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Cofinitely Rationally Closed Weak Supplemented Modules

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Abstract
The module M is cofinitely rationally closed weak supplemented module (denoted by CRCWS-module) if every cofinite rationally closed submodule of M has weak supplement submodule in M. The CRCWS- module is stronger generalize of closed cofinitely weak supplemented modules. In addition, we present a rationally closed weak supplemented modules (denoted by RCWS-module) which is a stronger generalize of closed weak supplemented modules. The relationships of our concepts to other related concepts was discussed and studied, and we give the conditions that make them equivalent were given. And also, we gave the necessary condition that makes the direct sum of RCWS-module (or CRCWS-module) is RCWS-module (CRCWS-module).

Keywords: Cofinitely rationally closed weak supplemented module, cofinite rationally closed submodule, closed cofinitely weak supplemented module.

المقاسات المكملة الضعيفة المغلقة بشكل راشد

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الخلاصة
المقاس M هي مقاسات المكملة الضعيفة المغلقة بشكل راشد (يشار اليها بواسطة CRCWS-module) اذا كانت كل مقاس جزئي مغلق بشكل راشد من M يمتلك مقاس جزئي مكمل ضعيف في M. ال CRCWS-module هي تعميم اقوى لمقاسات المكملة الضعيفة المغلقة بشكل نهائي. بالإضافة الى ذلك , نقدم مقاسات المكملة الضعيفة المغلقة الراشدة (يشار اليها بواسطة RCWS-module) وهي تعميم اقوى لمقاسات المكملة الضعيفة المغلقة. تمت مناقشة ودراسة علاقات مفاهيمنا بالمفاهيم الأخرى ذات الصلة ، وتم إعطاء الشروط التي تجعلها متكافئة. وقدمنا أيضا الشرط الضروري الذي يجعل الجمع المباشر RCWS-module (أو CRCWS-module) هو RCWS-module (أو CRCWS-module).

1. Introduction

Along this work, M is unitary R-module over a commutative ring R with identity.

According to Goodearl in [1], a submodule L is a rational submodule in a module M (symbolized by $L \leq_r M$ if for each $a, b \in M$ with $a \neq 0$ there exists $r \in R$ such that $rb \in L$

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and $ra \neq 0$. $Z(M) = \{a \in M | La = 0;$ for some essential left ideal L of $R\}$ is singular submodule of M. If $Z(M) = M$, then M is a singular module and if $Z(M) = 0$, then M is non-singular module.

Kasch in [2], was introduced the notion of a submodule L which is small (symbolized by $\leq_s M$), if $E \leq M$ such that $M = L + E$ implies $E = M$. Also, a submodule L is supplement of E in M (symbolized by sup-submodule or $L \leq_{sup} M$) , if $L + E = M$ and $L \cap E \leq_s L$.

Clark and others in [3], was introduced M is a supplemented module (symbolized by S-module), when every submodule of M has a sup-submodule. In our work recall the Jacabson radical of M (symbolized by $Rad(M)$), which is the sum of each $L \leq_s M$. A module M is $\bigoplus$-supplemented if for each B_1submodule of M, there is a sup-submodule B_2 (or ws-submodule) for B_1 which is a direct summand of M. In addition, introduced that a submodule L is weak supplement of E in M (symbolized by ws-submodule or $L \leq_{ws} M$), if $L + E = M$ and $L \cap E \leq_s M$. A module M is weak supplemented module (symbolized by WS-module), if any $E \leq M$ has a ws-submodule. For more details about the generalizations of supplemented module see [4]

Abbas and Ahmed in [5], introduced that a submodule P is rationally closed submodule of a module M (symbolized by $P \leq_{rc} M$ or rc-submodule), if P has no proper rational extension in M. In addition, M is rationally extending module (symbolized by RCS) if each $E \leq M$ is rational submodule in direct summand of M. For more details about the generalizations of RCS module see [6], [7], and [8]. A module M is rationally closed $\bigoplus$-supplemented (symbolized by RC-$\bigoplus$-supplemented) if every $N \leq_{rc} M$, has a sup-submodule which is direct summand, [8].

Qing and other in [9], introduced that a module M is closed weak supplemented (and symbolized by CWS-module) if every closed submodule of M has ws-submodule in M.

In Section 2: we introduced and study RCWS-modules, as well as we explain the relationship between CWS-module, RCWS-modulse and WS-module, also we give necessary condition to make these concepts are equivalent. Finally, we investigated the isomorphic between the RCWS-modules and another module.

In Section 3: first we present new concept of CRCWS-module such that we define a submodule K is cofinite rationally closed of M (and symbolized by crc-submodule or $K \leq_{crc} M$) if K cofinite submodule and rationally closed submodule of M. Also, we discuss the direct sum of CRCWS-modules.

2. Rationally closed weak supplemented modules

Definition 2.1. The module M is rationally closed weak supplemented (symbolized by RCWS-module), if every rc-submodule of M has a ws-submodule in M.

Remarks and examples 2.2.

1. Every weak supplemented module is RCWS-module. While, the converse is not necessarily true, since Z as Z-module is RCWS-module, but it is not weak supplemented.
2. Any RCWS-module is CWS-module. If M is non-singular then every CWS-module is RCWS-module.

3. Every RCS-module is RCWS-module. But the opposite is not achieved. Note a Z-module: $Z_4 \oplus Z_2$ is RCWS-module , but it is not RCS since $< (2,0), (2,1) >$ and $< (2,0) >$ are rc-submodule but it is not summand.
4. Every RC-$\oplus$-supplemented is RCWS-module. While the convers it is not necessary true, as in ($Z_8 \oplus Z_2$ as Z $-$module) is RCWS-module, but not RC-$\oplus$-supplemented.

From the above relationships, we get the following diagram:

$$\begin{array}{ccc} & & \textbf{WS-module} \\ & & \Downarrow \\ \textbf{RCS-module} & \Rightarrow & \textbf{RCWS-module} \\ & & \Downarrow \\ & & \textbf{CWS-module} \end{array}$$

Now, we will offer adequate conditions to make the reverse relationships above true:

Proposition 2.3. If M is a RCWS-module with $Rad(M) = 0$, then M is RCS-module.
Proof: Let M be a RCWS-module and $D \leq_{rc} M$, so there exists a ws-submodule W for D in M, such that $D \cap W \leq_s M$, but $Rad(M) = 0$, then $D \cap W = 0$. Hence, every rc-submodule is a summand of M, and therefore, an M is RCS. ∎

We know that if M is a semi simple then all those concepts are equivalent (weak supplemented, CWS-module and RCWS-module). Now, we give another condition for equivalent:

Proposition 2.4. Let M be a RCWS-module, for each $D \leq M$ there exists $W \leq_{rc} M$ such that $D = W + S$ or $W = D + S'$ for some $S, S' \leq_s M$. Then M is weak supplemented.

Proof: Assume that there exists $W \leq_{rc} M$ such that $+S' = W$, $S' \leq_s M$. But M is an RCWS, then there exists $F \leq_{ws} M$, such that $F + W = M$ and $F \cap W \leq_s M$. So, we have $M = F + D + S'$, but $S' \leq_s M$ then $M = F + D$ and $F \cap D \leq F \cap W \leq_s M$. Hence M is weak supplemented. ∎

In [3], we have, M is refinable module if for each $D, W \leq M$ with $M = D + W$. Then there exists a summand W' of M such that $W' \leq W$ and $W' + D = M$.
Proposition 2.5. Let M be a refinable module and $Z(M) = 0$. Suppose that for each $D \leq M$ there exists $W \leq_{rc} M$ (depending on D) such that $D = W + S$ or $W = D + S'$ for some $S,\ S'$ small submodule in M. Then the following concepts are equivalent:
1. M is $\oplus$-supplemented;
2. M is S-module;
3. M is WS-module;
4. M is RCWS-module;
5. M is CWS-module.

Proof: (1) $\Longrightarrow$ (2) it is clear.
(2) $\Longrightarrow$ (3) it is obvious.
(3) $\Longrightarrow$ (4) $\Longrightarrow$ (5) by Remarks and examples 2.2.
(5) $\Longrightarrow$ (1) it is follows from [9]. ∎

In a RCWS-module M, the submodule of M is not necessary to be RCWS-module. For example: if $M = Q \oplus Z_2$ as Z-module is RCWS-module (since it is weak supplemented by [3]), but the submodule $Z \oplus Z_2$ as Z-module is not RCWS-module (since $Rad\ (M) = 0$ and is not RCS module). In the next result we explain when is the submodule of RCWS -module is RCWS-module.

Proposition 2.6. Let $D \oplus F = M$ and M is RCWS-module, then D is RCWS-module.

Proof: Let $D \oplus F = M$ and $E \leq_{rc} D$, we have $E \leq_{rc} M$ by [5]. But M is RCWS-module, then E has ws-submodule $T \leq M$. Since $E \leq D \leq_{ws} M$ and $E \leq_{ws} M$ then by [4] $E \leq_{ws} D$. Hence, D is RCWS-module. ■

Now, we explain that the direct sum of RCWS-module need not be RCWS. If $N = Z \oplus Z_2$ as Z −module, we have Z and Z_2 are RCWS-module, but N is not RCWS-module. Now, we give some conditions to make the direct sum of RCWS-module it is RCWS-module.
Firstly, we named H is distributive submodule of M, if $H \cap (V_1 + V) = (H \cap V_1) + (H \cap V)$ for any $,V_1, V \subseteq M$. A module M is distributive module, when all its submodules are distributive [10].

Proposition 2.7. Let a module $M = M_1 \oplus M_2$ is distributive, then M_1 and M_2 are RCWS-modules if and only if M is RCWS.

Proof: Let M_1 and M_2 is RCWS-module and $\leq_{rc} M$, since M is distributive module, then we have $A = (A \cap M_1) \oplus (A \cap M_2)$. Hence, $A \cap M_1 \leq_{rc} A$ and $A \leq_{rc} M$, then $A \cap M_1 \leq_{rc} M$ by [11] and hence $A \cap M_i \leq_{rc} M_i$ $(i = 1,2)$. So, $A \cap M_i$ has ws-submodule $B_i \leq M_i$ $(i = 1,2)$. Hence, $[M = M_1 \oplus M_2 = B_1 + (A \cap M_1) \oplus B_2 + (A \cap M_2) = B_1 \oplus B_2 + (A \cap M_1) \oplus (A \cap M_2) = B_1 \oplus B_2 + A]$, and $(B_1 \oplus B_2) \cap A = (A \cap B_1) \oplus (A \cap B_2) \leq_s (M_1 \oplus M_2) = M$. Therefore, M is a RCWS-module. Directly from Proposition 2.7 the opposite direction is hold.

In the next result we show that the RCWS-module property is transmitted under the influence of isomorphism mapping.

Proposition 2.8. Every module isomorphic to RCWS-module is RCWS.

Proof: Let M be RCWS-module and $g: N \to M$ be an isomorphism. Let $0 \neq D \leq_{rc} N$ and $g(D) \leq_r W \leq M$. $D = g^{-1}g(D) \leq_r g^{-1}(W) \leq M$ by [11, for each monomorphism $g: N \to M$, if $D \leq_r N$. Then $g^{-1}(D) \leq_r M$]. But $D \leq_{rc} N$, then $D = g^{-1}(W)$ and $g(D) = W \leq_{rc} M$. Since M is a RCWS-module, thus $g(D)$ has a ws-submodule in M. Then by [12] D has a ws-submodule in N. Therefore, N is a RCWS-module. ■

Proposition 2.9. If g is an epimorphism from a RCWS-module N to a non-singular module M, then M is a RCWS-module.

Proof: Let N be a RCWS-module, $Z(M) = 0$ and $g: N \to M$ be an epimorphism. Let $D \leq_{rc} M$, but $Z(M) = 0$ then $D \leq_c M$ by [5] and [9] we have $g^{-1}(D) \leq_c N$. Then $F = g^{-1}(D) \leq_{rc} N$, but N is a RCWS-module. Then F has a ws-submodule W in N such that $M = g(F) + g(W) = D + g(W)$, $[\, g(F) = g(g^{-1}(D)) = D$ since g is epimorphism]. By [2] $g(F \cap W) \leq D \cap g(W) \leq_s M$. Hence, M is RCWS-module. ■

Recall that in [8], a module M is weakly supplement rationally extending module if every rc-submodule of M is ws-submodule in M. In the next proposition we give the relationship between weakly supplement rationally extending module and RCWS-module.

Proposition 2.10. A module M is an RCWS if and only if M is weakly supplement rationally extending module.

Proof: Let M be an RCWS-module and $U \leq_{rc} M$, so U has a ws-submodule B in M. Then $M = U + B$ and $U \cap B \leq_s M$. Hence, $U \leq_{ws} M$, and so M is weakly supplement rationally extending module. For opposite direction, let M be a weakly supplement rationally extending module and $U \leq_{rc} M$, then $U \leq_{ws} M$. So, we have U has a ws-submodule B in M. Hence, M is a RCWS-module. ■

3. Cofinitely rationally closed weak supplemented modules

Alizade, et al. in [13], introduced and named a submodule D of M is cofinite (for shortly, cof-submodule or $D \leq_{cof} M$), if M/D is finitely generated. A module M is called cofinitely supplemented, (for shortly cof-S-module) if each cof-submodule of M has a supplement in M. Recall that M is cofinitely-weak-supplemented (for shortly cof-WS-module) when all cof-submodule has a ws-submodule in M, [14].

As wall as, a module M is ⨁-cofinitely-supplemented if all cof-submodule of M has sup-submodule that is direct summand of M, [14].

Finally, M is closed cofinitely weak supplemented (for shortly, CCWS-module) if all cofinite closed submodule have ws-submodule in M, [15]. A submodule is cofinite closed if it is cofinite and closed submodule.

Definition 3.1. A module M is cofinitely rationally closed weak supplemented (for shortly, CRCWS-module, if any crc-submodule of M has a ws-submodule in M.

Remarks and examples 3.2.

1. Every cof-S-module and CWS-module are CRCWS. But the opposite is not necessarily true. For example Z as Z-module is CRCWS (since has only Z and (0) are crc-submodule has ws-submodule), but not cofinitely weak supplemented module (since not all cof-submodule has ws-submodule.
2. Every CRCWS-module is CCWS-module. The opposite is true when $Z(M) = 0$.
3. Every RCWS-module is CRCWS-module.
4. Every RCS-module is CRCWS. But the opposite is not necessarily true. For example: Z_{12} as Z-module is CRCWS-module (since is RCWS-module), but not RCS-module.

From the above relationships, we get the following diagram:

$$\begin{array}{ccccc} & & & & \textbf{RCS-module} \\ & & & & \Downarrow \\ & & & & \textbf{RCWS-module} \\ & & & & \Downarrow \\ \textbf{Cofinitely supplemented} & \Rightarrow & \textbf{cofinitely weak supplemented} & \Rightarrow & \textbf{CRCWS-module} \\ & & & & \Downarrow \\ & & & & \textbf{Closed cofinitely weak supplemented} \end{array}$$

Now we give a necessary condition to make the some concepts in Remarks and examples 3.2 are equivalents.

proposition 3.3. If a module M is non-singular, then all concepts are equivalent:
1. M is a cof-WS-module;
2. M is a CRCWS-module;
3. M is a CCWS-module.

Proof : (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) They are clear by Remarks and examples 3.2.
(3) $\Longrightarrow$ (1) It is clear by [15]. ■

A module named cofinitely-refinable (for shortly C-refinable), if every cof-submodule $C_1$of M and any $C_2 \leq M$ with $M = C_1 + C_2$, there exists a suumand F of M such that $F \leq C_1$ and $M = F + C_2$, [16].

Theorem 3.4. Let M be a C-refinable module and $Z(M) = 0$. Suppose that for each $D \leq_{cof} M$ there exist $W \leq_{rc} M$ (depending on D) such that $D = W + S$, or $W = D + S$ for some $S \leq_s M$. Then all following concepts are equivalent:
1. M is a $\bigoplus$-cofinitely supplemented;
2. M is a cofinitely supplemented;
3. M is a cof-WS-module;
4. M is a CRCWS-module;
5. M is a CCWS-module.

Proof: (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) They are clear by [15].
(3) $\Longrightarrow$ (4) $\Longrightarrow$ (5) They are clear by Remarks and examples 3.2.
(5) $\Longrightarrow$ (1) Its follows by [15]. ■

Proposition 3.5. Let M be a CRCWS-module (where, $Rad(M) = 0$), then every crc-submodule is a summand of M.

Proof: Let $W \leq_{crc} M$, since M is CRCWS-module then there exists a $D \leq_{ws} M$, such that $M = W + D$ and $W \cap D \leq_s M$, but $Rad(M) = 0$ implies $W \cap D = (0)$. So, we have $W \leq_{\oplus} M$.■

In the next result we show when the submodule of CRCWS-module is CRCWS-module.

Proposition 3.6. Every cofinite direct summand of CRCWS-module is CRCWS-module.

Proof: Let $D \leq_{crc} W$ and W each cofinite summand of M. Since $W \leq_{rc} M$ and $D \leq_{rc} W$, then $D \leq_{rc} M$ by [10]. So, we have $D \leq_{crc} M$, since $\frac{M/D}{N/D} \cong M/N$. But M is CRCWS-module then D has a ws-submodule U of M. Now, by (modular law $= W \cap (D + U) = D + (W \cap U)$) ,then $D \cap (W \cap U) = D \cap U \leq_s M$. Since W is summand of M and $D \cap U \leq W$, so $D \cap (W \cap U) \leq_s W$. Then W is CRCWS-module. ■

The following example illustrates that the direct sum of CRCWS-modules need not be a CRCWS-module.

Example 3.7. Let $M = Z[x] \oplus Z[x]$ as $Z[x]$-module, $Z[x]$ as $Z[x]$-module is CRCWS-module (since it is RCS-module by [8]). But M is not CRCWS-module (since by [9] is not CCWS-module).

Now, we give a necessary condition to make the direct sum of CRCWS-modules is CRCWS-modules.

Proposition 3.8. Let M_1 any R-module and M_2 is CRCWS-module with $= M_1 + M_2$, for each $D \leq_{crc} M$ and $D \cap M_2 \leq_{crc} M_2$. If any $D \leq_{crc} M$ with M_1 not contained in D has a weak supplement, then M is CRCWS-module.

Proof: Let $D \leq_{crc} M$ such that $M_1 \leq D$. Then $M = M_1 + M_2 = D + M_2$ has weak supplement 0. Since $D \cap M_2 \leq_{crc} M_2$ and M_2 is CRCWS-module then has $W \leq_{ws} M_2$ by [12]. Then $W + D = M$ and $W \cap D \leq_s M$. Then M is CRCWS-module. ■

Proposition 3.9. Let $Y = Y_1 \oplus Y_2$ an $R - module$ and Y_1, Y_2 are CRCWS-module. If we have $Y_j \cap (Y_i + D) \leq_{crc} Y_j$ and $Y_i \cap (D + W) \leq_{crc} Y_i$, when $W + (Y_j \cap (Y_i + D)$ and $W \cap (Y_j \cap (Y_i + D) \leq_s Y_j$, $j \neq i$, for each $D \leq_{rc} Y$ then Y is CRCWS-module.

Proof: Let $Y = Y_1 + (Y_2 + D)$ has (0) is a ws-submodule in Y, when $D \leq_{crc} Y$. But $Y_1$is CRCWS-module, and $Y_1 \cap (Y_2 + D) \leq_{crc} Y_1$, then $Y_1 \cap (Y_2 + D)$ has $W \leq_{ws} Y_1$.
Now, by [12] W is ws-submodule of $Y_2 + D$ in Y. Since Y_2 is CRCWS-module and $Y_2 \cap (D + W) \leq_{crc} Y_2$, there is ws-submodule V of $Y_2 \cap (D + W)$ in Y_2. So, by [12] $(W + V) + D = Y$ and $(W + V) \cap D \leq_s Y$. Therefor, Y is CRCWS-module.

4. Conclusions

In this work we reached the following conclusions: every RCWS-module is CWS-module, and the inverse we need a module be non-singular. A ⊕-supplemented, supplemented, weak supplemented, RCWS-module, CWS-module all this concept is equivalent when M is refinable and $Z(M) = 0$. A submodule of RCWS-module need not be RCWS-module. The direct sum of RCWS-module we need a module be distributive to make direct sum is RCWS-module. Also, every RCWS-module is CRCWS-module. Finally, a cof-WS- module, CRCWS-module and CCWS-module are equivalent when $Z(M) = 0$.

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