DOI: 10.24996/ijs.2024.65.9.32

ISSN: 0067-2904

New Formula on the Conjugate Gradient Method for Unconstrained Optimization and its Application

Ranen M. Sulaiman[1*], Zeyad M. Abdullah[2], Basim A. Hassan[1]
[1] *Department of Mathematics, College of Computers Sciences and Mathematics, University of Mosul, Iraq*
[2] *Department of Mathematics, College of Computers Sciences and Mathematics, University of Tikrit, Iraq*

Received: 9/6/2023 Accepted: 7/8/2023 Published: 30/9/2024

Abstract
In this paper, we have presented a new formula by applying the conjugacy condition for the second-order Taylor expansion. In comparison with classic conjugate gradient methods, the new formula uses both available gradient and function value information. The formula's global convergence findings are described. Numerical results demonstrate that this strategy is effective and its application.

Keywords: Conjugate gradient method, formula conjugate gradient, Global convergence.

صيغة جديدة لطريقة التدرج المترافق للأمثلية غير المقيدة وتطبيقها

رنين متي سليمان [1*], زياد محمد عبدالله [2] , د. باسم عباس حسن [1]
[1] قسم الرياضيات , كلية علوم الحاسوب والرياضيات , جامعة الموصل , العراق
[2] قسم الرياضيات , كلية علوم الحاسوب والرياضيات , جامعة تكريت , العراق.

الخلاصة
تمحور هذا البحث حول تقديم صيغة جديدة من خلال تطبيق شرط الترافق لتوسيع تايلور من الدرجة الثانية. وبالمقارنة مع طرائق التدرج المترافق التقليدية تستخدم الصيغة الجديدة معلومتي التدرج وقيمة الدالة المتاحة. وقد أثبتت هذه الصيغة تقاربها الشامل، كما بينت فعّالية استراتيجيتها بوضوح من خلال النتائج العددية مع تطبيقاتها.

1. Introduction

Numerous real-world applications have problems with non-linear optimization due to their complexity and size. The first-order methods are the best choice as a result. One of the first-order strategies that has continuously shown its effectiveness in tackling challenging constrained and unconstrained image processing problems which are the gradient methods,

Recently, a two-phase method was proposed in [1]. The combined benefits of the adaptive median filter and variational technique are to provide a two-phase system. For noise with salt and pepper, the adaptive-median filter is used in the first-phase. Let X and $A = \{1,2,3, \dots, M\} \times \{1,2,3, \dots, N\}, M = N = 265$ be the true image and index set of X, respectively, and let $N \subset A$ be the set of indices of the noise pixels found in the first phase. Then, the second phase figures up a workable way to decrease the functional as follows:

* Email: ranen.sulyman@uomosul.edu.iq

$$f_\alpha(u) = \sum_{(\kappa,j)\in N} \left[\left| u_{\kappa,j} - y_{\kappa,j} \right| + \frac{\beta}{2}(2 \times S^1_{\kappa,j} + S^2_{\kappa,j}) \right] \quad \text{..........(1)}$$

where β is the regularization parameter. $S^1_{\kappa,j} = 2\sum_{(m,n)\in p_{\kappa,j}\cap N^c} \varphi_\alpha\left(u_{\kappa,j} - y_{m,n}\right)$, $S^2_{\kappa,j} = \sum_{(m,n)\in p_{\kappa,j}\cap N} \varphi_\alpha\left(u_{\kappa,j} - y_{m,n}\right)$ while $\varphi_\alpha = \sqrt{\alpha + x^2}, \alpha > 0$ represents the edge-preserving potential function. Let $p_{\kappa,j}$ be the four pixels that are the nearest neighbors to the pixel at position $(\kappa, j) \in A$, $y_{\kappa,j}$ stands for the observed pixel value of the image at position $u_{\kappa,j} = \left|u_{\kappa,j}\right|_{(\kappa,j)\in N}$, and c stands for a column-vector of length F ordered lexicographically. In this case, c stands for N which is component count. The functional of problem (1) is not smooth due to the term $\left|u_{\kappa,j} - y_{\kappa,j}\right|$. Since it maintains the minimizer u near to the original picture y, it preserves the original image's unmodified pixels, it is generally agreed that this non-smooth component may be removed from (1). Thus, the smooth-functional that results is as follows:

$$f_\alpha(u) = \sum_{(\kappa,j)\in N} \left[2 \times S^1_{\kappa,j} + S^2_{\kappa,j}\right] \quad \text{..........(2)}$$

The iterative approach to obtain the minimal value of the following issues is addressed by unconstrained optimization algorithms:

$$f(u^*) = \min_{x\in R^N} f(u) \quad \text{..........(3)}$$

where $f: R^n \to R$ is a smooth function. For more details see [2].

Amongst numerous iterative methods for solving (3), the conjugate gradient methods constitute a popular type of method. Let g_i be the gradient of $f(x_i)$, then the iterative method to solve (3) is given by:

$$u_{i+1} = u_i + \alpha_i d_i \quad \text{..........(4)}$$

where α_i denotes the step-length and d_i denotes the search direction. α_i can be computed by:

$$\alpha_i = -\frac{g_i^T d_i}{d_i^T G d_i} \quad \text{..........(5)}$$

Alternatively, various line search conditions such as Wolfe conditions [3], [4,5] can be used:

$$f_{i+1} \le f_i + \delta \alpha_i g_i^T d_i \quad \text{..........(6)}$$

$$d_i^T g_{i+1} \ge \sigma d_i^T g_i \quad \text{..........(7)}$$

where $0 < \delta < \sigma < 1$, see [3], [6]. The search direction d_{i+1} is defined by:

$$d_{i+1} = -g_{i+1} + \beta_i d_i \quad \text{..........(8)}$$

where β_i is referred to as the conjugate gradient coefficient. The conjugate gradient coefficient is an essential factor in the conjugate gradient techniques. Other formulae to obtain the conjugate parameters are also given in [6]–[15] .

We focus our attention on the best-known the Hestenes-Stiefel (called HS method) method in which the formula is given as follows:

$$\beta_i^{HS} = \frac{y_i^T g_{i+1}}{d_i^T y_i} \quad \text{..........(9)}$$

where $y_i = g_{i+1} - g_i$. The most essential conjugate-gradient methods include the Fletcher-Reeves (called FR method) in which the formula is given by:

$$\beta_i^{FR} = \frac{g_{i+1}^T g_{i+1}}{g_i^T g_i} \qquad \text{..........(10)}$$

More details can be found in [16].

A little modification to the Hestenes-Stiefe to satisfy the descent property, specifically Wu and Chen (called WC method) [6] derived the following CG formula, in which the formula is defined by:

$$\beta_i^{WC} = \frac{y_{i+1}^T g_{i+1}}{d_i^T y_i} + \frac{2(f_i - f_{i+1}) + g_i^T v_i}{d_i^T y_i} \qquad \text{..........(11)}$$

The formula, which satisfies the Perry [12] conjugacy condition is specified by:

$$d_{i+1}^T y_i = -v_i^T g_{i+1} \qquad \text{..........(12)}$$

As a result, attempts have been undertaken to design conjugate gradient algorithms that are both globally and numerically efficient.
Furthermore, we state that the search directions d_{i+1} meet the adequate descent requirement if and only if:

$$g_{i+1}^T d_{i+1} \leq -c\|g_{i+1}\|^2 \qquad \text{..........(13)}$$

where c is a nonnegative constant. The convergence study of our proposed method is aided by the performance profile [13]. According to numerical measures, Wu and Chen [6] perform better than the family of current ones. The effective numerical results inspired our idea to consider other modifications which we believe will improve the numerical performance. Much effort in both analytic and computational has been devoted to identifying the best optimization method or even the best from the much wider class of optimization methods that are introduced in [14]–[19].
We use the second-order Taylor expansion to derive a novel parameter conjugate gradient. The derived approach investigates theoretical analysis as well as numerical outcomes.

2. Driving the new formula on the Conjugate Gradient method

Derivation from Taylor expansion. Take the second-order Taylor expansion around the point:

$$f(x_k) = f(x_{k+1}) - g_{i+1}^T v_i + \frac{1}{2} v_i^T G_{i+1} v_i \qquad \text{..........(14)}$$

To see this, the gradient of a second-order Taylor expansion form:

$$v_i^T g_i = v_i^T g_{i+1} - v_i^T G_{i+1} v_i \qquad \text{..........(15)}$$

Using exact line search $g_{i+1}^T v_i = 0$ and noting that $\alpha_i d_i = v_i$, we obtain:

$$f(x_k) = f(x_{k+1}) + \frac{1}{2} v_i^T G_{i+1} v_i \qquad \text{..........(16)}$$

From above equation and equation (5), which implies that:

$$d_i^T G\, v_i = -\frac{\alpha_i (g_i^T d_i)^2}{2(f_{i+1} - f_i)} \qquad \text{..........(17)}$$

From equation (15) and equation (12), we have:

$$d_{i+1}^T y_i = -\frac{\alpha_i^2 \left(g_i^T d_i\right)^2}{2\left(f_{i+1} - f_i\right)} - v_i^T g_i \quad \text{..........(18)}$$

In addition, $d_{i+1} = -g_{i+1} + \beta_i d_i$ and (18) imply that:

$$-g_{i+1}^T y_i + \beta_i d_i^T y_i = -\frac{\alpha_i^2 \left(g_i^T d_i\right)^2}{2\left(f_{i+1} - f_i\right)} - v_i^T g_i$$

$$\beta_i d_i^T y_i = \frac{\alpha_i^2 \left(g_i^T d_i\right)^2}{2\left(f_{i+1} - f_i\right)} - v_i^T g_i + g_{i+1}^T y_i \quad \text{..........(19)}$$

Which yields :

$$\beta_i = \frac{\alpha_i^2 \left(g_i^T d_i\right)^2 / 2\left(f_{i+1} - f_i\right)}{d_i^T y_i} - \frac{v_i^T g_i}{d_i^T y_i} + \frac{g_{i+1}^T y_i}{d_i^T y_i} \quad \text{..........(20)}$$

We call this formula the RZB. The RZB conjugate gradient algorithm is given as follows.

Algorithm RZB.

Initialization. Given $x_0 \epsilon R^n$ and the parameters $\varepsilon > 0, \delta\epsilon(0,1), \sigma\epsilon(\delta, 1)$, Set $i = 0$, $d_0 = g_0$.

Stage 1. If $\|g_i\| \leq \varepsilon$ then stop.

Stage 2. Compute α_i by a suitable line-search.

Stage 3. Let $x_{i+1} = x_i + \alpha_i d_i$ and compute β_i by (20).

Stage 4. Calculate the search direction $d_{i+1} = -g_{i+1} + \beta_i d_i$.

Stage 5. Let $i = i + 1$ and goto stage 2.

3. Convergence analysis

In this part, we turn to the convergence property of the RZB method. We assume that:

i. $f(x)$ is bounded on the set $\Psi = \{x \epsilon R^n : f(x) \leq f(x_0)\}$.

ii.. g is Lipschitz continuous, i.e. there exists a positive constant L such that:

$$\|g(z) - g(u)\| \leq L\|z - u\| , \forall z, u \epsilon R^n \quad \text{..........(21)}$$

Underneath these assumptions on the function, there exists a constant $\psi \geq 0$ such that $\|\nabla f(x)\| \leq \psi$. See [19].

A significant role can be demonstrated for the sufficient descent condition.

Theorem 1:

If d_{i+1} is generated by the new Algorithm, then $d_{i+1}^T g_{i+1} \leq -c\|g_{i+1}\|^2$ holds.

Proof:

Since $d_0 = -g_0$, we obtain $g_0^T d_0 = -\|g_0\|^2$. Let $g_i^T d_i < -c\|g_i\|^2$ for all $i \in n$. Now, multiplying (8) by g_{i+1}^T, we obtain:

$$d_{i+1}^T g_{i+1} \leq -\|g_{i+1}\|^2 + \left[\frac{\alpha_i^2 \left(d_i^T g_i\right)^2 / 2(f_{i+1} - f_i)}{d_i^T y_i} - \frac{v_i^T g_i}{d_i^T y_i} + \frac{g_{i+1}^T y_i}{d_i^T y_i}\right] d_i^T g_{i+1} \quad \text{..........(22)}$$

From (22) and (17), we have:

$$d_{i+1}^T g_{i+1} \leq -\|g_{i+1}\|^2 + \left[\frac{g_{i+1}^T y_i}{v_i^T y_i} + \frac{v_i^T g_{i+1}}{v_i^T y_i}\right] v_i^T g_{i+1} \qquad \text{..........(23)}$$

From above equation, we get:

$$d_{i+1}^T g_{i+1} \leq -\|g_{i+1}\|^2 + \frac{g_{i+1}^T y_i}{v_i^T y_i} + \frac{g_{i+1}^T y_i (v_i^T g_{i+1})(v_i^T y_i)}{(v_i^T y_i)^2} - \frac{(v_i^T g_{i+1})^2}{(v_i^T y_i)} \qquad \text{..........(24)}$$

Using the inequality:

$$u_i^T v_i \leq \frac{1}{2}\left[\|u_i\|^2 + \|s_i\|^2\right], u_i, s_i \in R^n \qquad \text{..........(25)}$$

to derive that, let

$$u_i = g_{i+1}(v_i^T y_i) \qquad , \qquad s_i = (v_i^T g_{i+1}) y_i$$
$$y_i^T g_{i+1}(v_i^T g_{i+1})(v_i^T y_i) \leq \frac{1}{2}\left[\|g_{i+1}\|^2 (v_i^T y_i)^2 + \|y_i\|^2 (v_i^T g_{i+1})^2\right] \qquad \text{..........(26)}$$

So, from (24) and (26), we obtain:

$$g_{i+1}^T d_{i+1} = g_{i+1}^T g_{i+1} + \frac{\frac{1}{2}\left[\|g_{i+1}\|^2 (v_i^T y_i)^2 + \|y_i\|^2 (v_i^T g_{i+1})^2\right]}{(v_i^T y_i)^2} - \frac{(v_i^T g_{i+1})^2}{(v_i^T y_i)^2} v_i^T y_i$$
$$\leq -\left[1 - \frac{1}{2}\right]\|g_{i+1}\|^2 + \left[\frac{1}{2}\|y_i\|^2 - (v_i^T y_i)\right]\frac{(v_i^T g_{i+1})^2}{(v_i^T y_i)^2} \qquad \text{..........(27)}$$
$$\leq -\frac{1}{2}\|g_{i+1}\|^2 + \left[\frac{1}{2}\|y_i\|^2 - (v_i^T y_i)\right]\frac{(v_i^T g_{i+1})^2}{(v_i^T y_i)^2}$$

Using (21), we get, $y_i^T y_i \leq L v_i^T y_i$ then:

$$g_{i+1}^T d_{i+1} \leq -\frac{1}{2}\|g_{i+1}\|^2 \leq -c\|g_{i+1}\|^2 \qquad \text{..........(28)}$$

This completes the proof.

Using the Wolfe conditions, the next lemma that is proved in [20] for any conjugate gradient method.

Lemma 1:
Suppose that assumptions (i) and (ii) hold and consider any conjugate gradient method (5) where $d_{i+1} = -g_{i+1} + \beta_1 d_i$ is supposed to be a descent-direction and α_i is selected by the strong-Wolfe line-search. If

$$\sum_{k>1} \frac{1}{\|d_{i+1}\|^2} = \infty, \qquad \text{..........(29)}$$

then

$$\lim_{k\to\infty}\left(inf\|g_{i+1}\|\right)=0. \qquad \text{..........(30)}$$

The proof can be found in [20].

The next theorem presents the convergence property result of the new Algorithm.

Theorem 2:
Assume that the previous assumptions are true. If f is a uniformly convex function on R^n that means if there is a constant $\mu > 0$ such that:

$$(\nabla f(z) - \nabla f(u))^T \geq \mu \|z-u\|^2 \ , \ \forall z, u \in R^n, \qquad ....(31)$$

then

$$\lim_{k\to\infty}\left(inf\|g_{i+1}\|\right) = 0. \qquad ....(32)$$

Proof:
From the definition of β_i and (17), it follows that:

$$\begin{aligned}|\beta_i| &= \left[\frac{\alpha_i^2\left(g_i^T d_i\right)^2/2(f_{i+1}-f_i)}{d_i^T y_i} - \frac{v_i^T g_i}{d_i^T y_i} + \frac{g_{i+1}^T y_i}{d_i^T y_i}\right] \\ &= \left|\frac{g_{i+1}^T y_i}{d_i^T y_i} - \frac{v_i^T g_{i+1}}{d_i^T y_i}\right|\end{aligned} \qquad ....(33)$$

By using (21), we get $\|y_k\| \leq L\|v_k\|$ and using (31), we obtain $v_i^T y_i \geq \mu\|v_k\|^2$. Applying the Cauchy inequality, of the above inequalities, then (33), we get:

$$\begin{aligned}|\beta_i| &\leq \frac{L\|v_i\|\|g_{i+1}\|}{\mu\|v_i\|^2} + \frac{\|v_i\|\|g_{i+1}\|}{\mu\|v_i\|^2} \\ &\leq \left(\frac{L+1}{\mu}\right)\frac{\psi}{\|v_i\|}\end{aligned} \qquad ....(34)$$

Therefore, using (34) in (8), we get:

$$\begin{aligned}\|d_{i+1}\| &\leq \|g_{i+1}\| + |\beta_i|\|v_i\| . \\ &\leq \psi + \left(\frac{L+1}{\mu}\right)\frac{\psi}{\|v_i\|}\|v_i\| \\ &\leq \psi + \frac{\psi L + \psi}{\mu} \leq \frac{\psi(\mu + L + 1)}{\mu}\end{aligned} \qquad ....(35)$$

Therefore, the equation (27) is true.

4. Numerical results

This section will outline the numerical tests with a few well-known optimization issues and their applications to issues with picture restoration. All of the tests are programmed in Matlab R2014a and they were run on a computer Intel (R) Core(TM) i7-6700 with a 2.50 GHz CPU and 4.00 GB of RAM.

1.Normal Unconstrained Optimization Problems.

The numerical performance of the RZB and the Fletcher-Reeves algorithms is evaluated in this section. Algorithms are implemented in Fortran language. The following parameters are used in our implementation $\delta = 0.001$ and $\sigma = 0.9$. We used the termination criterion as$\|g_{i+1}\| \leq 10^{-6}$.

The performance profile by Andrei [21,] is used to display the performance of the algorithms. The number of iterations, the number of objective function evaluations and the number of the restart are denoted by NI, NR and NF, respectively, and are used to compare numerical performance.

Throughout the numerical behavior comparisons with FR-CG, this method is shown to be efficient for the test problems.

Table 1: Using multiple test functions to compare different conjugate gradient algorithms with n=100.

		FR algorithm			RZB algorithm		
P. No.	**n**	**NI**	**NR**	**NF**	**NI**	**NR**	**NF**
1	100	19	12	35	18	9	34
2	100	47	18	93	38	21	80
3	100	43	18	88	32	16	71
4	100	32	15	52	13	7	26
5	100	10	6	27	13	8	31
6	100	95	33	150	100	27	151
7	100	25	11	43	22	6	44
8	100	32	13	64	12	6	25
9	100	15	6	25	18	10	29
10	100	37	8	67	44	19	65
11	100	12	5	25	10	6	19
12	100	15	9	31	8	6	17
13	100	89	32	174	71	33	169
14	100	124	41	231	54	8	98
15	100	71	35	110	32	13	60
16	100	101	40	217	79	50	174
17	100	108	41	174	103	32	157
18	100	11	6	26	10	7	24
19	100	32	12	65	24	13	56
20	100	130	49	196	111	36	170
21	100	121	65	218	90	30	139
22	100	74	21	123	83	25	129
23	100	69	50	1202	31	15	179
24	100	23	11	45	21	12	40
25	100	20	11	33	12	7	21
26	100	49	22	80	14	10	25
27	100	12	7	25	7	5	15
28	100	122	62	156	12	8	21
29	100	112	55	147	38	13	59
30	100	11	5	23	6	3	13
Total		1661	719	3945	1126	461	2141

Table 2: Using multiple test functions to compare different conjugate gradient algorithms with n=1000.

		FR algorithm			RZB algorithm		
P. No.	**n**	**NI**	**NR**	**NF**	**NI**	**NR**	**NF**
1	1000	38	23	65	34	19	62
2	1000	78	45	131	34	18	76
3	1000	46	19	92	35	20	79
4	1000	22	10	42	14	8	27
5	1000	24	16	191	22	13	47
6	1000	349	95	568	337	86	522
7	1000	46	28	741	1000	28	15
8	1000	77	46	129	13	7	26
9	1000	127	117	3531	112	104	3077
10	1000	73	27	115	69	30	108
11	1000	14	6	29	11	6	21
12	1000	8	6	17	7	5	15
13	1000	107	40	211	116	76	514
14	1000	445	196	711	173	33	303
15	1000	47	15	84	28	10	54
16	1000	101	40	214	77	49	167
17	1000	313	76	520	367	109	579
18	1000	16	12	125	8	6	21
19	1000	53	22	116	35	20	85
20	1000	364	119	593	353	108	552
21	1000	345	169	634	251	68	399
22	1000	370	88	616	304	74	506
23	1000	98	82	1967	39	24	397
24	1000	27	11	55	20	12	45
25	1000	19	11	35	9	6	18
26	1000	129	67	166	12	9	23
27	1000	11	7	23	7	5	15
28	1000	130	66	166	12	8	23
29	1000	110	54	145	31	9	54
30	1000	11	5	23	6	3	13
Total		3598	1518	12055	3536	973	7843

Problems-numbers indicant for:“1. is the Trigonometric, 2. is the Extended Rosenbrock, 3. is the Extended-White & Holst, 4. is the Extended-Beale, 5. is the Penalty, 6. is the Perturbed-Quadratic, 7. is the Generalized-Tridiagonal 1, 8. is Extended-Tridiagonal 1, 9. is the Extended Three Expo Terms, 10. is the Generalized Tridiagonal 2,11 is the Extended-Himmelblau, 12. is the Extended-PSC1,13. is the Extended-Maratos, 14. is the Quadratic-Diagonal-Perturbed, 15. is the Extended-Wood, 16. is the Extended-Hiebert, 17. is the-Quadratic QF1,18 is the Extended-Quadratic-Penalty QP1, 19. is Extended-Quadratic Penalty QP2, 20. is the Quadratic QF2, 21. is the DIXMAANE (CUTE), 22. is the Partial Perturbed-Quadratic, 23. is the EDENSCH-(CUTE), 24. is the LIARWHD-(CUTE), 25. is the DENSCHNA-(CUTE),26. is the DENSCHNC-(CUTE), 27. is the-DENSCHNB-(CUTE), 28.

is the Extended-Block-Diagonal, 29.is the Generalized-quartic GQ2, 30. is the HIMMELBH-(CUTE)".

2. Image Restoration Problems.

The aforementioned algorithms will be used to solve issues with picture restoration in this subsection. Here, the impulse noise-damaged original pictures are treated as objects. The results of applying the suggested BNN and FR methods on the test pictures, namely, Lena, House, Elaine and Cameraman are provided in Table 3 in order to assess how well algorithms work in practice. The technique was applied using Matlab, with the following stopping criteria:

$$\frac{|f(u_k)-f(u_{k-1})|}{|f(u_k)|} \leq 10^{-4} \quad \text{and} \quad \|f(u_k)\| \leq 10^{-4}(1+|f(u_k)|) . \qquad ....(36)$$

More details can be found in [1], [22]. There is modern research in the field of mathematics, see [23 -24].

The pictures that were put to the test are shown in Table 1, and the precise numerical results of our testing are presented in the format NI/NF/PSNR, where NI, NF, and PSNR stand for the number of iterations, function evaluations, and peak signal to noise ratio, or PSNR (peak signal to noise ratio), which is defined as:

$$PSNR = 10\, log_{10} \frac{255^2}{\frac{1}{MN}\sum_{i,j}(u_{i,j}^r - u_{i,j}^*)^2} \qquad ....(37)$$

where the restored and original pictures' pixel values are shown by $u_{i,j}^r$ and $u_{i,j}^*$, respectively

Table 3: Performance of FR and RZB methods.

Image	Noise level r (%)	FR Method			RZB-Method		
		NI	NF	PSNR (dB)	NI	NF	PSNR (dB)
Lena	50	82	153	30.5529	57	102	30.6121
	70	81	155	27.4824	59	100	27.3595
	90	108	211	22.8583	63	99	22.7329
House	50	52	53	30.6845	38	67	34.9191
	70	63	116	31.2564	46	74	31.0837
	90	111	214	25.287	52	82	25.1628
Elaine	50	35	36	33.9129	31	51	33.9109
	70	38	39	31.864	36	52	31.8718
	90	65	114	28.2019	41	77	28.1123
Cameraman	50	59	87	35.5359	40	71	35.5251
	70	78	142	30.6259	42	75	30.6805
	90	121	236	24.3962	56	95	24.7723

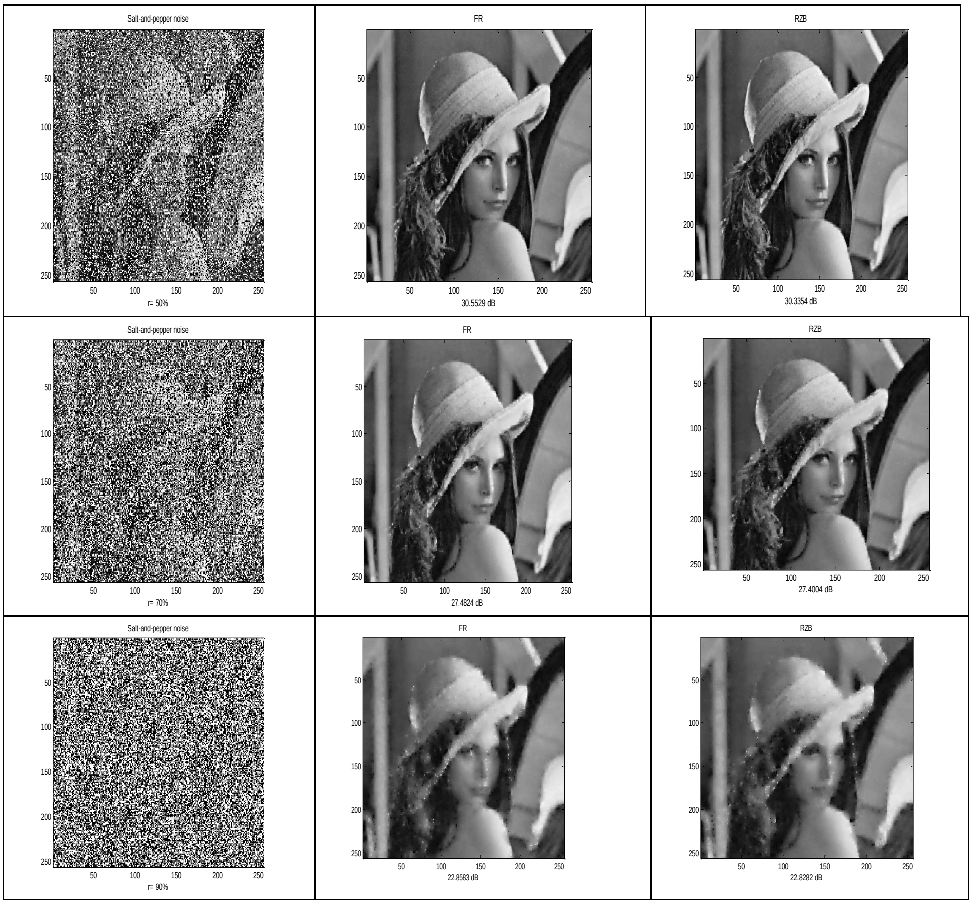

Figure 1: Demonstrates theresults of methods FR and RZB of 256 * 256 Lena image

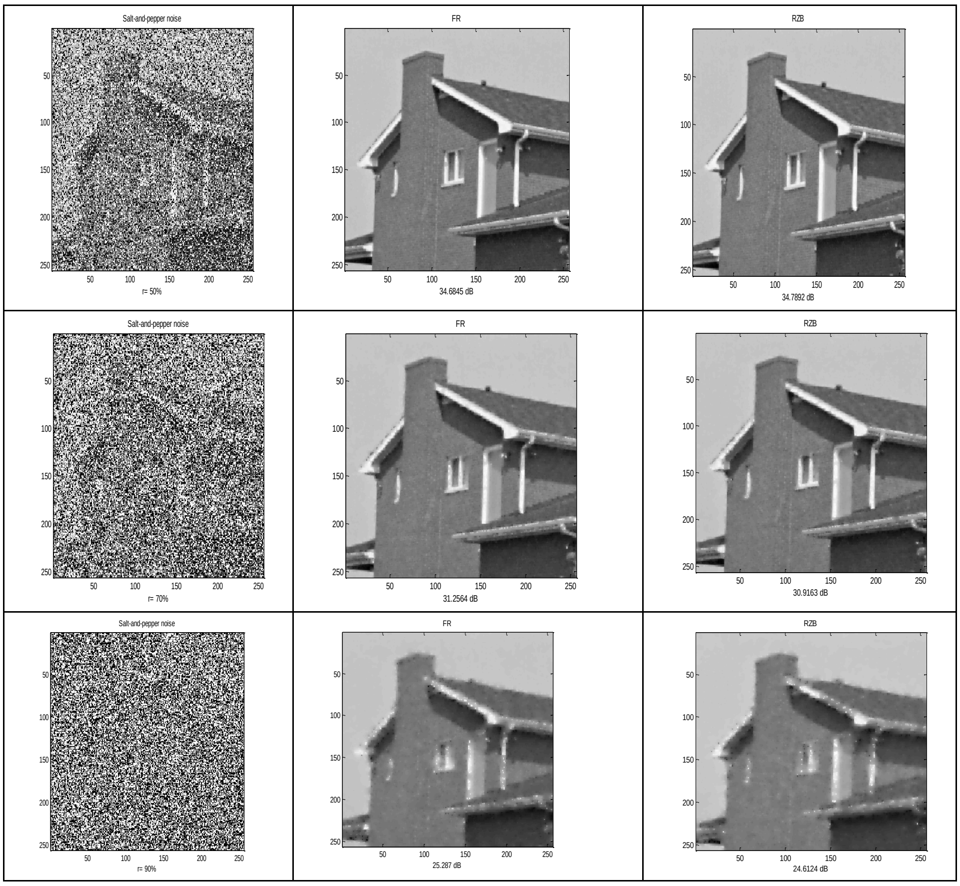

Figure 2: Demonstratesthe results of methods FR and RZB of 256 * 256 House image.

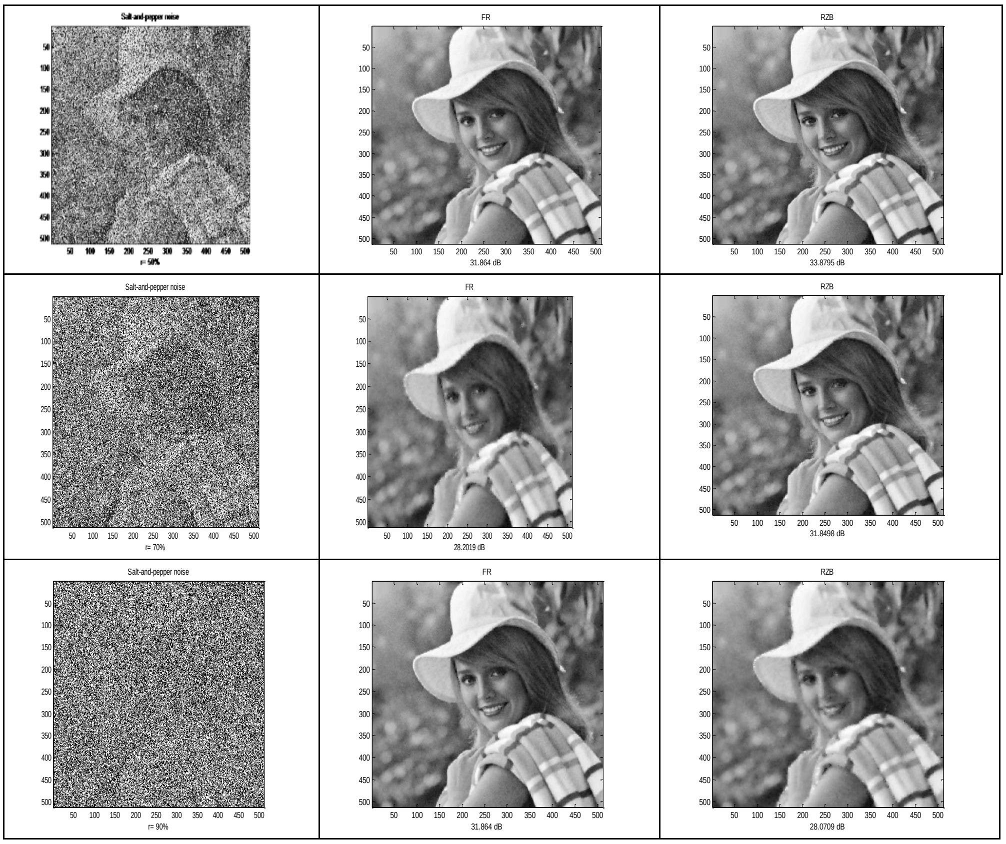

Figure 3: Demonstratesthe results of methods FR and RZB of 256 * 256 Elaine image.

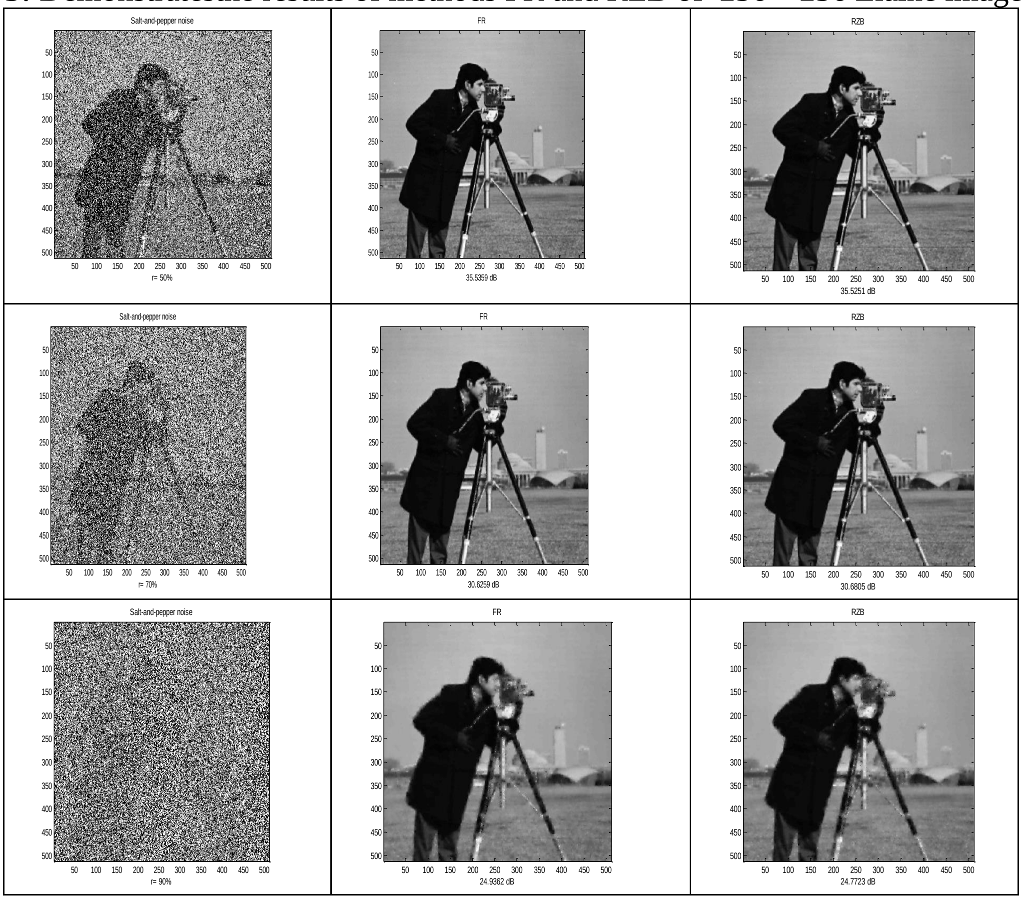

Figure 4: Demonstrates theresults of methods FR and RZB of 256 * 256 Cameraman image.

5. Conclusions

Using the second-order Taylor expansion for derivation a new formulae conjugate gradient. The revised formulas retained the goal function value but it is decreased the comparison to the conjugate gradient approaches. The global convergence results of the formula are discussed and numerical results are presented to show that this formula is very efficient. Experimental results in Table (2) confirm that the new algorithm is superior to the standard FR method. It is applied to remove burst noise in images, and it is also proved effective by compared with the standard FR method too.

References

[1] B. A. Hassan and H. M. Sadiq, "A new formula on the conjugate gradient method for removing impulse noise images," *Vestn. YuUrGU. Ser. Mat. Model. Progr*, vol. 15, no. 4, pp. 123–130, 2022, doi: doi.org/10.14529/mmp220412.

[2] W. J. Leong and M. A. Hassan, "Scaled memoryless symmetric rank one method for large-scale optimization," *Appl. Math. Comput.*, vol. 218, no. 2, pp. 413–418, Sep. 2011, doi: 10.1016/j.amc.2011.05.080.

[3] P. Wolfe, "Convergence Conditions for Ascent Methods," *SIAM Rev.*, vol. 11, no. 2, pp. 226–235, Apr. 1969, doi: 10.1137/1011036.

[4] P. Wolfe, "Convergence Conditions for Ascent Methods. II: Some Corrections," *SIAM Rev.*, vol. 13, no. 2, pp. 185–188, Apr. 1971, doi: 10.1137/1013035.

[5] C. Wu and G. Chen, "New type of conjugate gradient algorithms for unconstrained optimization problems," *J. Syst. Eng. Electron.*, vol. 21, no. 6, pp. 1000–1007, Dec. 2010, doi: 10.3969/j.issn.1004-4132.2010.06.012.

[6] Y. H. Dai and Y. Yuan, "A nonlinear conjugate gradient method with a strong global convergence property," *SIAM J. Optim.*, vol. 10, no. 1, pp. 177–182, 1999, doi: 10.1137/S1052623497318992.

[7] R. Fletcher, "Function minimization by conjugate gradients," *Comput. J.*, vol. 7, no. 2, pp. 149–154, Feb. 1964, doi: 10.1093/comjnl/7.2.149.

[8] C. Witzgall and R. Fletcher, "Practical Methods of Optimization," *Math. Comput.*, vol. 53, no. 188, p. 768, Oct. 1989, doi: 10.2307/2008742.

[9] Y. Liu and C. Storey, "Efficient generalized conjugate gradient algorithms, part 1: Theory," *J. Optim. Theory Appl.*, vol. 69, no. 1, pp. 129–137, 1991, doi: 10.1007/BF00940464.

[10] E. Polak and G. Ribiere, "Note sur la convergence de méthodes de directions conjuguées," *Rev. française d'informatique Rech. opérationnelle. Série rouge*, vol. 3, no. 16, pp. 35–43, 1969, doi: 10.1051/m2an/196903r100351.

[11] B. A. Hassan, K. Muangchoo, F. Alfara, A. H. Ibrahim, and A. B. Abubakar, "An improved quasi-Newton equation on the quasi-Newton methods for unconstrained optimizations," *Indones. J. Electr. Eng. Comput. Sci.*, vol. 22, no. 2, pp. 389–397, 2020, doi: 10.11591/ijeecs.v22.i2.pp389-397.

[12] D. F. Shanno, "Conjugate Gradient Methods with Inexact Searches," *Math. Oper. Res.*, vol. 3, no. 3, pp. 244–256, Aug. 1978, doi: 10.1287/moor.3.3.244.

[13] G. Zhou, "A descent algorithm without line search for unconstrained optimization," *Appl. Math. Comput.*, vol. 215, no. 7, pp. 2528–2533, Dec. 2009, doi: 10.1016/j.amc.2009.08.058.

[14] A. Hassan Ibrahim, P. Kumam, B. A. Hassan, A. Bala Abubakar, and J. Abubakar, "A derivative-free three-term Hestenes–Stiefel type method for constrained nonlinear equations and image restoration," *Int. J. Comput. Math.*, vol. 99, no. 5, pp. 1041–1065, May 2022, doi: 10.1080/00207160.2021.1946043.

[15] I. M. Sulaiman, N. A. Bakar, M. Mamat, B. A. Hassan, M. Malik, and A. M. Ahmed, "A new hybrid conjugate gradient algorithm for optimization models and its application to regression analysis," *Indones. J. Electr. Eng. Comput. Sci.*, vol. 23, no. 2, p. 1100, Aug. 2021, doi: 10.11591/ijeecs.v23.i2.pp1100-1109.

[16] B. A. Hassan and R. M. Sulaiman, "A new class of self-scaling for quasi-newton method based on the quadratic model," *Indones. J. Electr. Eng. Comput. Sci.*, vol. 21, no. 3, p. 1830, Mar. 2021, doi: 10.11591/ijeecs.v21.i3.pp1830-1836.

[17] H. N. Jabbar and B. A. Hassan, "Two-versions of descent conjugate gradient methods for large-scale unconstrained optimization," *Indones. J. Electr. Eng. Comput. Sci.*, vol. 22, no. 3, p. 1643, Jun. 2021, doi: 10.11591/ijeecs.v22.i3.pp1643-1649.

[18] B. A. Hassan, "A modified quasi-Newton methods for unconstrained optimization," *Ital. J. Pure Appl. Math.*, no. 42, pp. 504–511, 2019.

[19] S. Babaie Kafaki, "An eigenvalue study on the sufficient descent property of a modified Polak-Ribiére-Polyak conjugate gradient method," *Bull. Iran. Math. Soc.*, vol. 40, no. 1, 2014.

[20] Y. Dai, J. Han, G. Liu, D. Sun, H. Yin, and Y. Yuan, "Convergence Properties of Nonlinear Conjugate Gradient Methods," *SIAM J. Optim.*, vol. 10, no. 2, pp. 345–358, Jan. 2000, doi: 10.1137/S1052623494268443.

[21] N. Andrei, "An Unconstrained Optimization Test Functions Collection," *Adv. Model. Optim.*, vol. 10, no. 1, pp. 147–161, 2008.

[22] B. A. Hassan and H. Alashoor, "A New Type Coefficient Conjugate on the Gradient Methods for Impulse Noise Removal in Images," *Eur. J. Pure Appl. Math.*, vol. 15, no. 4, pp. 2043–2053, Oct. 2022, doi: 10.29020/nybg.ejpam.v15i4.4579.

[23] S. Makki**,** Emad Bakr, "Use Algebra of Group Action to Find Special Types of Caps in PG(3,13) ," Iraqi Journal of Science, 2023, Vol. 64, No. 8, pp: 492-5001 DOI: 10.24996/ijs.2023.64.8.31.

[24] O. Mahdi, A. Majeed, "Computational methods for solving nonlinear ordinary differential equations arising in engineering and applied sciences," Iraqi Journal of Science, 2023, Vol. 64, No. 8, pp: 4970-4991 DOI: 10.24996/ijs.2023.64.8.30.