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Common Fixed Points of Set –Valued Mappings on Partial b - Metric Spaces

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Abstract

In this work, the Banach contraction principle is generalized to a new formula. Then, it is used to give the guarantee conditions for the existence of common fixed point for set valued mappings defined on complete partial b-metric spaces. In addition, some results in connection with the existence of a fixed point for a set-valued mapping defined on a partial b-metric space are given. Some examples are established to illustrate our results.

Keywords: Partial b- metric space, Set-valued mapping, Common fixed point, Housdorff partial b-metric space and b-metric space.

النقاط الثابتة المشتركة للتطبيقات المتعددة في الفضاءات المترية الجزئية –b

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الخلاصة

في هذا العمل عممنا مبدأ باناخ الانكماشي الى صيغة جديدة واستخدمنا تلك الصيغة في اعطاء الشروط الضامنة لوجود النقطة الثابتة المشتركة للدوال المتعددة المعرفة على الفضاءات المترية الجزئية (b) . بعض النتائج التي تخص وجود النقطة الثابتة للتطبيق المتعدد درست باستخدام نفس الصيغة المطورة لمبدأ باناخ. درست بعض الامثلة التوضيحية للنتائج التي تم التوصل اليها.

1- Introduction

Mohammad Imdad et al. in 2014 [1] proved some new common fixed point theorems in metric spaces for weakly compatible mapping that satisfy an implicit relation under different sets of conditions.

The existence and uniqueness of coincidence and common fixed points for a pair of self-mappings defined on generalized metric spaces with a graph was proved by Karim Chaira et al. in 2019 [2].

In 2021 Yusra J. and Samira N. [3] studied the coincidence points theorems for two pairs mappings defined on a non-empty subset in a metric space by using the weakly compatible between these mappings .

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Ishak A. Damjano et al. in 2010 [4] proved some common fixed point theorems for self-mappings on ordered cone metric spaces in case the cone is not necessarily normal.
Czerwik in 1993 [5] introduced b-metric spaces by modifying the last property of the metric space. In 2019, Hessian et al. [6] established rational condition in b-metric spaces. In the same year, Qawaqneh H. et al. [7] studied some applications in b-metric space. Nikola Savanovi´c et al. in 2022[8] studied a hybrid pair of single and set-valued mappings and discussed the common fixed points in b-metric spaces. Mitrovi´ c Z. et al. in 2020 [9] presented some results about common fixed-point for quasi-contraction nonlinear mappings in b-metric spaces.
Badr A. et al. in 2021[10] discussed common fixed point for the Proinov type mapping via simulation function on b- metric spaces. Lili C. et al. in 2022 [11] devoted the existence and uniqueness of common fixed points for two mappings in complete b-metric spaces by virtue of the new functions F and θ.

In 1994, Matthews [12] introduced the notion of the partial metric space by adding the non-self-distance property and proved the Banach contraction principle in this space. In 2020 [13], the existences of a fixed point for set valued mappings with some modifications for the Banach construction principle in the Mathews' space was studied.

In 2013 Calogero V. and Francesca V. [14] defined new implicit relations of contractive conditions in partial metric spaces to study common fixed point theorems for two self-mappings. Sigh S. et al. in 1989 [15] gave new ideas for common fixed points of hybrid pair for single-valued and set-valued mappings.
Some theorems for the existence of a solution on the Voltera integral on dynamic programming in Mathews' space can be found in [16].

In [17] and [18], the integral inequality with some new classes is used to give the guaranteed conditions for the existence of a fixed point for set valued mappings on complete partial metric spaces.

In 2014, Shukla [19] defined a partial b-metric space by connecting with the Czerwiks space and Mathews' space.

In our work, we give some results and applications about a fixed point for a set valued mappings and a common fixed point for two set- valued mappings defined on the partial b-metric spaces.

2- Some Basic Concepts

The definitions of b-metric, partial metric space and partial b-metric space will be recalled. (see [5],[12],[19]).

Definition2-1[5]

The space (Y, d, s) is a b-metric space where $Y \neq \emptyset$ and $d: Y\, x\, Y \rightarrow [0, +\infty)$ if the following conditions satisfy:

$(Mb1)$ $d(u, v) = 0$ if and only if $u = v$ $for\ all$ $u, v \in Y$;

$(Mb2)$ $d(u,\ v) = d\,(v, u)$ for all $u, v \in Y$;

$(Mb3)$ $d(u, v) \leq s\,[\,d\,(u, z) + d\,(z\,, v)]$, where $s \geq 1$ and $u, v, z \in Y$.

Definition 2-2 [12]

The space (Y, p) is called a partial metric space where a set $Y \neq \emptyset$ and if a function p : $Y \times Y \rightarrow [0, +\infty)$ satisfies :

$p1$) Non-negativity and a self-distance , $0 \leq p\ (a, a) \leq p(a, b), a, b \in Y$,
$p2$) Indistancy implies equality, if $p\ (a, a) = p(b, b) = p(a, b)$ then $a = b\ ; a, b \in Y$,
$p3$) Symmetry, $p\ (a, b) = p\ (b, a)$,
$p4$) Triangular property, $p\ (a, c) \leq p\ (a, b) + p(b, c) - p(b, b)$. Where $a, b, c \in Y$.

Definition 2-3 [19]

A partial b-metric space(Y, pb) is a set Y and a function $pb : Y \times Y \rightarrow [0, +\infty)$ such that for all $u, v, w \in Y$, the following conditions hold:

$(Pb.1)$ $u = v$ if and only if $pb\ (u, u) = pb(u, v) = pb(v, v)$;
$(Pb.2)$ $Pb\ (u, u) \leq pb(u, v)$;
$(Pb.3)$ $pb(u, v) = pb\ (v, u)$;
$(Pb.4)$ $Pb\ (u, u) \leq s[pb(u, w) + pb(w, v)] - pb(w, w)$, $s \geq 1$ and $u, v, w \in Y$.

Definition 2-4[19]

The sequence $\{ y_n \}$ in (Y, Pb) is to be :

1- Converges if $\lim_{n \to \infty} pb(y_n, y) = pb(y, y)$, $y \in Y$
2- Cauchy if $\lim_{n,m \to \infty} pb(y_n, y_m)$ exists and is finite.
3- (Y, Pb) is complete if every Cauchy sequence satisfies:
$\lim_{n,m \to \infty} pb(y_n, y_m) = \lim_{n \to \infty} pb(y_n, y) = pb(y, y)$, $y \in Y$

Definition 2-5 [20]

Let $CBpb\ (Y)$ be all closed bounded subsets of (Y, pb), then for all $M, N \in CBpb(Y)$, Housdorff partial b-metric space which denoted by Hpb on (Y, pb) is defined by :

$Hpb\ (M, N) = \max \{\partial pb(M, N), \partial pb(N, M)\}$ such that
$\partial pb(M, N) = \sup \{ \beta pb(u, N) : u \in M\}$,
$\partial pb(N, M) = \sup \{ \beta pb(v, M) : v \in N\}$,
$\beta pb(u, N) = \inf\{pb(u, \grave{u}), \grave{u} \in N\}$.

3- Main Results

The Banach contraction principle is modified and used to prove the existence of a common fixed point for set- valued mappings on partial b- metric spaces.

Two essential lemmas in our work will be recalled.

Lemma 3-1[21]

Let (Y, pb) be a partial b-metric space and M, N are closed bounded subsets of Y.For all $u \in M$, then there exists $y = y(u) \in N$ and $k > 1$ with $pb(u, y) \leq k\partial pb(M, N)$.

For more details, see [21].

Lemma 3-2[21]

Let (Y, pb) be a partial b-metric space , and M, N, W three closed bounded subsets in Y then

a- $Hpb\ (M, M) \leq Hpb\ (M, N)$,
b- $Hpb\ (M, N) = Hpb\ (N, M)$,
c- $Hpb\ (M, N) \leq s[Hpb\ (M, W) + Hpb\ (W, N)] - \inf_{w \in W} pb(w, w)$.

For more information, see[22].

Two Set-valued mappings $G, W : Y \rightarrow CB(Y)$ defined on (Y, Pb) can have a common fixed point under a modification contraction condition as will be proved.

Theorem 3-3
Let (Y, pb) be a complete partial b- metric space and $G, W : Y \rightarrow CB(Y)$ be two set-valued mappings defined on Y which satisfy the contraction condition $Hpb(Gu, Wv) \leq k\, L(u, v)$ where $L(u, v) = \max\{ pb(u, v), pb(u, Gu), pb(v, Wv)\}$ and $k \in (0,1)$.Then, G, W have a common fixed point.

Proof:
Assume that $u_0 \in Y$, $u_1 \in Wu_0$,then the following two cases will be discussed :
First :
If $(u_0, u_1) = 0$, then $u_0 = u_1$, so $u_0 \in Wu_0$, hence u_0 is a fixed point for W, G .
Second :
If $L(u_0, u_1) > 0$, then by using Lemma (3-1) with $\rho > 0$ such that $\rho + k < 1$ there is $u_2 \in Gu_1$ such that

$$pb(u_2 , u) \leq Hpb(Gu_1 , Wu_0) + \rho L(u_0 , u_1). \quad (1)$$

By the same way , there is $u_3 \in Gu_2$ such that

$$pb(u_3 , u_2) \leq Hpb(Gu_2 , Wu_1) + \rho L(u_1 , u_2). \quad (2)$$

Thus, one can construct a sequence $\{u_m\}$ in Y such that
a- $u_{m+1} \in Wu_m$, $u_{m+2} \in Gu_{m+1}$.
b- $L(u_{m+1} , u_m) > 0$.
c- $pb(u_{m+1} , u_m) \leq Hpb\, (Gu_m, Wu_{m-1}) + \rho L(u_m , u_{m-1})$.
d- $pb\, (u_{m+2} , u_{m+1}) \leq Hpb\, (Gu_{m+1}, Wu_m) + \rho L(u_{m+1} , u_m)$

However, we have $Hpb\, (Gu_{m+1}, Wu_m) \leq kL(u_{m+1} , u_m)$

Thus, $pb(u_{m+2} , u_{m+1}) \leq kL(u_{m+1} , u_m) + \rho L(u_{m+1} , u_m) \leq (k + \rho)L(u_{m+1} , u_m)$.

Let $\omega = k + \rho < 1$.
Hence

$$\begin{aligned} pb(u_{m+1} , u_m) &\leq \omega\, L(u_{m-1} , u_m) \\ &= \omega \max\{ pb(u_{m-1} , u_m), pb(u_{m-1}, Gu_{m-1}), pb(u_m, Wu_m)\} \\ &\leq \omega \max\{ pb(u_{m-1} , u_m), pb(u_{m-1}, u_m), pb(u_m, u_{m+1})\} \\ &= \omega \max\{ pb(u_{m-1}, u_m), pb(u_m, u_{m+1})\}. \end{aligned}$$

That is $pb(u_{m+1} , u_m) \leq \omega \max\{ pb(u_{m-1}, u_m), pb(u_m, u_{m+1})\}$.

Now, the maximum value will be discussed as follows:
First: if $\max\{ pb(u_{m-1}, u_m), pb(u_m, u_{m+1})\} = pb(u_m , u_{m+1})$,

Then $pb(u_{m+1} , u_m) \leq \omega pb(u_m, u_{m+1}) < pb(u_{m+1} , u_m)$ that is contradiction.

Second: if $\max\{ pb(u_{m-1}, u_m), pb(u_m, u_{m+1})\} = pb(u_{m-1}, u_m)$,

then $pb(u_{m+1} , u_m) \leq pb(u_{m-1} , u_m)$.
Thus,

$$\begin{aligned} pb(u_{m+1} , u_m) &\leq \omega\, pb(u_{m-1} , u_m) \\ &\leq \omega^2 pb(u_{m-1} , u_{m-2}) \end{aligned}$$

$\leq \omega^3 pb(u_{m-2}, u_{m-3})$
$\vdots$
$\leq \omega^n pb(u_1, u_0)$.

Since $\omega = k + \rho < 1$, then $\{u_m\}$ represents Cauchy sequence.
Since (Y, pb) be a complete partial b- metric space, then $\{u_m\}$ converges to a point u in Y.
That is $\lim_{n\to\infty} pb(u_m, u) = pb(u, u)$.
Now:

$$pb(u_{m+2}, Wu) \leq Hpb(Gu_{m+1}, Wu))$$
$$\leq \omega\, L(u_{m+1}, u)$$
$$= \omega \max\{ pb(u_{m+1}, u), pb(u_{m+1}, Gu_{m+1}), pb(u, Wu)\}$$
$$\leq \omega \max\{ pb(u_{m+1}, u), pb(u_{m+1}, u_{m+2}), pb(u, Wu)\}.$$

By the last property of the partial b-metric spaces, one can get the following:

$$pb(u_m, Wu) \leq s[pb(u_m, u) + pb(u, Wu)] - pb(u, u)$$
$$\leq s[pb(u_m, u) + pb(u, Wu)]$$

But, $\lim_{m\to\infty} pb(u_m, Wu) = pb(u, Wu)$.
Then, $pb(u, Wu) \leq \omega pb(u, Wu)$.
Hence, $(u, Wu) = 0$, hence $u \in Wu$.
That is u is a fixed point for W.

Now, by following the same previous steps for G, we get:

$$pb(u_{m+2}, Gu) \leq Hpb(Gu_{m+1}, Wu))$$
$$\leq \omega\, L(u_{m+1}, u)$$
$$= \omega \max\{ pb(u_{m+1}, u), pb(u_{m+1}, Gu_{m+1}), pb(u, Wu)\}$$
$$\leq \omega \max\{ pb(u_{m+1}, u), pb(u_{m+1}, u_{m+2}), pb(u, Wu)\}$$
$$pb(u_m, Gu) \leq s[pb(u_m, u) + pb(u, Gu)] - pb(u, u) \leq s[pb(u_m, u) + pb(u, Gu)].$$

But, $\lim_{m\to\infty} pb(u_m, Gu) = pb(u, Gu)$.
Then, $pb(u, Gu) \leq \omega pb(u, Gu)$.
Hence, $(u, Gu) = 0$, hence $u \in Gu$.
That means u is fixed point for G.
Subsequently, u is common fixed point for W and G.
The same contraction condition can be used to get another important result.

Theorem 3-4

Let (Y, pb) be a complete partial b-metric space, $T : Y \to CB(Y)$ is a set-valued mapping that satisfies the following condition $Hpb(Tu, Tv) \leq k\, L(u, v)$, $L(u, v) = \max\{ pb(u, v), pb(u, Tu), pb(v, Tv)\}$, $k \in (0,1)$, then T has a fixed point.

Proof:

Let $u_0 \in Y$, $u_1 \in Tu_0$, then two cases will be discussed:

First : if $L(u_0, u_1) = 0$, then $u_0 = u_1$, so $u_0 \in Tu_0$, hence u_0 is a fixed point for T.
Second :
If $L(u_0, u_1) > 0$, then by using Lemma (3-1), with $\rho > 0$ such that $\rho + k < 1$ there is $u_2 \in Tu_1$ such that

$$pb(u_2, u) \leq pb(Tu_1, Tu_0) + \rho L(u_0, u_1). \quad (3)$$

As the same way, there is $u_3 \in Tu_2$ such that

$$pb(u_3, u_2) \leq Hpb(Tu_2, Tu_1) + \rho L(u_1, u_2). \quad (4)$$

Thus, one can construct a sequence $\{u_m\}$ in Y such that
a- $u_m \in Tu_{m-1}$, $u_{m+1} \in Tu_m$.
b- $L(u_m, u_{m-1}) > 0$.
c- $pb(u_m, u_{m-1}) \leq Hpb\,(Tu_{m-1}, Tu_{m-2}) + \rho L(u_{m-1}, u_{m-2})$.
d- $pb(u_{m+1}, u_m) \leq Hpb\,(Tu_m, Tu_{m-1}) + \rho L(u_m, u_{m-1})$.

But, $Hpb\,(Tu_m, Tu_{m-1}) \leq kL(u_m, u_{m-1})$.
Thus, $pb(u_{m+1}, u_m) \leq kL(u_m, u_{m-1}) + \rho L(u_m, u_{m-1}) \leq (k+\rho)L(u_m, u_{m-1})$.
Let $\omega = k + \rho < 1$. Hence,

$$\begin{aligned} pb(u_m, u_{m-1}) &\leq \omega\, L(u_{m-2}, u_{m-1}) \\ &= \omega \max\{pb(u_m, u_{m-1}), pb(u_{m-2}, Tu_{m-2}), pb(u_{m-1}, Tu_{m-1})\} \\ &\leq \omega \max\{pb(u_{m-2}, u_{m-1}), pb(u_{m-2}, u_{m-1}), pb(u_{m-1}, u_m)\} \\ &= \omega \max\{pb(u_m, u_{m-1}), pb(u_{m-1}, u_m)\}. \end{aligned}$$

That is, $pb(u_m, u_{m-1}) \leq \omega \max\{pb(u_m, u_{m-1}), pb(u_{m-1}, u_m)\}$.

Now, the maximum value will be discussed as follows:
First: if max $= pb(u_{m-1}, u_m)$, then $pb(u_m, u_{m-1}) \leq \omega pb(u_{m-1}, u_m) < pb(u_m, u_{m-1})$, contradiction.

Second: if max $= pb(u_m, u_{m-1})$, then $pb(u_m, u_{m-1}) \leq pb(u_{m-2}, u_{m-1})$
Thus,

$$\begin{aligned} pb(u_m, u_{m-1}) &\leq \omega\, pb(u_{m-2}, u_{m-1}) \\ &\leq \omega^2 pb(u_{m-2}, u_{m-3}) \\ &\leq \omega^3 pb(u_{m-3}, u_{m-4}) \\ &\vdots \\ &\leq \omega^n pb(u_1, u_0). \end{aligned}$$

Since $\omega = k + \rho < 1$, then $\{u_m\}$ is Cauchy sequence, and thus it converges to some point u in Y.
That is $\lim_{n\to\infty} pb(u_m, u) = pb(u, u)$.
Now:

$$\begin{aligned} pb(u_{m+1}, Tu) &\leq Hpb(Tu_m, Tu)\,) \\ &\leq \omega\, L(u_m, u) \\ &= \omega \max\{pb(u_m, u), pb(u_m, Tu_m), pb\,(u, Tu)\} \\ &\leq \omega \max\{pb(u_m, u), pb(u_m, u_{m+1}), pb(u, Wu)\}. \end{aligned}$$

By the last property for partial b-metric spaces, one can get :

$$pb(u_{m-1}, Tu) \leq s[\,pb\,(u_{m-1}, u) + pb\,(u, Tu)] - pb\,(u, u)$$
$$\leq s[\,pb(u_{m-1}, u) + pb\,(u, Tu)].$$

But, $\lim_{m\to\infty} pb(u_{m-1}, Tu) = pb(u, Tu)$. Then, $pb(u, Tu) \leq \omega pb(u, Tu)$.

Hence, $(u, Tu) = 0$, that is $u \in Tu$ which means u is a fixed point for T.

Another important result in complete partial b- metric space can be discussed in the following theorem.

Theorem 3-5

Let (Y,pb) be a complete partial b-metric space , $G:Y \to CB(Y)$ be a set -valued mapping. for any $u \in Y$ there is $y \in I_a^u = \{ \mathrm{y} \in \mathrm{G(y)}: \mathrm{a\,pb}(u,\mathrm{y}) \leq \mathrm{pb}(\mathrm{u},\mathrm{G(u)})\}$, $a \in (0,1)$, $I_a^u \subset Y$ satisfying
$pb(y,G(y)) \leq k\,pb(u,y)$, $k \in (0,1)$, then G has a fixed point in Y providing $k < a, g(y) = p(y, Gy)$ is a lower semi-continuous.

Proof:

For any initial point $y_0 \in Y$, there exists $y_1 \in I_a^{y_0}$ such that $pb(y_1, Gy_1) \leq kpb(y_0, y_1)$ and for $y_1 \in Y$, there is $y_2 \in I_a^{y_1}$ such that $pb(y_2, G\, y_2) \leq k\,pb(y_1, y_2)$.
Similarly , one can get a sequence $\{y_n\}_{n=0}^{\infty}$ such that $y_{n+1} \in I_a^{y_n}$ and

$$pb(y_{n+1}, G\,y_{n+1}) \leq k\,pb(y_n, y_{n+1}) \quad , \quad n = 0,1,2,\ldots \tag{5}$$

Since, $y_{n+1} \in I_a^{y_n}$, then, $apb(y_n, y_{n+1}) \leq pb(y_n, Gy_n)$
That is ,

$$pb(y_n, y_{n+1}) \leq \frac{1}{a} pb(y_n, Gy_n) \quad , \ n = 0, 1, 2, \ldots \tag{6}$$

By the inequalities (5) and (6), one can get the following inequality

$$pb(y_n, y_{n+1}) \leq \frac{k}{a}\, pb(y_{n-1}, Gy_n) \ , \ n = 0, 1, 2, \ldots$$

Now,

$$\begin{aligned} pb(y_n, y_{n+1}) &\leq \frac{k}{a}\, [\frac{k}{a}\ pb(y_{n-2}, Gy_{n-2})] \\ &\leq (\frac{k}{a})^2\, [\frac{k}{a}\, pb(y_{n-3}, Gy_{n-3})] \\ &\ \vdots \\ &\leq (\frac{k}{a})^n\, pb(y_0, G\,y_0) \ . \end{aligned}$$

For any $n, m \in N$, one can get

$$\begin{aligned} &pb(y_n, y_{(n+m)}) \\ &\leq s[\,pb(y_{\,n}, y_{\,(n+1)}) + pb(y_{\,(n+1)}, y_{\,(n+2)})] - pb(y_{\,(n+1)}, y_{\,(n+1)}) \\ &+ s[p(y_{\,(n+3)}, y_{\,(n+4)}) \\ &+ p(y_{\,(n+4)}, y_{\,(n+5)})] - p(y_{\,(n+4)}, y_{\,(n+4)}) + \cdots + s[p(y_{\,(n+m-2)}, y_{\,(n+m-1)}) \\ &+ p(y_{\,(n+m-1)}, y_{\,(n+m)})] \\ &- p(y_{\,(n+m-1)}, y_{\,(n+m-1)}) \ . \end{aligned}$$

Thus,

$$\begin{aligned} pb(y_{(n)}, y_{(n+m)}) &\leq [(\frac{k}{a})^{n} + (\frac{k}{a})^{n+1} + \cdots + \left(\frac{k}{a}\right)^{n+m-1}\Big]\, pb(y_0, G\,y_0) \\ &\leq \frac{(\frac{k}{a})^{n}}{1-\frac{k}{a}}\ pb(y_0, G\,y_0) \to 0 \ as \ n \to \infty \quad \text{due to} \quad \frac{k}{a} < 1 \ . \end{aligned}$$

$$\lim_{n,m\to\infty} pb(y_n, y_m) = \lim_{n\to\infty} pb(y_n, y_0) = pb(y_0, y_0) \ .$$

Furthermore, Since $\{g(y_n)\}_{n=0}^{\infty} = \{p\,b(y_n, Gy_n)\}_{n=0}^{\infty}$ is decreasing and bounded below, hence it converges to zero . By the lower semi-continuous for g ,

$$0 \leq g(y_0) \leq \lim_{n\to\infty} g(y_n) = 0 \ .$$

Hence, $g(y) = 0$ and then $p\,b(y_0, Gy_0) = 0.$
That is $y_0 \in \bar{G} \to y_0 = G\,y_0$.

Now, since $I_a^y \subset Gy$, one can conclude the following corollary .

Corollary 3-6

If (Y,pb) is complete partial b-metric space, $H: Y \to CB(Y)$ such that $pb(y,Hy) \leq kpb(u,y)$
for any $u \in Y, y \in Hu$ *and* $k \in (0,1)$, then H has a fixed point in Y.

4- Some Applications

For applications of the previous theorems, two examples will be studied in this section.

Example 4-1

For $Y = \left\{\frac{1}{5}, \frac{1}{5^2}, \frac{1}{5^3}, \dots, \frac{1}{5^n}, \dots\right\} \cup \{0,1\}$, Set $p_b(k,l) = \max\{k,l\}$ *for all* $k,l \in Y$
(Y,p_b) is a partial b-metric space with $s = 2$.
Now, define:

$$F(l) = \begin{cases} \left\{\frac{1}{5^i}, \frac{1}{5^{i+1}}\right\}, & l = \frac{1}{5^i}, i = 1,2,\dots \\ \{1\} & l = 1 \end{cases}$$

Take $w = \frac{1}{5} \in Y \Longrightarrow F(w) = F\left(\frac{1}{5}\right) = \left\{\frac{1}{5^i}, \frac{1}{5^{i+1}}\right\}, i = 1,2,\dots$
And $u = 1 \in Y \Longrightarrow F(u) = F(1) = \{1\}$.
$pb\left(w, F(w)\right) = \inf\{Pb\ (w, w_0), w_0 \in F(w)\} = \inf\{\max\{w, w_0\} = \frac{1}{5}$.
$P_b\left(u, F(u)\right) = \inf\{P_b\ (u, u_0), u_0 \in F(u)\} = 1.$
$H\left(F(w), F(u)\right) = \dfrac{1}{5}\ k \max\{Pb(u,w), Pb\left(u, F(u)\right),, Pb(\ w, F(v))\}$
$\max\{Pb(u,w), Pb\left(u, F(u)\right), Pb(\ w,\ F(\ v))\} = 1$.
So, if $k = \frac{2}{3} \in (0,1)$.

Then, $H\left(F(\ w\ ), F(u)\right) \leq\ k \max\{Pb\ (\ u,\ w\ ),, Pb\left(\ u, F(u)\right), Pb(\ w, F(v))\}$
Which means F has a fixed point in Y.

Example 4-2

Let $Y = \left\{\frac{1}{2}, \frac{1}{4}, \dots, \frac{1}{2^n}, \dots\right\} \cup \{0, 1\}$
$Pb(k,l) = max\{k,l\}$, for all $k,l \in Y$; Y is Complete Partial b- Metric Space.

define $G: Y \to CB(Y)$ as follows:

$$G(k) = \begin{cases} \left\{\frac{1}{2^n}, 1\right\}, & k = \frac{1}{2^n}, \quad n = 0,1,2,\dots \\ \left\{0, \frac{1}{2}\right\} & , k = 0 \end{cases}$$

Let $k = \frac{1}{2^n}$, $l = 0$, then $Pb(k,l) = max\{k,l\} = \frac{1}{2^n}$;

$Hpb(G(k), G(l)) = max\{\frac{1}{2^n}, 0\} = \frac{1}{2^n}$
$g(k) = pb(k, G\ k) = \inf\{pb(k,q), q \in Gk\} = inf\{max\ (k,q), q \in Gk\}$
Then

$$g(k) = \begin{cases} \frac{1}{2^n} & k = \frac{1}{2^n}, \quad n = 1,2,3,\dots \\ 0 & k = 0 \\ 1 & k = 1 \end{cases}$$

Hence, g is continuous.

Furthermore, by definition of $I_a^k = \{ q \in G(k): apb(k,q) \leq pb\big(k, G(k)\big) \}$;
There exists y $\in I_{0.2}^{q}$ such that $pb\ (q, G\ (q)\ ) = \frac{1}{2}\ pb(\ k, q)$.
Hence, by Theorem 3-5, the existence of a fixed point is guaranteed. ■

5-Conculosions

We have been shown in our work, the set valued mapping can have a fixed point in complete partial b-metric space if it is satisfied specific condition. Also, we have been proved that two set valued mappings can have a common fixed point in complete partial b-metric space if they are satisfied specific condition. Some applications are illustrated the previous results.

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