On Goldie extending modules

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Abstract

Let R be an associative ring with identity and let M be a unital left R- module. E. Akalan, G. Birkenmeier and A. Tercan introduced the following concept. An R-module M is called a $\mathcal{G}$-extending module if for each submodule X of M, there exists a direct summand D of M such that $X \cap D$ is essential in both X and D. The main purpose of this work is to develop the $\mathcal{G}$-extending modules

Keywords: $\mathcal{G}$-extending modules.

حول مقاسات التوسع من النمط – $\mathcal{G}$

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الخلاصة:

لتكن R حلقة تجميعية ذات عنصر محايد و M مقاس احادي ايسر. اكلان ، بركنيماير و تركان قدموا مفهوم مقاس توسع من النمط $-\mathcal{G}$. يدعى المقاس M مقاس التوسع من النمط $-\mathcal{G}$ اذا كان لكل مقاس جزئي X من M ، توجد مركبة مجموع مباشر D من M بحيث ان $X \cap D$ جوهري في كل من X و D. الغرض الرئيسي من هذا البحث هو تطوير مفهوم المقاسات التوسيعية من المنط $-\mathcal{G}$.

Introduction:

For a module M, consider the following relations on the set of submodules of M : (i) $X \alpha Y$ if and only if there exists a submodule A of M such that $X \leq_e A$ and $Y \leq_e A$; (ii) $X \beta Y$ if and only if $X \cap Y \leq_e X$ and $X \cap Y \leq_e Y$. Note that β is an equivalent relation and is equivalent to a relation defined in [1] for right ideals of rings. It can be seen that a module M is extending (CS- module), if and only if for each submodule X of M, there exists a direct summand D of M such that $X \alpha D$, see [2] . Motivated by the characterization of the extending condition by α and Goldie and smith's use of β in [1] and [3]. A module M is called $\mathcal{G}$-extending (i.e. , Goldie extending) if and only if for each submodule X of M ,there exists a direct summand D of M such that $X \beta D$,see [2]. Clearly that every extending module is $\mathcal{G}$-extending.

In this paper , we give some results on α , β, and $\mathcal{G}$-extending modules.

In section one , we study the basic properties of α and β. We prove that if $f: M \longrightarrow N$ is an R-homomorphism such that $Y \beta B$, then $f^{-1}(Y)\ \beta\ f^{-1}(B)$, where $Y, B \leq N$, see (1.1). Also we give sufficient conditions under which a direct summand of a $\mathcal{G}$-extending module is $\mathcal{G}$- extending.

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In section two of the paper we give some characterizations of $\mathcal{G}$-extending module. For example we show that an R - module M is $\mathcal{G}$-extending if and only if for each submodule A of M , there exists a direct summand D of M such that $M=D\oplus D'$, for some submodule D' of M , $A\,\beta\, D$ and D' is a complement of A. See proposition (2.1).

1 - $\mathcal{G}$-extending modules

Following [2] , we say that an R-module M is $\mathcal{G}$-extending (i.e. , Goldie extending) if for each submodule X of M ,there exists a direct summand D of M such that $X\,\beta\, D$. In this section, we study the basic properties of β and give sufficient conditions under which a direct summand of a $\mathcal{G}$-extending module is $\mathcal{G}$-extending , we begin in remark:

Remark (1.1): Let M be an R- module. Then.

1- $A\,\beta\, M$ if and only if $A\leq_e M$.

2- $A\,\beta\,\{0\}$ if and only if $A=0$.

3- Let X and Y be submodules of M, then $X\,\beta\, Y$ if and only if each nonzero element in X has a nonzero multiple in Y and each nonzero element in Y has a nonzero multiple in X.

4- Let A_1 ,A_2,B_1 and B_2 be submodules of M if $A_1\,\beta\, B_1$ and $A_2\,\beta\, B_2$,then $(A_1\cap A_2)\,\beta\,(B_1\cap B_2)$.

5- Let $f: M\longrightarrow N$ be an R- homomorphism such that $Y\,\beta\, B$, then $f^{-1}(Y)\ \beta\, f^{-1}(B)$, where $Y, B\leq N$.

Note: The converse of (5) is not true in general as the following example show : Consider Z as Z-module and let $A=B=2Z$ clearly $2Z\ \beta\ Z$ but $0=\frac{2Z}{2Z}$ is not related to $\frac{Z}{2Z}$ by β .

Note: Let $f: M_1\longrightarrow M_2$ be an epimorphism if $X\,\beta\, Y$, then it is not necessary that $f(X)\,\beta\, f(Y)$ as the following example show:-

Let $\pi :Z \longrightarrow \frac{Z}{2Z}$ be the natural epimorphism clearly $2Z\ \beta\, Z$ but $\pi(2Z)$ is not related to $\pi(Z)$ by β

Proposition (1.2): Let X and Y be submodules of an R- module M if $X\,\beta\, Y$, then $\frac{X}{X\cap Y}$ and $\frac{Y}{X\cap Y}$ are singular.

Proof: Assume that $X\,\beta\, Y$, then $X\cap Y\leq_e X$ and $X\cap Y\leq_e Y$. Now consider the following short exact sequences:

$$0\longrightarrow X\cap Y \xrightarrow{i} X \xrightarrow{\pi 1} \frac{X}{X\cap Y} \longrightarrow 0$$

$$0\longrightarrow X\cap Y \xrightarrow{j} Y \xrightarrow{\pi 2} \frac{Y}{X\cap Y} \longrightarrow 0$$

Where i, j are the inclusion maps and π_1, π_2 are the natural epimorphisms.Since $X\cap Y\leq_e X$ and $X\cap Y\leq_e Y$, then we get $\frac{X}{X\cap Y}$ and $\frac{Y}{X\cap Y}$ are singular ,by [4 , p. 32].

By the second isomorphism theorm , we have the following corollary:

***C*orollary (1.3):** Let A and B be submodules of an R- module M such that $A\,\beta\, B$, then $\frac{A+B}{A}$ and $\frac{A+B}{B}$ are singular.

Proposition (1.4): Let $\{A_\alpha , B_\alpha \mid \alpha\in\wedge\}$ be a family of submodules of an R- module M such that $\{A_\alpha \mid \alpha \in\wedge\}$ is independent family in M and $A_\alpha\ \beta\ B_\alpha$,$\forall\ \alpha\in\wedge$ .Then $\{B_\alpha \mid \alpha\in\wedge\}$ is independent family in M and $(\bigoplus_{\alpha\in\wedge} A_\alpha)\ \beta\ (\bigoplus_{\alpha\in\wedge} B_\alpha)$.

Proof: We use the induction on the number of elements of $\wedge$. Suppose that the family has only two elements. , i. e. $\{A_1 , A_2\}$ is independent family in M , $A_1\ \beta\ B_1$ and $A_2\ \beta\ B_2$. Then by (1.1), we get: $(A_1\cap A_2)\,\beta\,(B_1\cap B_2)$. But $A_1\cap A_2= 0$, therefore $B_1\cap B_2= 0$, by remark (1.1). Hence $\{B_1,B_2\}$ is independent family.

By [4] ,we have $(A_1 \cap B_1) \oplus (A_2 \cap B_2) \leq_e (A_1 \oplus A_2)$ and $(A_1 \cap B_1) \oplus (A_2 \cap B_2) \leq_e (B_1 \oplus B_2)$. But $(A_1 \cap B_1) \oplus (A_2 \cap B_2) \leq (A_1 \oplus A_2) \cap (B_1 \oplus B_2) \leq (A_1 \oplus A_2)$,therefore $(A_1 \oplus A_2) \cap (B_1 \oplus B_2) \leq_e (A_1 \oplus A_2)$. Similarly we can show $(A_1 \oplus A_2) \cap (B_1 \oplus B_2) \leq_e (B_1 \oplus B_2)$. Hence $(A_1 \oplus A_2)\ \beta\ (B_1 \oplus B_2)$.
Now assume that the result is true for the case when the index set with n-1 elements. Now let { A_1, A_2, , A_n } be an independent family and assume that $A_i\ \beta\ B_i$, $\forall\ i = 1, \dots, n$. since $(A_n \cap (A_1 \oplus A_2 \oplus \dots \oplus A_{n-1}))\ \beta\ (B_n \cap (B_1 \oplus B_2 \oplus \dots \oplus B_{n-1}))$, and $(A_n \cap (A_1 \oplus A_2 \oplus \dots \oplus A_{n-1})) = 0$, then $\{B_1, B_2, \dots, B_n\}$ is an independent family and $(A_1 \oplus A_2 \oplus \dots \oplus A_{n-1} \oplus A_n)\ \beta\ (B_1 \oplus B_2 \oplus \dots \oplus B_{n-1} \oplus B_n)$, by using the previous cases. Finally, let $\{A_\alpha \mid \alpha \in \wedge\}$ be an independent family and $A_\alpha\ \beta\ B_\alpha$,$\forall\ \alpha \in \wedge$. Let M be a nonzero submodule of $\bigoplus_{\alpha \in \wedge} A_\alpha$ and let $0 \neq x$ be a nonzero element in M. So $x = a_1 + a_2 + \dots + a_n$,where $a_i \in A_{\alpha_i}$, $\forall\ i = 1, \dots, n$. Hence $M \cap (A_{\alpha_1} + A_{\alpha_2} + \dots + A_{\alpha_n}) \neq 0$. But $(\bigoplus_{i=1}^{n} A_{\alpha_i})\ \beta\ (\bigoplus_{i=1}^{n} B_{\alpha_i})$, therefore $(M \cap (A_{\alpha_1} \oplus A_{\alpha_2} \oplus \dots \oplus A_{\alpha_n}) \cap (B_{\alpha_1} \oplus B_{\alpha_2} \oplus \dots \oplus B_{\alpha_{n-1}} \oplus B_{\alpha_n})) \neq 0$.Hence $M \cap (\bigoplus_{\alpha \in \wedge} A_\alpha) \cap (\bigoplus_{\alpha \in \wedge} B_\alpha) \neq 0$ which means that $(\bigoplus_{\alpha \in \wedge} A_\alpha) \cap (\bigoplus_{\alpha \in \wedge} B_\alpha) \leq_e (\bigoplus_{\alpha \in \wedge} A_\alpha)$. In similar way one can prove that $(\bigoplus_{\alpha \in \wedge} A_\alpha) \cap (\bigoplus_{\alpha \in \wedge} B_\alpha) \leq_e (\bigoplus_{\alpha \in \wedge} B_\alpha)$.

Thus $(\bigoplus_{\alpha \in \wedge} A_\alpha)\ \beta\ (\bigoplus_{\alpha \in \wedge} B_\alpha)$

Note: Let A_1, A_2, B_1 and B_2 be submodules of an R- module If $A_1\ \beta\ B_1$ and $A_2\ \beta\ B_2$, then it is not necessary that $(A_1 + A_2)\ \beta\ (B_1 + B_2)$ as the following example show:-

Consider the Z- module $Z \oplus Z_2$. let $A_1 = A_2 = Z(\bar{2}, \bar{0})$ and $B_1 = Z(\bar{1}, \bar{0})$, $B_2 = Z(\bar{1}, \bar{1})$. One can easily show that $A_1\ \beta\ B_1$ and $A_2\ \beta\ B_2$. But $(A_1 + A_2)$ is not related to $(B_1 + B_2)$, where there exists a nonzero submodule $K = \{\bar{0}\} \oplus Z_2$ of $(B_1 + B_2)$ such that $(A_1 + A_2) \cap K = \{(\bar{0}, \bar{0})\}$.

Proposition (1.5): Let $\{M_\alpha \mid \alpha \in \wedge\}$ be a family of R- modules and let A_α and B_α be submodules of M_α ,$\forall\ \alpha \in \wedge$. Then $(\bigoplus_{\alpha \in \wedge} A_\alpha)\ \beta\ (\bigoplus_{\alpha \in \wedge} B_\alpha)$ if and only if $A_\alpha\ \beta\ B_\alpha$,$\forall\ \alpha \in \wedge$.

Proof: Suppose that $(\bigoplus_{\alpha \in \wedge} A_\alpha)\ \beta\ (\bigoplus_{\alpha \in \wedge} B_\alpha)$. Then $(\bigoplus_{\alpha \in \wedge} A_\alpha) \cap (\bigoplus_{\alpha \in \wedge} B_\alpha) \leq_e (\bigoplus_{\alpha \in \wedge} A_\alpha)$ and $(\bigoplus_{\alpha \in \wedge} A_\alpha) \cap (\bigoplus_{\alpha \in \wedge} B_\alpha) \leq_e (\bigoplus_{\alpha \in \wedge} B_\alpha)$. But $(\bigoplus_{\alpha \in \wedge} A_\alpha) \cap (\bigoplus_{\alpha \in \wedge} B_\alpha) = \bigoplus_{\alpha \in \wedge} (A_\alpha \cap B_\alpha)$,therefore $\bigoplus_{\alpha \in \wedge} (A_\alpha \cap B_\alpha) \leq_e (\bigoplus_{\alpha \in \wedge} A_\alpha)$ and $\bigoplus_{\alpha \in \wedge} (A_\alpha \cap B_\alpha) \leq_e (\bigoplus_{\alpha \in \wedge} B_\alpha)$. Hence $(A_\alpha \cap B_\alpha) \leq_e A_\alpha$ and $(A_\alpha \cap B_\alpha) \leq_e B_\alpha$, $\forall\ \alpha \in \wedge$.
Conversely, assume that $A_\alpha\ \beta\ B_\alpha$,$\forall\ \alpha \in \wedge$.Then $(A_\alpha \cap B_\alpha) \leq_e A_\alpha$ and $(A_\alpha \cap B_\alpha) \leq_e B_\alpha$, $\forall \alpha \in \wedge$,then $\bigoplus_{\alpha \in \wedge} (A_\alpha \cap B_\alpha) \leq_e (\bigoplus_{\alpha \in \wedge} A_\alpha)$ and $\bigoplus_{\alpha \in \wedge} (A_\alpha \cap B_\alpha) \leq_e (\bigoplus_{\alpha \in \wedge} B_\alpha)$,by[4]. So $(\bigoplus_{\alpha \in \wedge} A_\alpha) \cap (\bigoplus_{\alpha \in \wedge} B_\alpha) \leq_e (\bigoplus_{\alpha \in \wedge} A_\alpha)$ and $(\bigoplus_{\alpha \in \wedge} A_\alpha) \cap (\bigoplus_{\alpha \in \wedge} B_\alpha) \leq_e (\bigoplus_{\alpha \in \wedge} B_\alpha)$.Thus $(\bigoplus_{\alpha \in \wedge} A_\alpha)\ \beta\ (\bigoplus_{\alpha \in \wedge} B_\alpha)$.

Proposition(1.6): Let M be a $\mathcal{G}$-extending R-module and let N be a submodule of M such that the intersection of N with any direct summand of M is a direct summand of N. Then N is $\mathcal{G}$-extending**.**

Proof: Let $X \leq N \leq M$. Since M is $\mathcal{G}$-extending, then there exists a direct summand D of M such that $X\ \beta\ D$. By our assumption $N \cap D$ is a direct summand of N. By remark (1.1), we get $(N \cap X)\ \beta\ (N \cap D)$. Hence N is $\mathcal{G}$-extending

Let M be an R- module , we say that M satisfy **summand intersection property (SIP)** if the intersection of any two direct summands of M is a direct summand of M. See [5].

Corollary (1. 7): Let M be a $\mathcal{G}$-extending module with the SIP , then every direct summand of M is $\mathcal{G}$-extending.

Corollary (1.8): Let M be a $\mathcal{G}$-extending module if M is free over PID, then every direct summand of M $\mathcal{G}$-extending.

Let M be an R- module and let $N \leq M$. Recall that N is said to be **fully invariant** submodule if $f(N) \leq N, \forall f \in End(M)$. See [6].

Proposition(1.9): Every fully invariant submodule of $\mathcal{G}$-extending module is $\mathcal{G}$-extending.

Proof: Let M be a $\mathcal{G}$-extending module , let N be a fully invariant submodule of M and X be a submodule of N. Since M is $\mathcal{G}$-extending,then there exists a direct summand D of M such that $X \beta D$. i.e. there exists a submodule D' of M such that $M=D \oplus D'$.Now consider the projection map $\pi : M \longrightarrow D$,then $(1-\pi): M \longrightarrow D'$. Claim that $N=(N \cap \pi (M)) \oplus ((I-\pi)(M) \cap N)$, to show that, let $x \in N$, then $x=a+b$, $a \in D$ and $b \in D'$ .Now $\pi (x)=\pi (a+b) =a$ and $(1-\pi)(x)=b$. But N is fully invariant, therefore $\pi (x)=a \in f(N) \cap N$ and $(1-\pi)(x)= b \in (1-\pi)(M) \cap N$. Thus $N=(N \cap \pi (M)) \oplus ((I-\pi)(M) \cap N)= (N \cap D) \oplus (N \cap D')$ .Since $X \beta D$, then $X=(X \cap N) \beta (N \cap D)$.Thus by proposition(1.6) N is $\mathcal{G}$-extending

Let M be an R- module M. Recall that M is said to be

multiplication module if for each submodule X of M there exists an ideal I of R such that $X = I M$, see [7].

It is known that every submodule of a multiplication module is fully invariant , see [7] .

Corollary (1.10): Let M be a multiplication $\mathcal{G}$-extending R- module, then every submodule of M is $\mathcal{G}$-extending.

Let M be an R- module. Recall that M is said to be **Distributive module** if $A \cap (B+C)=(A \cap B)+(A \cap C)$,for all A,B,C submodules of M. See [8].

Proposition (1.11): Let M be a distributive $\mathcal{G}$-extending R – module, then every submodule of M is $\mathcal{G}$-extending.

Proof: Let N be a submodule of M and let X be a submodule of N. Since M is $\mathcal{G}$-extending, then there exists a direct summand D of M such that $M=D \oplus D'$, for some submodule D' of M and $X \beta D$. But M is distributive, therefore $N=(N \cap D) \oplus (N \cap D')$. By proposition (1.6) $X = (X \cap N) \oplus (D \cap N)$. Hence N is $\mathcal{G}$-extending.

Proposition (1.12): Let M be a $\mathcal{G}$-extending R-module and let N be a submodule of M such that $Z(\frac{M}{N})=0$, then N is a direct summand of M.

Proof: Since N is a submodule of M and M is $\mathcal{G}$-extending, then there exists a direct summand D of M such that $\frac{N}{N \cap D}$ and $\frac{D}{N \cap D}$ are singular, byremark (1.1). By the second isomorphism theorem $\frac{D}{N \cap D} \cong \frac{N+D}{N}$ Since $\frac{M}{N}$ is nonsingular, then $\frac{D}{N \cap D}$ is nonsingular. But the only submodule which is singular and nonsingular is the zero submodule. Therefore $D= N \cap D$. But D is closed ,Therefore $N=D$, by [4] . Thus N is a direct summand of M.

Using an argument similar to that used in the proof of propositions. (1.1), (1.2) , (1.3) , (1.4) and (1.5) one can get the following proposition for the relation α .

Proposition (1.13): Let M be an R- module. Then.

1- Let A_1 , A_2, , B_1 and B_2 be submodules of an R- module M . If $A_1 \alpha B_1$ and $A_2 \alpha B_2$, then $(A_1 \cap A_2) \alpha (B_1 \cap B_2)$.

2- Let $f: M \longrightarrow N$ be an R- homomorphism such that $A \alpha B$, then $f^{-1}(A) \alpha f^{-1}(B)$, where $A , B \leq N$.

3- Let A and B be submodules of an R- module M . If $A\,\alpha\, B$, then there exists a submodule C of M such that $\frac{C}{A}$ and $\frac{C}{B}$ are singular.

4- Let $\{A_i\ ,\ B_i \mid i \in I\}$ be a family of submodules of an R- module M such that $\{A_i \mid i \in I\}$ is independent family in M and $A_i\ \alpha\ B_i$, $\forall\ i \in I$. Then $\{B_i \mid i \in I\}$ is independent family in M and $(\bigoplus_{i\in I} A_i)\ \alpha\ (\bigoplus_{i\in I} B_i)$

5- . Let $\{M_i \mid i \in \wedge\}$ be a family of R- modules and let A_i and B_i be submodules of M_i , $\forall\ \alpha \in \wedge$. Then $(\bigoplus_{i\in I} A_i)\ \alpha\ (\bigoplus_{i\in I} B_i)$ if and only if $A_i\,\alpha\, B_i$, $\forall\ i \in I$.

2- A characterizations of $\mathcal{G}$-extending modules.

In this section, we give various characterizations of $\mathcal{G}$-extending modules. And give some sufficient conditions on M under which the direct sum of $\mathcal{G}$-extending modules is a $\mathcal{G}$-extending, we start this section by the following proposition:

Proposition (2.1): Let M be an R- module, then M is $\mathcal{G}$-extending if and only if for each submodule A of M , there exists a direct summand D of M such that $A\,\beta\, D$ and D' is a complement of A , where $M=D\oplus D'$, $D'\leq M$.

Proof: Suppose that M is $\mathcal{G}$-extending and let A be a submodule of M . Then there exists a direct summand D of M such that $A\,\beta\, D$. Let $M=D\oplus D'$, D' is a submodule of M. Since $A\cap D \leq_e A$, then $0=A\cap D\cap D' \leq_e A\cap D'$. So $A\cap D'=0$. Let B be a submodule of M such that $D' \leq B$ and $A\cap B= 0$. Since $A\cap D \leq_e D$, then $0=A\cap B\cap D\leq_e B\cap D$. So $D\cap B=0$. But D' is a complement of D , therefore $B=D'$. Thus D' is a complement of A. The converse is clear.

Theorem (2.2): Let M be an R- module. Then M is $\mathcal{G}$-extending if and only if for each direct summand A of the injective hull $E(M)$ of M , there exists a direct summand D of M such that $(A\cap M)\,\beta\, D$.

Proof: $(\Rightarrow)$ Clear.

$(\Leftarrow)$ Let A be a submodule of M and let B be a complement of A, then by [4, Prop. 1.3, P.17] , $A\oplus B\leq_e M$. Since $M \leq_e E(M)$, then $A\oplus B\leq_e E(M)$. Thus $E(A)\oplus E(B) = E(A\oplus B) = E(M)$. By our assumption, there exists a direct summand D of M such that $(E(A)\cap M)\,\beta\, D$. But $A\leq_e E(A)$, therefore $(A\cap M) \leq_e (E(A)\cap M)$ and hence $(A\cap M)\,\beta\,(E(A)\cap M)$. But β is transitive, therefore $(A\cap M)\ \beta\, D$.

Let M be an R- module, then M is $\mathcal{G}$-extending if and only if for each submodule A of M , there exists a direct summand D of M such that $A\,\beta\, D$ and D' is a complement of A , where $M=D\oplus D'$, $D'\leq M$.

In general the direct sum of $\mathcal{G}$-extending modules need not be $\mathcal{G}$-extending as the following example show.

Let $R = Z[x]$ be a polynomial ring of the ring of integers Z and let $M = Z[x] \oplus Z[x]$. Since Z is integral domain , then $Z[x]$ is integral domain and hence $Z[x]$ is $\mathcal{G}$-extending. Now , since $Z[x]$ is nonsingular , then M is nonsingular. But M is not extending module , by [9 , P. 109] , therefore M is not $\mathcal{G}$-extending , by [2].

Now , we give sufficient conditions under which the direct sum of $\mathcal{G}$-extending modules is a $\mathcal{G}$-extending.

Proposition (2.3): Let $M=M_1\oplus M_2$ be a distributive module , Then M is $\mathcal{G}$-extending if and only if M_1 and M_2 are $\mathcal{G}$-extending.

Proof: $(\Rightarrow)$ suppose that M is $\mathcal{G}$-extending. Since M is distributive , then by Prop. (1.11) M_1 and M_2 are $\mathcal{G}$-extending.

($\Leftarrow$) Let $M = M_1 \oplus M_2$ be a distributive module , M_1 and M_2 are $\mathcal{G}$-extending and let $A \leq M$. Since M is distributive, then $A = A \cap M = A \cap (M_1 \oplus M_2) = (A \cap M_1) \oplus (A \cap M_2)$. Since M_1 , M_2 are $\mathcal{G}$-extending , then there exists a direct summand D_1 of M_1 and direct summand D_2 of M_2 such that $(A \cap M_1)\, \beta\, D_1$ and $(A \cap M_2)\, \beta\, D_2$. Hence $A = (A \cap M_1) \oplus (A \cap M_2))\ \beta\ (D_1 \oplus D_2)$ and $D_1 \oplus D_2$ is a direct summand of M , by Prop. (1.4). Thus M is $\mathcal{G}$-extending.

Proposition(2.4): Let $M = \bigoplus_{i \in I} M_i$ be an R-module ,where M_i is a submodule of M ,$\forall\ i \in I$. If M_i is $\mathcal{G}$-extending , for each $i \in I$ and every closed submodule of M is fully invariant , then M is $\mathcal{G}$-extending.

Proof: Let A be a closed submodule of M and $\pi_i : M \longrightarrow M_i$ be the natural projection on M_i , for each $i \in I$. Let $x \in A$, then $x = \sum x_i$, $i \in I$, $x_i \in M$

$\pi_i(x) = x_i$. By our assumption, A is fully invariant and hence $\pi_i(A) \leq A \cap M_i$. So, $\pi_i(x) = x_i \in A \cap M_i$ and hence $x \in \bigoplus_{i \in I} (A \cap M_i)$. Thus $A \leq \bigoplus_{i \in I} (A \cap M_i)$. But $\bigoplus_{i \in I} (A \cap M_i) \leq A$, therefore $A = \bigoplus_{i \in I} (A \cap M_i)$, $\forall\ i \in I$ .Since $A \cap M_i \leq M_i$ and M_i is $\mathcal{G}$-extending , then there exists direct summands D_i of M_i such that $(A \cap M_i)\, \beta\, D_i$. By Prop. (2.1.9) $A = (\bigoplus_{i \in I} (A \cap M_i))\ \beta\ (\bigoplus_{i \in I} D_i)$, for each $i \in I$. Thus M is $\mathcal{G}$-extending.

Proposition(2.5): Let M_1 and M_2 be $\mathcal{G}$-extending modules such that $annM_1 + annM_2 = R$, then $M_1 \oplus M_2$ is $\mathcal{G}$-extending.

Proof: Let A be a submodule of $M_1 \oplus M_2$. Since $annM_1 + annM_2 = R$, then by the same way of the proof of [6,prop.4.2,CH.1] $A = B \oplus C$, where B is a submodule of M_1 and C is a submodule of M_2. Since M_1 and M_2 are $\mathcal{G}$-extending , then there exists a direct summands D_1 of M_1 and D_2 of M_2 such that $B\, \beta\, D_1$ and $C\, \beta\, D_2$, hence $A = (B \oplus C)\, \beta\ (D_1 \oplus D_2)$, by Prop. (1.4). Thus M is $\mathcal{G}$-extending.

Proposition (2.6): Let R be a PID , then the following statements are equivalent:

1- $\bigoplus_I R$ is $\mathcal{G}$-extending, for every index set I.

2- Every projective R- module is $\mathcal{G}$-extending.

Proof: (1)$\Rightarrow$(2) Let M be a projective R- module , then by [10, Corollary (4.4.4), p.89] ,there exists a free R- module F and an epimorphism $f : F \longrightarrow M$. Since F is free, then $F \cong \bigoplus_I R$, for some index set I. Now consider the following short exact sequence:

$$0 \longrightarrow Kerf \xrightarrow{\ i\ } \bigoplus_I R \xrightarrow{\ f\ } M \longrightarrow 0$$

Where i is the inclusion map . Since M is projective , then the sequence splits .Thus $\bigoplus_I R = Kerf \oplus M$. Since $\bigoplus_I R$ is free and R is PID, then by corollary (1.8) M is $\mathcal{G}$-extending.

(2)$\Rightarrow$(1) Clear.

By the same argument ,we can prove the following:

Proposition(2.7): Let R be a PID , then the following statements are equivalent:

1- $\bigoplus_I$ R is $\mathcal{G}$-extending, for every finite index set I.

2- Every finitely generated projective R- module is $\mathcal{G}$-extending.

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