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I-Nearly Primary Submodules

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Abstract
Let R be a commutative ring with identity and I be a fixed ideal of R and M be an unitary R-module. A proper submodule N of M is said to be I-nearly primary if for each $a \in R,\ x \in M$ with $ax \in N - IN$, then either $x \in N + J(M)$ or $a \in \sqrt{[N + J(M): M]}$.

Keywords: Primary submodules, weakly primary submodules, nearly primary submodules, I- primary submodules.

المقاسات الجزئية الابتدائية تقريبا من النمط−I

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الخلاصة
لتكن R حلقة ابدالية ذات عنصرمحايد، وليكنI مثالي من R ، M مقاسا احاديا معرفا على R. يقال ان المقاس الجزئي الفعلي N من M هو مقاسا جزئيا ابتدائيا تقريبا من النمط−I اذ كان a ينتمي الى R، x ينتمي الى M بحيث ax ينتمي الى $N - IN$ فانه يؤدي الى اما x ينتمي الى المقاس الجزئي $N +$ $J(M)$ او a ينتمي الى المثالي $\sqrt{[M + J(M): M]}$.

Introduction

Throughout this paper, R represents an associative ring with nonzero identity and I a fixed ideal of R and M be a unitary R-module. A proper submodule N of M is called a primary submodule if whenever $r \in R$ and $x \in M$ with $rx \in N$ implies that $r \in \sqrt{[N: M]}$ or $x \in N$, [1]. Then, many generalizations of primary submodules were studied such as weakly primary submodules in [2], ψ-primary submodules in [3], and nearly primary submodules in [4]. The authors in [5] introducing the notions I- prime and I- primary submodules. A proper submodule N of M is called I- prime submodule if $rx \in N - IN$ for all $r \in R, x \in M$ implies that either $r \in [N: M]$ or $x \in N$. A proper submodule N of M is called I- primary submodule if $rx \in N - IN$ for all $r \in R,\ x \in M$ implies that either $r \in \sqrt{[N: M]}$ or $x \in N$. In this paper, we define and study I- nearly primary submodules which are generalizations of weakly primary submodules and nearly primary submodules to I- nearly primary submodules. We generalize some basic properties of primary and nearly primary to I- nearly primary submodules and give some characterizations of I- nearly primary submodules.

1- Main result

Definition 1.1: Let I be an ideal of R and M an R–module. A proper submodule N of M is called I-nearly primary submodule, if $r\,x \in N - IN$ for all $r \in R, x \in M$ implies that either $x \in N + J(M)$ or $r^n \in [N + J(M): M]$ for some $n \in Z_+$, where $J(M)$ is the Jacobson radical of M.

For example: Consider the ring of integers Z and the Z-module Z_{12}. Take I = 4Z as an ideal of Z and N = $(\bar{4})$ be a submodule of Z_{12} generated by

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4. Then N is an I-nearly primary submodule of Z_{12} since N − IN = $(\bar{4}) - 4Z.(\bar{4}) = (\bar{4}) - (\bar{4}) = \emptyset$.

Proposition 1.2.

1)Let N, K be two are submodules of a R-module M. If N is I-nearly primary in M such that $J(M) \subseteq J(K)$, then N is I-nearly primary in K.

2)$I_1 \subseteq I_2$. Then N is I_1 - nearly primary implies N is I_2- nearly primary.

Proof (1): Suppose that $a \in R$ and $m \in K$ with $am \in N - IN$. Since N is I-nearly primary submodule of M, so either $m \in N + J(M)$ or $a^n \in [N + J(M): M]$ for some $n \in Z_+$. But $J(M) \subsetneq J(K)$, so either $m \in N + J(M)$ or $a^n \in [N + J(M): M]$. Therefore N is I-nearly primary submodule of K.

2: Suppose that N is I_1 - nearly primary. Let $a \in R, m \in M$ with $am \in N - I_2N$ Since $I_1 \subseteq I_2$, $N - I_2N \subseteq N - I_1N$. Then $am \in N - I_1N$. But N is I_1 - nearly primary. So $m \in N + J(M)$ or $a^n \in [N + J(M): M]$ for some $n \in Z_+$. Thus N is I_2- nearly primary.

The following theorem gives a useful characterization for I-nearly primarysubmodule.

Theorem 1.3 : Let N a proper submodule of an R–module M. Then N is I-nearly primary submodule in M if and only if for any ideal J of R and submodule K of M such that $JK \subseteq N - IN$, we have $J \subseteq \sqrt{[N + J(M): M]}$ or $K \subseteq N + J(M)$.

Proof: Suppose that N is I-nearly primary submodule of M, and $JK \subseteq N - IN$ for some ideal J of R and submodule K of M . If $J \nsubseteq \sqrt{[N + J(M): M]}$ and $K \nsubseteq N + J(M)$,so there exists $r \in J - \sqrt{[N + J(M): M]}$ and $x \in K - [N + J(M)]$ such that $rx \in N - IN$. By assuming N is I-nearly primary submodule in M, either $x \in N + J(M)$ or $r \in \sqrt{[N + J(M): M]}$which is a contradiction. Hence $J \subseteq [N + J(M): M]$ or $K \subseteq N + J(M)$. Conversely suppose that $rm \in N - IN$ for $r \in R$ and $m \in M$. Then $(r)(m) = (rm) \subseteq N - IN$. So by assumption, $(r) \subseteq \sqrt{[N + J(M): M]}$ or $(m) \subseteq N + J(M)$. Therefore $r \in \sqrt{[N + J(M): M]}$or $m \in N + J(M)$. Thus$N$ is I-nearly primary submodule of M.

Recall that a subset S of a ring R is called multiplicatively closed subset of R if $1 \in S$ and $ab \in S$ for every $a, b \in S$. Now, let M be a R-module and S be a multiplicatively closed subset of $R$and let R_s be the set of all fractional $r / s$where $r \in R$ and $s \in S$ and M_s be the set of all fractional x / s where $x \in M$ and $s \in S$. For $x_1, x_2 \in M$and $s_1, s_2 \in$ S,$x_1/s_1 = x_2/s_2$ if and only if there exists $t \in S$ such that $t(s_1x_2 - s_2x_1) = 0$.

So, we can make M_s in to R_s–module by setting $x/s + y/t = (tx + sy)/st$ and $r/t \, . \, x/s = rx/ts$ for every $x, y \in M$ and $s, t \in S$, $r \in R$. So M_s is he module of fractions.

Recall that if N is a submodule of an R–module M and S be a multiplicatively closed subset of R so $N_s = \{n / s : n \in N, s \in S\}$ be a submodule of the R_s– *module* M_s , see [6].

The quotient and localization of primary submodules are again primary submodules. But in case of I- nearly primary submodules, we give a condition under which the localization becomes true as we see in the following theorem.

Theorem 1.4: If N is an I-nearly primary submodule of an R-module M. Then

1) Suppose that $N_s \neq M_s$ and $(IN)s \subseteq IsNs$. Then N_s is an I_s -nearly primary in M_S.

2) Suppose that $K \subseteq N$ and $N/K + J(M/K) = N + J(M)/K$, then N/K is an I-nearly primary submodule of M/K.

Proof .1: For all $r/s \in R_s$, $x/t \in Ms$, let $r/s . x/t \in Ns - Is\,Ns \subseteq Ns - (IN)s = (N - IN)s$. Then$rx/st = m/u$ for $m \in N - IN$ and $u \in S$. So for some $v \in S, vurx = vstm \in N - IN$. As N is I-nearly primary submodule, hence either $vur \in \sqrt{[N + J(M): M]}$or $x \in N + J(M)$. So $ruv/suv = r/s \in \sqrt{[N + J(M): M]_s} = \sqrt{[N_s + J(M_s): M_s]}$by [7] or $x/t \in [N + J(M)]_s = N_s + J(M_s)$. Hence N_s is I_s-nearly primary submodule of an R_s _module Ms.

Proof.2: Let $a \in R, m \in M$ such that $a(m + K) = am + K \in N/K - I(N/K)$. Then $am + K \in [N - IN]/K$. So $am \in N - IN$. Since N is I-nearly primary submodule of M, so either $m \in N + J$(M) or $a \in \sqrt{[N + J(M): M]}$. Therefore

$m + K \in [N + J(M)/K = N/K + J(M/K)$or $a \in \sqrt{[N/K + J(M/K) : M/K]}$.Therefore N/K is I-nearly primary submodule of M/K. Therefore $m + K \in [N + J(M)]/K = N/K + J(M/K)$ or $a \in \sqrt{[N/K + J(M/K) : M/K]}$.Therefore N/K is I- nearly primary submodule of M/K.

Theorem 1.5: If N is I-nearly primary in M and $[N:M]N \nsubseteq IN$, then N is a nearly primarysubmodule M.

Proof: Suppose that $[N:M]N \nsubseteq IN$,we show that N is a nearly primary. Let $a \in R$ and $y \in M$ such that $ay \in N$. If $ay \notin IN$, since N is I- nearly primary gives $y \in N + J(M)$ or $a \in \sqrt{[N + J(M):M]}$. So let $ay \in IN$. Now, assume that $aN \nsubseteq IN$,let $an \notin IN$ where $n \in N$.Then $a(y + n) \in N - IN$, so $a \in \sqrt{[N + J(M):M]}$ or $(y + n) \in N + J(M)$. Hence $a \in \sqrt{[N + J(M):M]}$ or $y \in N + J(M)$. So we can assume that $aN \subseteq IN$. Assume that $y[N:M] \nsubseteq IN$. So $\exists b \in [N:M]$ and $yb \notin IN$. So $(a + b)y \in N$. Therefore $y \in N + J(M)$ or $(a + b) \in \sqrt{[N + J(M):M]}$. Then $y \in N + J(M)$ or $a \in \sqrt{[N + J(M):M]}$. So we can assume that $y[N:M] \subsetneq IN$. Since $[N:M]N \nsubseteq IN$, there exist $r \in [N:M]$ and $x \in N$ with $rx \notin IN$. Then $(a + r)(y + x) \in N - IN$. Then $(a + r) \in \sqrt{[N + J(M):M]}$ or $(y + x) \in N + J(M)$. Hence $a \in \sqrt{[N + J(M):M]}$ or $y \in N + J(M)$. So N is a nearly primary submodule M.

Corollary 1.6: If N is 0- nearly primaryin M and $[N:M]N \neq 0$, then N is an nearly primary submodule of M.

Propostion 1.7: Let N be a submodule of an $R -$ module M.

1- If N is I-nearly primary and $J(M) \subseteq N$, then N is a I-primary .

2- If M is a local module and N is a maximal I- nearly primary in M, then N is a I- primary in M.

3- If M is an semisimple module and N is a I- nearly primaryin M, then N is a I- primary in M.

Proof (1): The proof is trivial.

(2). Let $a \in R, m \in K$ such that $am \in N - IN$. Since N is I-nearly primary in M, so either $m \in N + J(M)$ or $a \in \sqrt{[N + J(M):M]}$. Since N is a maximal and M is alocal, so $J(M) = N$ by [8].Hence either $m \in N$ or $a \in \sqrt{[N:M]}$.Therefore N is I-primary in M.

(3). Suppose $a \in R,\ m \in K$ suchthat $am \in N - IN$. Since N is I-nearly primaryin M, so either $m \in N + J(M)$ or $a \in \sqrt{[N + J(M):M]}$. But M is semisimple, so $J(M) = 0$ by [9]. Then either $m \in N$ or $a \in \sqrt{[N:M]}$.Therefore N is I-primary in M.

Theorem 1.8: Let M be an R- module and N be a submodule of M.Then, the following statements are equivalent:

(1) N is a I-nearly primary in M .

(2) $[\mathrm{N:r}] = [\mathrm{N} + \mathrm{J(M)}] \cup [\mathrm{IN}:r]$ for each $r \in R \backslash \sqrt{[N + J(M):M]}$

(3) $[\mathrm{N:r}] = \mathrm{N} + \mathrm{J(M)}$ or $[\mathrm{N:r}] = [\mathrm{IN}:r]$ for each $r \in R \backslash \sqrt{N + J(M):M]}$.

Proof:

$(1) \to (2)$: Let $r \notin \sqrt{[N + J(M):M]}$ and $m \in [\mathrm{N:r}]$. So $rm \in N$. If $rm \notin IN$. So $m \in N + J(M)$,since N is I-nearly primary in M. If $rm \in IN$,then $m \in [\mathrm{IN:r}]$. So $[N:r] \subsetneq [\mathrm{N} + \mathrm{J(M)}] \cup [\mathrm{IN:r}]$. Now since $\mathrm{IN} \subsetneq \mathrm{N}$, the other inclusion is hold.

$(2) \to (3)$: Because if a submodule is a union of two submodules, then it is equal to one of them.

$(3) \to (1)$: Suppose $rm \in N - IN$ for each $r \in R, m \in M$. If $\mathrm{r} \notin \sqrt{[\mathrm{N} + \mathrm{J(M)}:\mathrm{M}]}$, then. Since $rm \notin IN$, so $m \notin [IN:r]$. But $rm \in N$, so $m \in [N:r]$. Hence $[N:r] = N + J(M)$. Therefore $m \in N + \mathrm{J(M)}$. Thus N is an I-nearly prime submodule of M.

Theorem 1.9: Suppose M_1 be an R_1-module and M_2 be an R_2-module. Then we have:

(1) If N_1 is an I_1- nearly primary submodule of M_1 such that $IN_1 \times M_2 \subsetneq I(N_1 \times M_2)$ and $J(M_1) \times M_2 \subsetneq J(M_1 \times M_2)$,then $N_1 \times M_2$ is an I- nearly primary in $M_1 \times M_2$.

(2) If N_2 is an I_2- nearly primary in M_2 such that $IN_2 \times M_1 \subsetneq I(N_2 \times M_1)$ and $J(M_2) \times M_1 \subsetneq J(M_2 \times M_1)$, then $M_1 \times N_2$ is an I- nearly primary in $M_1 \times M_2$

Proof:1) Assume that N_1 is an I_1- nearly primaryin M_1.Suppose $(a,b) \in R_1 \times R_2, (m_1,m_2) \in M$ with $(a,b)(m_1,m_2) = (am_1,\ bm_2) \in N_1 \times M_2 - I(N_1 \times M_2)$, and $N_1 \times M_2 - I(N_1 \times M_2) \subsetneq N_1 \times M_2 - IN_1 \times M_2 = (N_1 - IN_1) \times M_2$.We have $am_1 \in N_1 - IN_1$ but N_1 is I_1- nearly primary submodule of M_1. Then $a^n \in [N_1 + J(M_1):M_1]$ for some positive integer n or $m_1 \in N_1 + J(M_1)$. *So* $(a,b)^n = (a^n,b^n) \in [N_1 + J(M_1):M_1] \times (M_2:M_2) = \left[(N_1 + J(M_1)) \times M_2 :_{R_1 \times R_1} M_1 \times M_2\right] = \left[N_1 \times M_2 + J(M_1) \times M_2 :_{R_1 \times R_1} M_1 \times M_2\right] \subseteq \left[N_1 \times M_2 + J(M_1 \times M_2) :_{R_1 \times R_1} M_1 \times M_2\right]$ forsome

$n \in Z_+$or $(m_1, m_2) \in [N_1 + J(M_1)] \times M_2 = N_1 \times M_2 + J(M_1) \times M_2 \subseteq N_1 \times M_2 + J(M_1 \times M_2)$.
Hence $N_1 \times M_2$is an I-nearly primary submodule of $M_1 \times M_2$.

Proposition 1.10: Let $I_1$and I_2 ideals of R_1andR_2 respectively with $I = I_1 \times I_2$. Then :

1. If N_1, N_2 submodules of M_1 and M_2 respectively such that $I_iN_i = N_i$ for $i = 1,2$, then $N_1 \times N_2$ is an I-nearly primary in $M_1 \times M_2$.
2. If N_1 is primary in M_1, then $N_1 \times M_2$ is an I-nearly primary in $M_1 \times M_2$.
3. If N_1 is an I_1-nearly primary in M_1 and $I_2M_2 = M_2$, then $N_1 \times M_2$ is an I-nearly primary in $M_1 \times M_2$.
4. If N_2 is a primary in M_2, then $M_1 \times N_2$ is an I-nearly primary in $M_1 \times M_2$.
5. If N_2 is an I_2-nearly primary in M_2 and $I_1M_1 = M_1$.

Proof 1. Since $I_1N_1 = N_1$ and $I_2N_2 = N_2$. Then $I_1N_1 \times I_2N_2 = (I_1 \times I_2)(N_1 \times N_2) = I(N_1 \times N_2) = N_1 \times N_2$. So $N_1 \times N_2 - I(N_1 \times N_2) = \emptyset$. Thus there is nothing to prove.

2. Let N_1 be a primary in M_1. So $N_1 \times M_2$ is a primary in$M_1 \times M_2$, [5] and hence I- nearly primary in$M_1 \times M_2$.

3. Let N_1 is an I_1- nearly primary in M_1such that $I_2M_2 = M_2$. Suppose that $(r_1, r_2) \in R$, $(m_1, m_2) \in M$ and $(r_1, r_2)(m_1, m_2) = (r_1m_1, r_2m_2) \in N_1 \times M_2 - I(N_1 \times M_2) = N_1 \times M_2 - (I_1 \times I_2)(N_1 \times M_2) = (N_1 \times M_2 - (I_1N_1 \times I_2M_2) = (N_1 \times M_2 - (I_1N_1 \times M_2) = (N_1 - IN_1) \times M_2$. Then $r_1m_1 \in N_1 - IN_1$ and $N_1$is I_1- nearly primary submodule of M_1, so $r_1^n \in [N_1 + J(M_1): M_1]$ for some $n \in Z_+$or $m_1 \in N_1 + J(M_1)$. Therefore$(r_1, r_2)^n = (r_1^n, r_2^n) \in [N_1 + J(M_1): M_1] \times R_2 = \left[(N_1 + J(M_1)) \times M_2 :_{R_1 \times R_1} M_1 \times M_2\right] = \left[N_1 \times M_2 + J(M_1) \times M_2 :_{R_1 \times R_1} M_1 \times M_2\right] \subseteq \left[N_1 \times M_2 + J(M_1 \times M_2) :_{R_1 \times R_1} M_1 \times M_2\right]$ for some $n \in Z_+$or $(m_1, m_2) \in [N_1 + J(M_1)] \times M_2 = N_1 \times M_2 + J(M_1) \times M_2 \subseteq N_1 \times M_2 + J(M_1 \times M_2)$. So $N_1 \times M_2$is an$I$-nearly primaryin$M_1 \times M_2$.

The proof of (3) and (4) is similar to part (2) and (3) respectively.

Theorem 1.11. Let M be an R-module and let N be a proper submodule of M such that$N/IN + J(M/IN) = N + J(M) / IN$. Then N is an I-nearly primary in M iff N/INis an 0-nearly primary in M/IN.

Proof:⟹) Let N is an I-nearly primary in M. Let $a \in R, m \in M$ such that$0 \neq am + IN = a(m + IN) \in N/IN$ in M/IN. Then $am \in N - IN$. Since N is I-nearly primary submodule of M, so either $m \in N + J(M)$ or $a \in \sqrt{[N + J(M): M]} = \sqrt{[(N + J(M))/IN : M/IN]}$. Therefore $m + IN \in [N + J(M)]/IN = N/IN + J(M/IN)$ or $a \in \sqrt{[N/IN + J(M/IN) : M/IN]}$.Hence N/IN is $0-$ nearly primary in M/IN.

⟸) Let N/IN is an 0-nearly primary in M/IN . Let $a \in R, m \in M$ such that $am \in N - IN$. So $0 \neq a(m + IN) = am + IN \in N/IN$. But N/IN is an 0-nearly primary in M/IN. Thus $m + IN \in N/IN + J(M/IN) = [N + J(M)]/IN$ or $a \in \sqrt{[\frac{N}{IN} + J\left(\frac{M}{IN}\right): M/IN]} = \sqrt{[(N + J(M))/IN : M/IN]}$ and so$m \in N + J(M)$or $a \in \sqrt{[N + J(M): M]}$. Hence N is an 0-nearly primary.

Theorem1.12: If M is an R-module and I is an ideal of R, then the following statements are equivalent.

1-IM is an I-nearly primary in M;

2- For $x \in [M - (IM + J(M))]$; $[IM : x] = [I(IM): x] \cup \sqrt{[IM + J(M): M]}$;

3- For$x \in [M - (IM + J(M))]$, $[IM : x] = [I(IM) : x]$ or$[IM: x] = [IM + J(M) : M]$.

4-If $JD \subsetneq IM - I(IM)$, then $J \subsetneq \sqrt{[IM + J(M): M]}$or $D \subsetneq IM + J(M)$ for each an ideal J of R and submodule D of M.

Proof. (1) → (2): Suppose $that$ $x \in [M - (IM + J(M))]$, $r \in [IM: x]$.So $rx \in IM$.If $rx \notin I(IM)$, but IM is a I-nearly primary submodule and $x \notin IM + J(M)$, so $r \in \sqrt{[IM + J(M): M]}$. If $rx \in I(IM)$, so $r \in [I(IM): x]$. So $[IM : x] \subsetneq \sqrt{[IM + J(M): M]} \cup [I(IM) : x]$. On other hand, $I(IM) \subsetneq IM$, then $[I(IM): x] \subsetneq [IM : x] \cup \sqrt{[IM + J(M): M]}$.

(2)→(3): It is clear that because if an idea is a union of two ideals, then it is equal to one of them.

(3)→(4): Suppose that $JD \subsetneq IM$. Let$J \nsubseteq \sqrt{[IM + J(M): M]}$ and $D \nsubseteq IM + J(M)$. Aussme$x \in D$. If$x \notin IM + J(M)$. So$Jx \subsetneq IM$ and hence $J \subsetneq [IM: x]$. But$J \nsubseteq \sqrt{[IM + J(M): M]}$, so $J \subsetneq [IM: x] =$

$[I(IM): x]$.Thus, $xJ \subsetneq I(IM)$, so $DJ \subseteq I(IM)$. Suppose that $x \in IM$. let $y \in D - IM$. Then $(x + y) \in D - IM$. Hence $yJ \subseteq I(IM)$, $(x+y)J \subseteq I(IM)$. Let $r \in J$. Then $x = (x+y)r - yr \in I(IM)$. So $xJ \subsetneq I(IM)$. Thus $JD \subsetneq I(IM)$.
(4) → (1): By theorem (1.3).
Recall that a proper submodule L of an R-module M is called small in M if for every proper submodule K of M , L+K$\neq M$, [9].
Theorem 1.13: Suppose N is a submodule of a R-moduleM_1 and P is a submodule of an R-module M_2.
1- If $N \oplus P$ is an I-nearly primary and small submodule of $M = M_1 \oplus M_2$such that $J(M_1 \oplus M_2) \subsetneq J(M_1) \oplus M_2$and $J(M_1 \oplus M_2) \subsetneq M_1 \oplus J(M_2)$, then N and P are I-nearly primary inM_1, M_2respectively.
2- Let N be a small submodule of an R-module $M_1$and $J(M_1) + M_2$is small in M. If N is I-nearly primary, then $N \oplus M_2$ is an I-nearly primary submodule of $M = M_1 \oplus M_2$.
Proof. (1). Let $am_1 \in N - IN$ where $a \in R, m_1 \in M_1$.Then $a(m_1, 0) \in (N \oplus P) - IN \oplus P)$. Since $N \oplus P$ is small, so $N + P \subsetneq J(M)$ by [9]. But $N + P$ is an I-nearly primary, then either $(m_1, 0) \in N \oplus P + J(M) = J(M_1) \oplus J(M_2)$ and so $m_1 \in J(M_1) \subsetneq N_1 + J(M_1)$or $a^n \in [N \oplus P + J(M): M]$ =$[J(M) : M] \subsetneq [J(M_1) \oplus M_2 : M_1 \oplus M_2]$ for some $n \in Z_+$and so $a^n \in [J(M_1): M_1] \subsetneq [N + J(M_1): M_1]$.It follows that either $m_1 \in N_1 + J(M_1)$ or $a^n \in [N + J(M_1): M_1]$. Hence N is I-nearly primary in M_1.
By a similar proof, N_2 is an I-nearly primary in M_2.
(2). Let $a(m_1, m_2) \in (N \oplus M_2) - I(N \oplus M_2)$, where $a \in R, (m_1, m_2) \in M$. Then $am_1 \in N - IN$. Since N is an I-nearly primary and small in M_1, then either $m_1 \in N + J(M_1) = J(M_1)$ or $a^n \in [N + J(M_1): M_1]$ for some $n \in Z_+$ by [9]. So that
If $m_1 \in N + J(M_1) = J(M_1)$, then $(m_1, m_2) \in J(M_1) \oplus M_2 \subsetneq J(M) \subsetneq N \oplus M_2 + J(M)$.
If $a^n \in [N + J(M_1): M_1]$ for some $n \in Z_+$and since N is small in M_1, then $a^n \in [J(M_1) \oplus M_2: M]$. But $J(M_1) \oplus M_2$ is small in M_2, so $[J(M_1) \oplus M_2 : M] \subsetneq [J(M) : M] \subsetneq [N \oplus M_2 + J(M) : M]$ by [10], so $N \oplus M_2$is an I-nearly primarysubmodule of $M = M_1 \oplus M_2$.
Corollary 1.14: Suppose that N and P are two submodules of an R-modules M_1 , M_2 respectively.
1- If $N \oplus P$ is an I-nearly primary in a hollow$R$-module $M_1 \oplus M_2$ with $J(M_1 \oplus M_2) \subsetneq J(M_1) \oplus M_2$and $J(M_1 \oplus M_2) \subsetneq M_1 \oplus J(M_2)$,then$N$ and$P$ are I-nearly primary inM_1, M_2respectively.
2-If N a I-nearly primary submodule of a hollow R-module $M_1$and M_2be any R-module such that $M = M_1 \oplus M_2$ is a hollow R-module, then $N \oplus M_2$is an I-nearly primary submodule of $M = M_1 \oplus M_2$.
Proof (1): Since $M_1 \oplus M_2$ is a hollow module, so every submodules are small by [10] . Hence the result follows direct of (1.13, 1).
(2): Since M_1, $M_2$are two hollow modules, so every submodules of them are small by [10]. Hence the result follows direct of (1.13, 2).

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