then

$$\left|M^{a,b,c}_{\tau_1,\lambda_1+1,\tau_2,\lambda_2}f(z) - 1\right| < B. \quad (3.45)$$

Corollary 3.16: Let $\phi \in \Phi[\Omega, B]$. If the condition (3.44) holds, and

$$\left|\phi\left(M^{a,b,c}_{\tau_1,\lambda_1+1,\tau_2,\lambda_2}f(z), M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z), M^{a,b,c}_{\tau_1,\lambda_1-1,\tau_2,\lambda_2}f(z), M^{a,b,c}_{\tau_1,\lambda_1-2,\tau_2,\lambda_2}f(z); z\right) - 1\right| < B, \quad (3.46)$$

then

$$\left|M^{a,b,c}_{\tau_1,\lambda_1+1,\tau_2,\lambda_2}f(z)-1\right|<B. \tag{3.47}$$

4. Third-order differential superordination and sandwich results

This section, discuss corresponding third-order differential superordination features associated with the admissible classes of the linear operator $M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z)$, due to its key role to establish sandwich theorems.

Definition 4.1: Let $\Omega \subseteq \mathbb{C}$ and $\mu \in \Lambda$. The class of admissible functions $\Phi[\Omega,\mu]$ containing those functions $\Upsilon: \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ that satisfy the following admissibility condition:

$$\Upsilon(\gamma_1,\gamma_2,\gamma_3,\gamma_4;z) \notin \Omega$$

whenever

$$\gamma_1=\mu(s), \quad \gamma_2=\frac{s\acute{\mu}(s)+a\mu(s)}{j(a+1)},$$

$$Re\left(\frac{(a+1)(a+2)\gamma_3-(2a+1)(a+1)\gamma_2+a^2\gamma_1}{(a+1)\gamma_2-a\gamma_1}\right)\le\frac{1}{j}Re\left(\frac{s\acute{\acute{\mu}}(s)}{\acute{\mu}(s)}+1\right),$$

and

$$Re\left(\frac{(a+1)(a+2)[(a+3)\gamma_4-3(a+2)\gamma_3+3(a+1)\gamma_2-a\gamma_1]}{(a+1)\gamma_2-a\gamma_1}\right)\le\frac{1}{j^2}Re\left(\frac{s^2\acute{\acute{\acute{\mu}}}(s)}{\acute{\mu}(s)}+1\right),$$

where $z \in \mathbb{U}, s \in \partial U/E(\mu)$ and $j \in \mathbb{N}/\{1\}$.

Theorem 4.2: Let $\Upsilon \in \acute{\Phi}[\Omega,\mu], f \in A$ and $M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z) \in \Lambda_1$ such that

$$Re\,\frac{s\acute{\acute{\mu}}(s)}{\acute{\mu}(s)}\ge 0, \quad \left|\frac{M^{a+1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z)-M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z)}{\acute{\mu}(s)}\right|\le j\left|\frac{1}{a+1}\right|, \tag{3.48}$$

and

$$\left\{\Upsilon\left(M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z),M^{a+1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z),M^{a+2,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z),M^{a+3,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z);z\right): z\in\mathbb{U}\right\}, \tag{3.49}$$

are univalent. In addition,

$$\Omega \subset \left\{\Upsilon\left(M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z),M^{a+1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z),M^{a+2,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z),M^{a+3,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z);z\right): z\in\mathbb{U}\right\}, \tag{3.50}$$

then

$$\mu(z) \prec M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z). \tag{3.51}$$

Proof. The proof can be obtained in similar way to the proof of Theorem 3.2, with considering Theorem 2.4 as a basic tool. ■

As a direct consequence of Theorem 4.2, the following theorem discuss the case $\Omega = h(\mathbb{U})$ for some $h(z)$ that mapping $\mathbb{U}$conformally onto Ω.

Theorem 4.3: Let $h(z)$ be analytic function in $\mathbb{U}$, $\Upsilon \in \acute{\Phi}[\Omega,\mu], f \in A$ and $M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z) \in \Lambda_1$ satisfies the conditions (3.48) and (3.49). In addition,

$$h(z) \prec \Upsilon\left(M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z),M^{a+1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z),M^{a+2,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z),M^{a+3,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z);z\right), \tag{3.52}$$

then

$$\mu(z) \prec M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z). \tag{3.53}$$

Definition 4.4: Let $\Omega \subseteq \mathbb{C}, \mu \in$. The class of admissible functions $\Phi[\Omega,\mu]$ containing those functions $\Upsilon: \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ that satisfy the following admissibility condition:

$$\Upsilon(\gamma_1,\gamma_2,\gamma_3,\gamma_4;z) \notin \Omega$$

whenever

$$\gamma_1 = \mu(s), \qquad \gamma_2 = \frac{\tau_1 s\acute{\mu}(s) + \lambda_1\mu(s)}{\sigma j},$$

$$Re\left(\frac{\sigma(\gamma_3 - 2\gamma_2)(\sigma - 1) + \lambda_1^2\gamma_1}{\tau_1\lambda_1(\gamma_2 - \gamma_1) + \tau_1^2\gamma_2}\right) \le \frac{1}{j} Re\left(\frac{s\acute{\acute{\mu}}(s)}{\acute{\mu}(s)} + 1\right),$$

and

$$\begin{aligned} Re\Bigg(\frac{1}{\tau_1^3}[\sigma(\sigma-1)(\sigma-2)\gamma_4 - \sigma(\sigma-1)^2\gamma_3 \\ + [\sigma(\sigma-1)(\sigma+\lambda_1-1) - 3\sigma\lambda_1(\sigma-2) + 3\sigma\tau_1 - 2\sigma]\gamma_2 \\ - [\sigma(\sigma-1) + 3\lambda_1^2(\sigma-2) + 3\tau_1\lambda_1 - 2\lambda_1]\gamma_1]\Bigg) \le \frac{1}{j^2} Re\left(\frac{s^2\acute{\acute{\acute{\mu}}}(s)}{\acute{\mu}(s)} + 1\right), \end{aligned}$$

where $z \in \mathbb{U}, s \in \partial\mathbb{U}/E(\mu), j \in \mathbb{N}/\{1\}$ and $\sigma = \tau_1 + \lambda_1$.

Theorem 4.5: Let $\Upsilon \in \acute{\Phi}[\Omega,\mu]$, $f \in A$ and $M^{a,b,c}_{\tau_1,\lambda_1+1,\tau_2,\lambda_2}f(z) \in \Lambda_1$ such that

$$Re\frac{s\acute{\mu}(s)}{\acute{\mu}(s)} \ge 0, \quad \left|\frac{M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z) - M^{a,b,c}_{\tau_1,\lambda_1+1,\tau_2,\lambda_2}f(z)}{\acute{\mu}(s)}\right| \le j\left|\frac{\tau_1}{\sigma}\right|, \tag{3.54}$$

and

$$\left\{\Upsilon\left(M^{a,b,c}_{\tau_1,\lambda_1+1,\tau_2,\lambda_2}f(z), M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z), M^{a,b,c}_{\tau_1,\lambda_1-1,\tau_2,\lambda_2}f(z), M^{a,b,c}_{\tau_1,\lambda_1-2,\tau_2,\lambda_2}f(z);z\right): z \in \mathbb{U}\right\} \tag{3.55}$$

are univalent. In addition,

$$\Omega \subset \Upsilon\left\{\left(M^{a,b,c}_{\tau_1,\lambda_1+1,\tau_2,\lambda_2}f(z), M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z), M^{a,b,c}_{\tau_1,\lambda_1-1,\tau_2,\lambda_2}f(z), M^{a,b,c}_{\tau_1,\lambda_1-2,\tau_2,\lambda_2}f(z);z\right): z \in \mathbb{U}\right\}, \tag{3.56}$$

Then

$$\mu(z) \prec M^{a,b,c}_{\tau_1,\lambda_1+1,\tau_2,\lambda_2}f(z). \tag{3.57}$$

Proof. The proof can be obtained in similar way to the proof of Theorem 3.7, with considering Theorem 2.5 as a basic tool. ■

If we assume $\Omega = h(\mathbb{U})$ for some $h(z)$ that mapping $\mathbb{U}$ conformally onto Ω, then the following theorem obtained directly from Theorem 4.5.

Theorem 4.6: Let $h(z)$ be analytic function in $U, \Upsilon \in \acute{\Phi}[\Omega,\mu]$, $f \in A$ and $M^{a,b,c}_{\tau_1,\lambda_1+1,\tau_2,\lambda_2}f(z) \in \Lambda_1$ satisfies the conditions (3.54) and (3.55). In addition,

$$h(z) \prec \Upsilon\left(M^{a,b,c}_{\tau_1,\lambda_1+1,\tau_2,\lambda_2}f(z), M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z), M^{a,b,c}_{\tau_1,\lambda_1-1,\tau_2,\lambda_2}f(z), M^{a,b,c}_{\tau_1,\lambda_1-2,\tau_2,\lambda_2}f(z);z\right), \tag{3.58}$$

then

$$\mu(z) \prec M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z). \tag{5.59}$$

The next theorems, establish sandwich results involving the operator $M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z)$, by collect Theorem 3.2 and Theorem 4.3, then Theorem 3.7 and Theorem 4.6 respectively

Theorem 4.7: Let $h_1(z)$ and $\mu_1(z)$ are analytic functions in $\mathbb{U}$, $h_2(z)$ is univalent in $\mathbb{U}$ and $\mu_2(z) \in \Lambda_1$ with $\mu_1(0) = \mu_2(0) = 1$. Also, let $\Theta \in \Phi[\Omega, \mu] \cap \acute{\Phi}[\Omega, \mu]$, $f \in A$ and $M_{\tau_1,\lambda_1,\tau_2,\lambda_2}^{a,b,c} f(z) \in \Lambda_1 \cap A$. In addition

$$\Theta\left(M_{\tau_1,\lambda_1,\tau_2,\lambda_2}^{a,b,c} f(z), M_{\tau_1,\lambda_1,\tau_2,\lambda_2}^{a+1,b,c} f(z), M_{\tau_1,\lambda_1,\tau_2,\lambda_2}^{a+2,b,c} f(z), M_{\tau_1,\lambda_1,\tau_2,\lambda_2}^{a+3,b,c} f(z); z\right),$$

are univalent in $\mathbb{U}$. Suppose that the conditions (3.1) and (3.48) are satisfied, also

$$h_1(z) \prec \Theta\left(M_{\tau_1,\lambda_1,\tau_2,\lambda_2}^{a,b,c} f(z), M_{\tau_1,\lambda_1,\tau_2,\lambda_2}^{a+1,b,c} f(z), M_{\tau_1,\lambda_1,\tau_2,\lambda_2}^{a+2,b,c} f(z), M_{\tau_1,\lambda_1,\tau_2,\lambda_2}^{a+3,b,c} f(z); z\right) \prec h_2(z),$$

then

$$\mu_1(z) \prec M_{\tau_1,\lambda_1,\tau_2,\lambda_2}^{a,b,c} f(z) \prec \mu_2(z).$$

Theorem 4.8: Let $h_1(z)$ and $\mu_1(z)$ are analytic functions in $\mathbb{U}$, $h_2(z)$ is univalent in $\mathbb{U}$ and $\mu_2(z) \in \Lambda_1$ with $\mu_1(0) = \mu_2(0) = 1$. Also, let $\Theta \in \Phi[\Omega, \mu] \cap \acute{\Phi}[\Omega, \mu]$, $f \in A$ and $M_{\tau_1,\lambda_1+1,\tau_2,\lambda_2}^{a,b,c} f(z) \in \Lambda_1 \cap A$. In addition

$$\Theta\left(M_{\tau_1,\lambda_1+1,\tau_2,\lambda_2}^{a,b,c} f(z), M_{\tau_1,\lambda_1,\tau_2,\lambda_2}^{a,b,c} f(z), M_{\tau_1,\lambda_1-1,\tau_2,\lambda_2}^{a,b,c} f(z), M_{\tau_1,\lambda_1-2,\tau_2,\lambda_2}^{a+3,b,c} f(z); z\right)$$

are univalent in $\mathbb{U}$. Suppose that the conditions (3.20) and (3.54) are satisfied, also

$$h_1(z) \prec \Theta\left(M_{\tau_1,\lambda_1+1,\tau_2,\lambda_2}^{a,b,c} f(z), M_{\tau_1,\lambda_1,\tau_2,\lambda_2}^{a,b,c} f(z), M_{\tau_1,\lambda_1-1,\tau_2,\lambda_2}^{a,b,c} f(z), M_{\tau_1,\lambda_1-2,\tau_2,\lambda_2}^{a+3,b,c} f(z); z\right) \prec h_2(z),$$

then

$$\mu_1(z) \prec M_{\tau_1,\lambda_1+1,\tau_2,\lambda_2}^{a,b,c} f(z) \prec \mu_2(z).$$

5. Conclusion

This article devoted to construct an admissible class of the seven-parameter Mittag-Leffler operator $M_{\tau_1,\lambda_1,\tau_2,\lambda_2}^{a,b,c} f(z)$, and assume this class to study third order differential subordination and subordination. Generally, the results of this paper divided into two main parts, the first one concerning relation (1.6) as an essential tool in their investigations, and the second part involving relation (1.7). Each of these two parts establish third order differential subordination and its corresponding superordination. Further, some new sandwich theorems are derived by collecting some considering results linked to our mentioned operator $M_{\tau_1,\lambda_1,\tau_2,\lambda_2}^{a,b,c} f(z)$.

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