Kolmogorov Turbulent Simulations of Photon Limited Images of Binary Stars

Athil Salman Farhan* and Ali Talib Mohammed
Department of Astronomy and Space, College of Science, University of Baghdad, Baghdad, Iraq.

Abstract
The autocorrelation function calculations have been carried out on photon-limited computer-simulated images of binary stars that recorded through kolmogorov atmospheric turbulence. The effect of the parameters of photon limited binary star on the variation of signal to noise, signal to background ratios, number of images that processed and the magnitude of binary stars are studied and mathematic equations are given to investigate this effect. The result indicates that signal to background ratio of photon limited images of a binary star is independent of the total number of recorded photons.

Keywords: Turbulent atmosphere, Fourier transform, optical imaging, adaptive optics.

نمذجة النجوم الثنائية ذات الفوتونات المحدودة بأستخدام احصائية كولموكوروف

أثيل سلمان فرحان * ، علي طالب محمد
قسم الفلك والفضاء، كلية العلوم ، جامعة بغداد، بغداد، العراق

الخلاصة:
تم حساب دالة الترابط الذاتي التي تم اجرائها على النجوم الثنائيه ذات الفوتونات المحدودة التي سجلت من خلال اضطراب غلاف جوي يخضع الى موديل كولموكوروف. أن تأثير معاملات هذه النجوم الفوتونية على نسبة الأشارة الى الخلفية والأشارة الى الضوضاء قد تم حسابها وتمثيلها بمعادلات رياضية. النتائج اظهرت ان نسبة الأشارة الى الأرضيه لا تعتمد على عدد الفوتونات المسجلة.

1. Introduction:

The nature of the wavefront perturbations from a distant star that introduced by the atmosphere is firstly studied by the Russian mathematician Andrei Kolmogorov that developed by Tatarski [1]. The angular resolution or "seeing" is limited by atmospheric turbulence and not by the theoretical diffraction limit of the telescope [2].The kolmogorov model is supported by many of experimental measurements and is heavily used in simulations of astronomical imaging [3- 8]. The model assumes that the perturbations in the wavefront are due to the variations in the refractive index of the atmospheric layers that the wavefront should travel through.

These variations introduce phase fluctuations. The amplitude fluctuations have not significant effect on the structure of the images that taken by large optical telescope [3].

*Email:atheel.salman@yahoo.com

Up to now, there are no publications in the literatures that studied photon limited images of an astronomical object using kolmogorov turbulent model [9-13]. For this reason, the analytical model of imaging a reference and binary star in the presence of such model is presented.

2. Mathematical Formulations:

The response of an incoherent optical system for the spatial frequency could be demonstrated in terms of the optical transfer function (OTF) by [15].

$$T(u,v)=\frac{\iint_{-\infty}^{+\infty} B(x,y)B^{*}(x-\bar{\lambda} fu,y-\lambda fv)dxdy}{\iint_{-\infty}^{+\infty}|B(x,y)|^{2}dxdy} \quad (1)$$

Where B and B^* are the complex pupil function and its complex conjugate, f is the focal length of the lens, λ is a wavelength and u, v are spatial frequency variables. If we assume the wavefront from a reference star is a plane wave and $B(x, y)$ is a circular function of a unity magnitude, then diameter of these functions depend on the diameter of the optical telescope. The wavefront from a reference star is distorted by the presence of atmospheric turbulence and therefore eq. (1) becomes:

$$T(u,v)=\frac{\iint_{-\infty}^{+\infty} B(x,y)B^{*}(x-\lambda fu,y-\lambda fv)U(x,y)U^{*}(x-\lambda fu,y-\lambda fv)dxdy}{\iint_{-\infty}^{+\infty}|B(x,y)|^{2}|U(x,y)|^{2}dxdy} \quad (2)$$

U is described a perturbed complex wavefront of a reference star that introduced by atmospheric turbulence. The modulation transfer function (MTF) is given by:

$$MTF(u,v)=|T(u,v)| \quad (3)$$

The phase screen is related to its kolmogorov spectrum by the following Fourier transform equations [16]:

$$H(k)=\iint_{-\infty}^{+\infty} H(r)\exp\big(-i2\pi(r.k)\big)\,dr \quad (4)$$

$$H(r)=\iint_{-\infty}^{+\infty} H(k)\exp\big(i2\pi(r.k)\big)\,dk \quad (5)$$

$H(r)$ could be presented as a two-dimensional phase screen in the aperture of the telescope, and r is a two-dimensional coordinates. The phase screens that assume a pure kolmogorov spectrum are not stationary, i.e., $\langle H(r)^2\rangle$ increases without limit as $|r| \to \infty$. Phase screens are scaled by multiplying the phase by $(r/r_o)^{5/6}$. The phase structure function is defined by [16]:

$$D(|r|)=\langle (H(r')-H(r'+r))^2\rangle \quad (6)$$
$$=6.88(|k|/r_o)^{5/3}$$

where r_o is the Fried parameter [17][18] and k is a distance that measured from the center of an array. The kolmogorov power spectral density (PSD) is given by:

$$\phi(k)=0.023\, r_o^{-5/3}|k|^{-11/3} \quad (7)$$

3. Computational methods:

Computer simulations are conducted to assess the quality of an astronomical images that taken by an optical telescope in the presence of kolmogorov atmospheric turbulence. The strength of atmospheric turbulence (sc) is given by:

$$sc=r_o/R\,, \qquad 0<sc\le 1 \quad (8)$$

The simulations consist the followings:

(1) The complex wavefront of a reference star $U(x, y)$ that passed through the kolmogorov turbulence is generated by assigning a normal random distribution with zero mean and unit variance to its real and imaginary parts using different realizations.
(2) The phase structure function, $D(|r|)$, as given by eq. (6) is computed according to:

$$k = [(i - nc)^2 + (j - nc)^2]^{1/2} \quad (9)$$

(3) Power spectrum density (PSD) that defined in eq. (7) is also computed:

$$PSD(i, j) = \phi(k) \quad (10)$$

(4) Set PSD (0, 0) =0 (set piston to zero, i.e the value of center of an array is equal to zero).
(5) Multiply the result of step (1) by the square root of PSD.
(6) perform inverse Fourier transform and the real part represents the kolmogorov phase screen, $\varphi(x, y)$.
Lastly, the perturbed complex wave front of a reference star is represented by,

$$U(x, y) = e^{-j\left[\frac{2\pi}{\lambda}\varphi(x,y)\right]} \quad (11)$$

Multiply eq. (11) by the telescope aperture, B(x,y), of radius R and of unity magnitude. |MTF| is computed by taking the absolute of eq. (2) and the psf is taken to be the absolute of the inverse Fourier transform of eq. (2).

We extend our study to include the effect of kolmogorov turbulence on binary stars. The binary star is taken to be a two impulse delta functions separated by a certain distance from the center of an array .This means that each star is a one pixel extent and of unity magnitude as shown below:

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & elsewhere \end{cases} \quad (12)$$

The separation is chosen with respect to the ratio "$separation/D$" (D=2R). This ratio is taken to be 0.1. This will produce a binary star that has a separation to be just within the full extent of the base of the psf of the optical telescope in use (no turbulence) [11] .This binary star is then convolved with the psf of the telescope/atmosphere system in which the turbulent atmosphere are modeled by kolmogorov statistic. The power spectrum and autocorrelation function at different values of r_o are then computed.

4. Results and discussion:

We consider the telescope aperture is infinitely large and the observations at very bad seeing condition (i.e, $r_o = 1$). The Kolmogorov phase screen function, $\varphi(x, y)$, its plot, and its corresponding surface plot are shown in figure - 1.

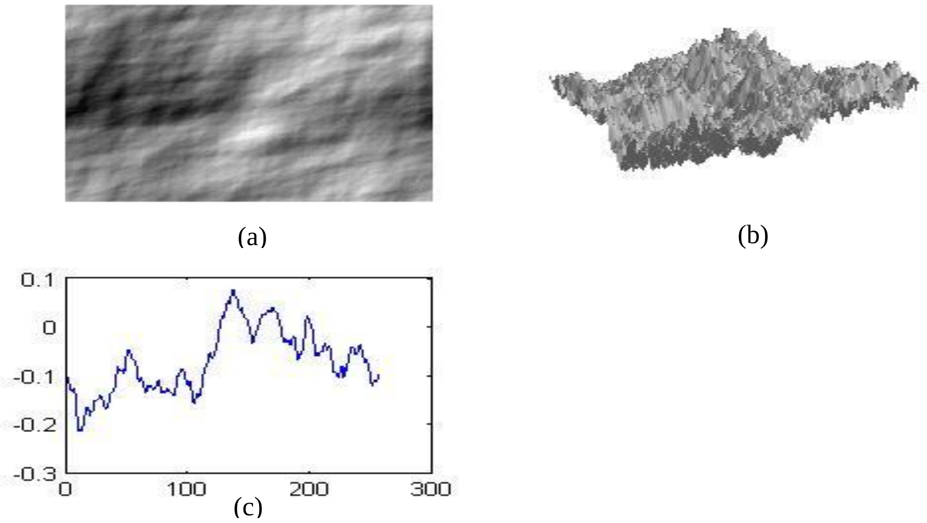

Figure 1 - $\varphi(x,y)\ at\ r_o = 1$; a-Image; b-Surface plot of (a); c-Cross section through the center of (a).

The cross section through the center of $\varphi(x,y)$ for different values of r_o are shown in figure -2.

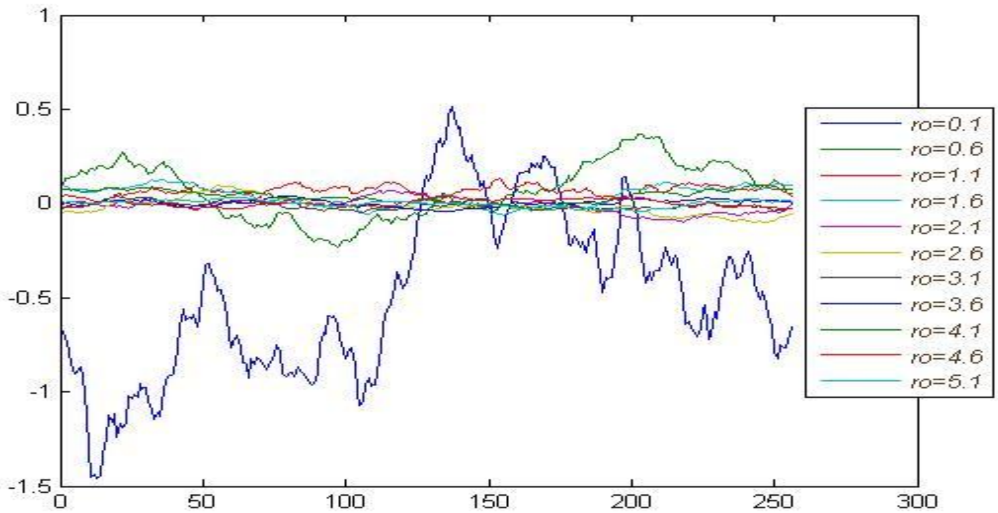

Figure 2- Cross section through the center of $\varphi(x,y)$ at different values of r_o.

It is quite clear from this figure that as r_o increases, the fluctuations in $\varphi(x,y)$get smoother and approaching constant value. It should be pointed out here that the phase becomes perfectly constant at $r_o \to \infty$.

Now our aim is to observe a reference star that taken by infinitely large optical telescope. This star is taken to be a dirac delta function, as given by eq. (12). In this case, the two dimensional dirac delta function could be represented by a matrix of size 256 by 256 pixels of zero values except the center that has a value of one.

Psf is computed via the inverse Fourier transform of eq. (11) and the MTF is computed by taking the absolute Fourier transform of psf.

The binary star is taken to be a two dirac delta function separated by a distant of 5 by 5 pixels from the center of an array. The binary star is convolved with the computed psf and the result is normalized to one at its maximum value. This result is then subjected to poisson distribution as given by:

$$f = \lambda^L \exp(-\lambda)/L! \qquad (13)$$

Where L is a nonnegative integer and λ is a parameter that define both the mean and the variance of the distribution.
In our model we set $\lambda^L = q$ and λ inside the exponentiation is taken to be the value of the normalized binary star. The result is a photon limited image at a certain value of q.
The poisson distribution is appropriate for applications that involve counting the number of times a random event occurs in a given amount of time, distance, area, ets. This is an approximate application for receiving photon from astronomical objects especially at low-light level. For poisson distribution, λ must be equal or less than one and above that, the physical distribution changes toward normal distribution.
Now λ is taken to be the value of each pixel of a normalized binary star multiplied by q parameter (q must be less than one for poisson distribution). The output is a matrix of binary star represents the number of photon that received at each pixel. This binary star is then Fourier transformed and the absolute squared is performed. The results demonstrate the power spectrum of this binary star at a certain value of q. The autocorrelation function is then computed by taking the inverse Fourier transform of the power spectrum.
The results of implementing such processing using different values of q, number of images, and magnitude of binary star at $r_o = 1$ are shown in figure - 3- 6.

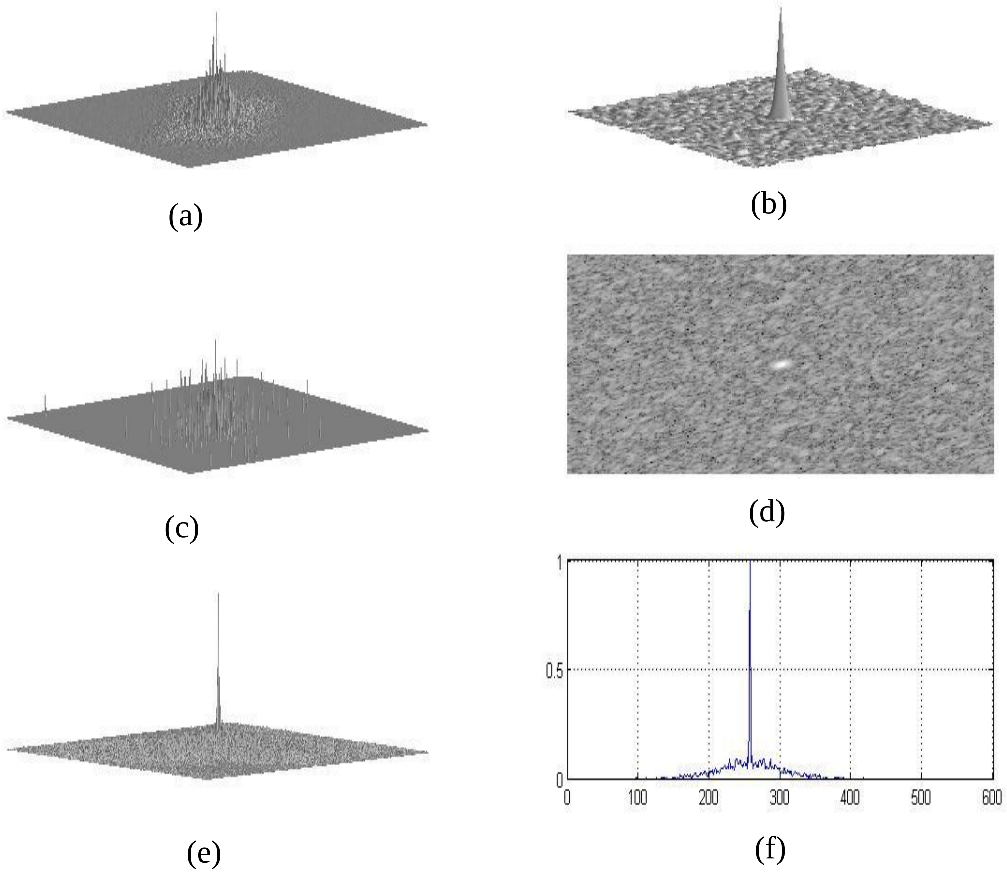

Figure 3- Psf, Binary star of equal magnitude [mag (1,1)], $r_o = 1$, q=1 and number of images=1; a-Psf; b-MTF; c-Binary star ($< n_{ph} >= 470$); d-Fringes of (c); e-Surface plot of the autocorrelation function; f-Cross section through (e).

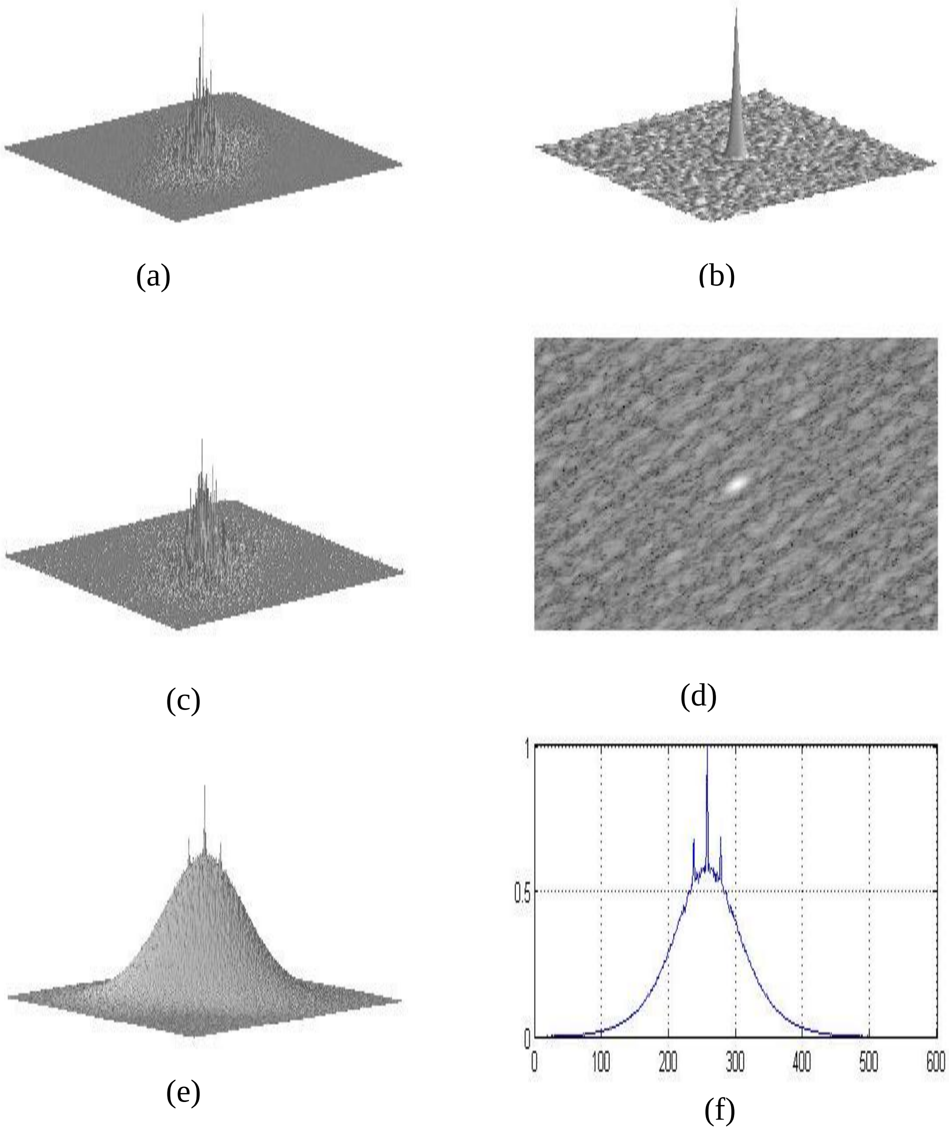

Figure 4 - Psf, Binary star of mag (1,1), $r_o = 1$, q=50 and number of images=1; a-Psf; b-MTF; c-Binary star ($< n_{ph} >= 23334$); d-Fringes of (c); e- Surface plot of the autocorrelation function; f-Cross section through (e).

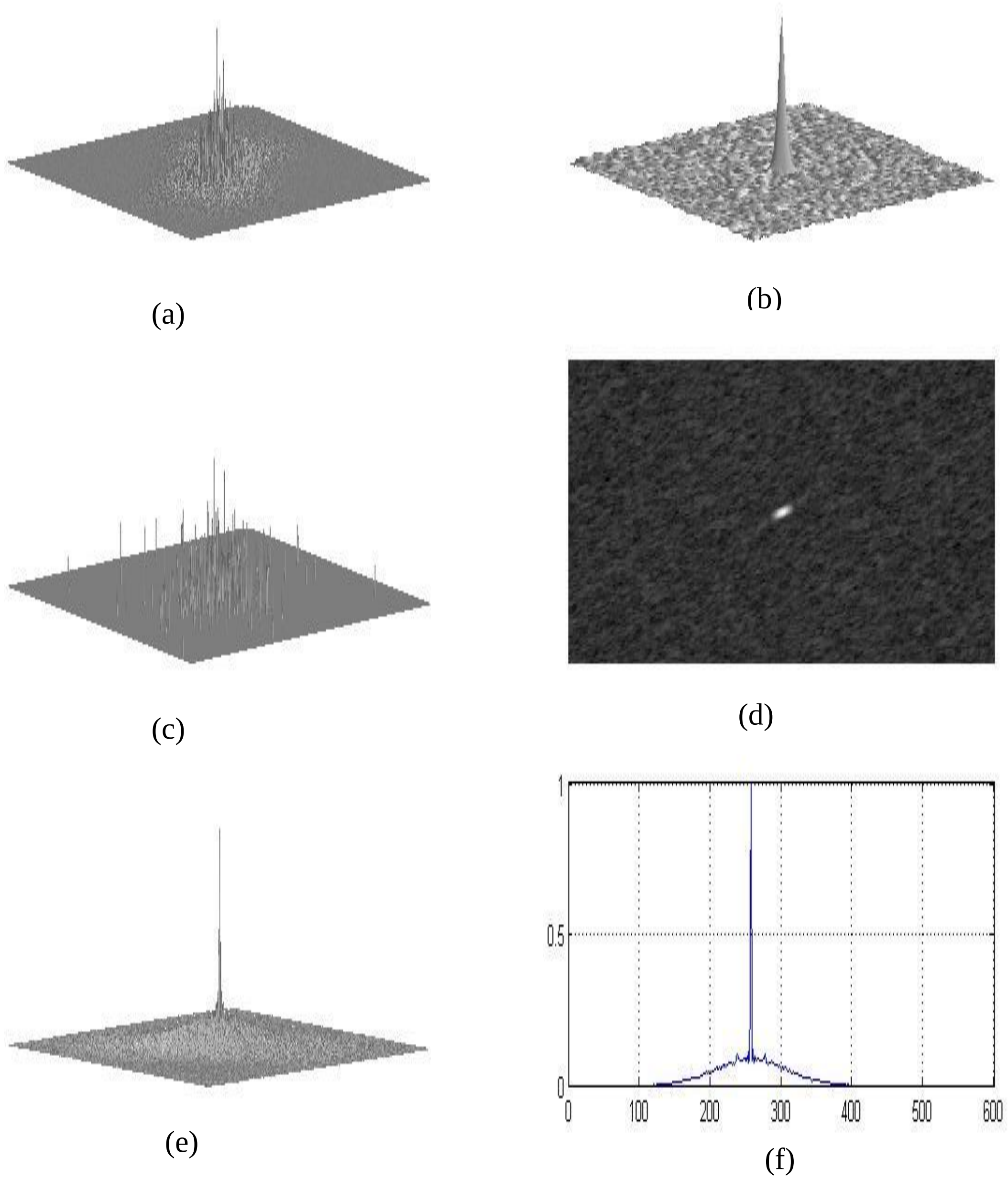

Figure 5- Psf, Binary star of mag (1,1), $r_o = 1$, q=1 and number of images=10; a-Psf; b-MTF; c-Binary star ($< n_{ph} >= 578$); d-Fringes of (c); e- Surface plot of the autocorrelation function; f-Cross section through (e).

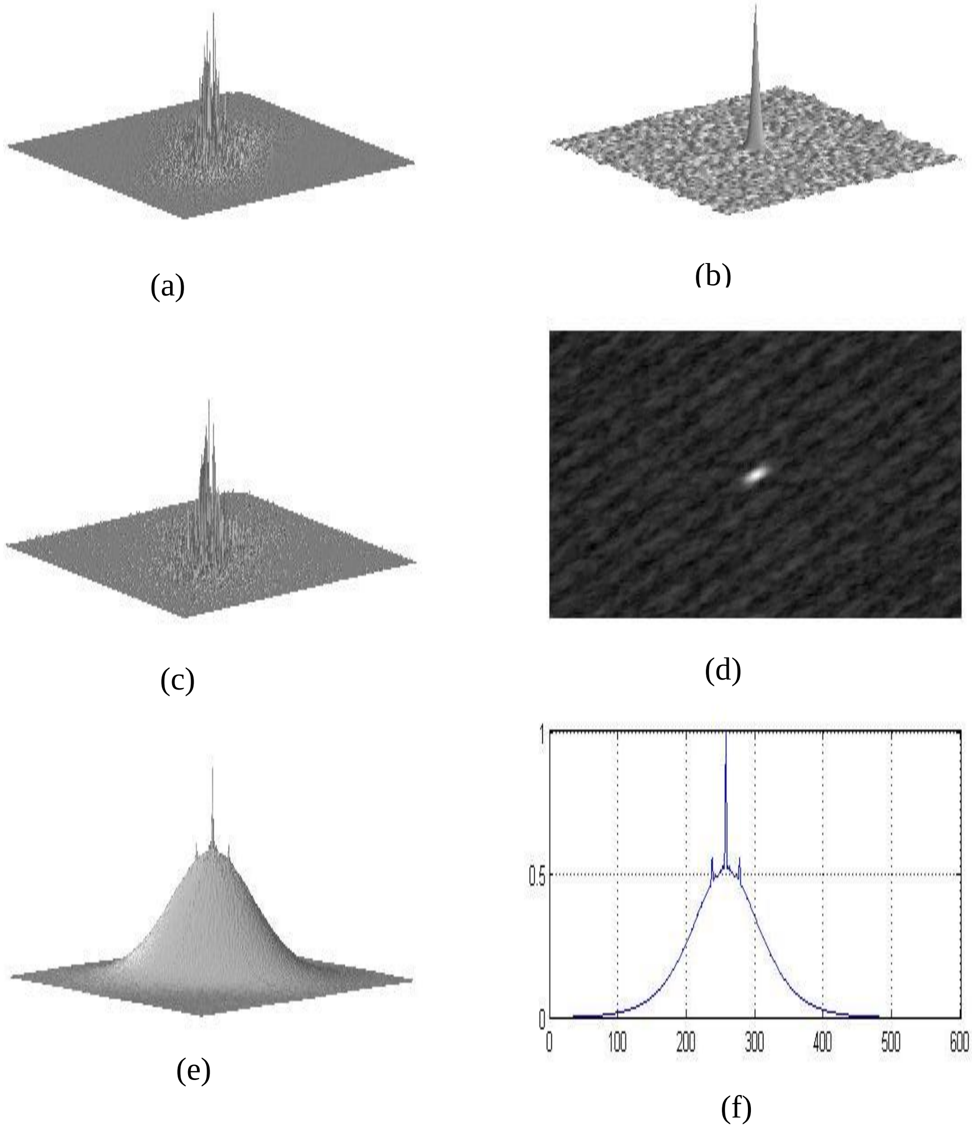

Figure 6- Psf, Binary star of mag (1,0.25), $r_o = 1$, q=50 and number of images=10; a-Psf; b-MTF; c-Binary star ($< n_{ph} >= 17489$); d-Fringes of (c); e- Surface plot of the autocorrelation function; f-Cross section through (e).

SNR and SBR are calculated by taking the value of the pixel that is located at (Nc+sep,Nc+sep) to be S+B. Where (Nc,Nc) is the index of the central array that demonstrate autocorrelation function. The background is computed from averaging all points that located at a distance of 10 pixels from the center of an array. The noise is calculated as the standard deviation of these points.
Figure - 7&8 represent the SNR and SBR as a function of q that used to generate photon limited image of a binary star.

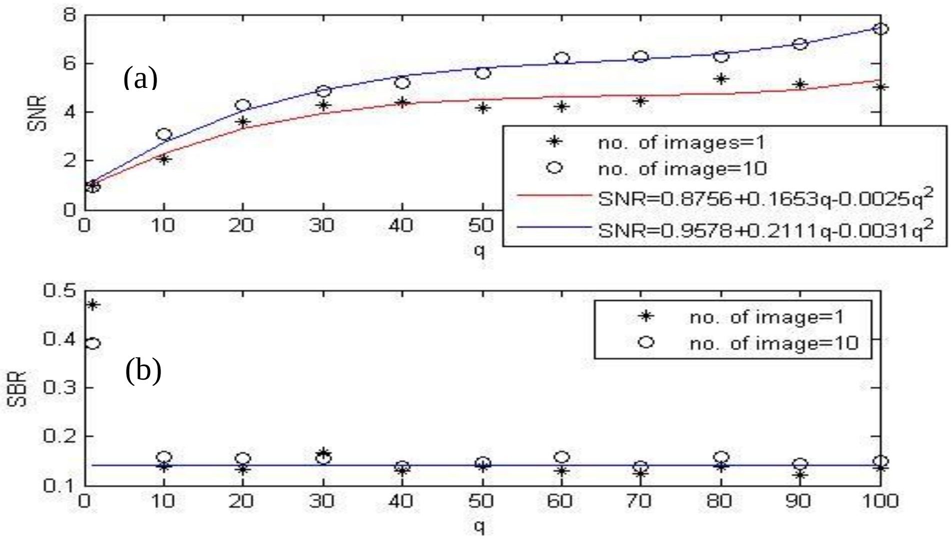

Figure 7- a- SNR as a function of q ($r_o = 1, mag\ (1,1)$) b- SBR as a function of q ($r_o = 1, mag\ (1,1)$).

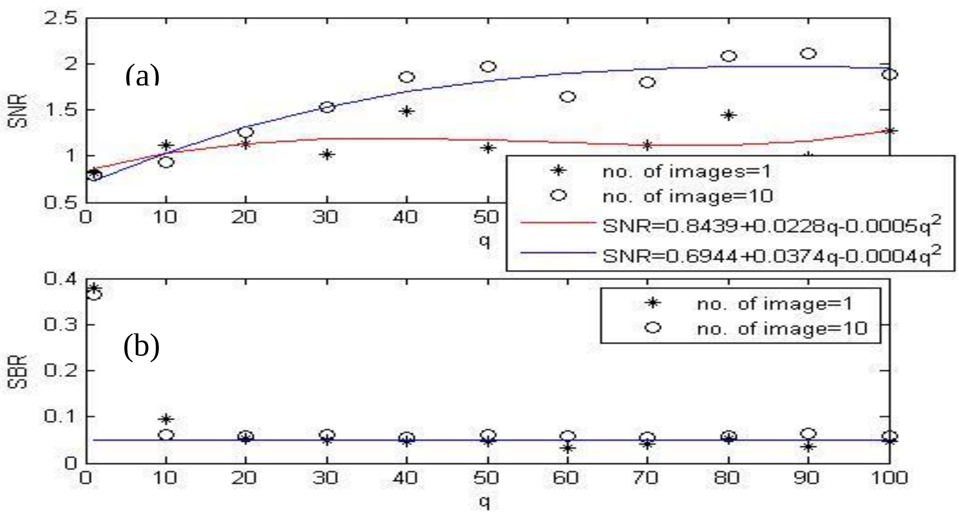

Figure 8- a- SNR as a function of q ($r_o = 1, mag\ (1, 0.25)$) b- SBR as a function of q ($r_o = 1, mag\ (1, 0.25)$).

5. Conclusion:

The results that obtained from modeling the turbulent atmosphere as a kolmogorov statistics and then generating photon limited images of a binary star using different values of q, number of images, and magnitude difference indicate the following:

1.The mathematical behaviors of SNR & SBR of photon limited images of binary star for mag (1,1) and mag (1,0.25) are demonstrate and shown in figure - 7 & figure - 8 respectively.

2. SBR is independent of the total number of recorded photons of the limited image of binary star.

3. SBR for mag (1,1) is about 0.15 while SBR for mag (1,0.25) is about 0.05. This is because the signal in the autocorrelation function is decreases while the background remains unchanged.

4.SNR for mag(1,1) is about 5 times that of SNR for mag (1,0.25).

References:

1. Tatarski, V.I. **1961**.*Wave propagation in a turbulent medium*, McGraw-Hill, New York.

2. Rostami, M. Michailovich, O. and Wang. **2012**. *Image Deblurring using Derivative Compressed Sensing for Optical Image. Application*, 21 (7),PP:3139-3149.

3. Tubbs, R.N. **2003**. *Lucky exposure: Diffraction limited astronomical imaging through the atmosphere*", Ph.D. thesis, Cambridge University.
4. Buscher, D.F. , Armstrong, J.T. , Hummel, C.A. , Quirrenbach, A. Johnston, M.D. and Denston, K.J. **1995** . Interferometric seeing measurements on Mt. Wilson: Power spectra and outer scal. *Applied Optics* 34, pp:1081-1096.
5. Osborn, J. **2010**. Profiling the turbulent atmosphere and novel correction techniques for imaging and photometry in astronomy. Ph.D. thesis, Department of Physics, Durham University.
6. Jime'nez, C.O.F. **2006**. Understanding the atmospheric turbulence structure parameter, Cn, in the littoral regime. University of Puerto Rico Mayaguez Campus.
7. Burger, L. and Litvin, I.A. **2008**. Andrew forbes simulation atmospheric turbulence using phase-only spatial modulator. *South African J. Science,* 104, pp:129-134.
8. Vereshaging, N.K. and Vitanyi, P.M.B. **2004**. Kolmogrov's structure functions and model selection. *IEEE Transaction Theory,* p:50.
9. Dainty, J.C. **1974**. The transfer functions, signal to noise ratio, and limiting magnitude in stellar speckle interferometry. *Mon. Not. R. Ast. Soc.* 169, pp:631-41.
10. Mavroidis, T. ,Dainty, J.C. and Northcott, M.J. **1990**. Imaging of coherently illuminated objects through turbulence: Plane-wave illumination. *J. Opt. Soc. Am*. A. 7, pp: 348-355.
11. Mohammed, A.T. **2012**. Imaging through Atmospheric Turbulence using Kolmogorov and Gaussian Statistics. *J. physical Science and Application,* ol.2(4),pp:128-138.
12. Dainty, J.C. **1978**. Computer simulations of speckle interferometery of binary stars in the photon counting mode. *Mon. Not. R. Ast. Soc.* 183, pp:223-36.
13. Mohammed, A.T. , Rashid, N.M. and Sadik, A.R. **1990**. Computer simulations of astronomical objects as seen by ground based optical telescopes. *Optical and Lasers in Engineering* 12, pp: 233-243.
14. Mavroidis, T., Dainty, J.C. and Northcott, M.J. **1990**. Imaging of coherently illuminated objects through turbulence: Plane-wave illumination. *J. Opt. Soc. Am. A.* 7 , pp:348-355.
15. Goodman, J.W. .**2000**.*Statistical optics a Wiley-Interscience publication*.
16. . Lane, R.G. , Glindemann, A. and Dainty, J.C. **1992**. Simulation of a Kolmogorov phase screen. *Waves in Random Media,* 2, pp:209-224.
17. Fried, D.L. **1965** .Statistics of a geometric representation of wave front distribution.*J. Opt. Soc. Am.* 55, pp:1427-1435.
18. Fried, D.L. **1966**. Resolution through a randomly inhomogeneous medium for very long and very short exposures. *J. Opt. Soc. Am.* 56, pp:1372-1379.