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I-Semiprime Submodules

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Abstract
Let R be a commutative ring with identity and I a fixed ideal of R and M be an unitary R-module.We say that a proper submodule N of M is I-semi prime submodule if $a \in R,\ x \in M$ with $a^2x \in N - IN$ implies that $a\,x \in N$. In this paper, we investigate some properties of this class of submodules. Also, some characterizations of I-semiprime submodules will be given, and we show that under some assumptions I-semiprime submodules and semiprime submodules are coincided.

Keywords: Prime submodules, weakly semiprime submodules, semiprime submodules, I-semiprime submodules.

المقاسات الجزئية شبه الاولية من النمط- I

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الخلاصة
لتكن R حلقة ابدالية ذات عنصرمحايد، وليكن I مثالي من R ، M مقاسا احاديا معرفا على R. في هذا البحث ، نقول ان المقاس الجزئي N من M هو مقاس جزئي شبه اولي من النمط- I اذ كان r ينتمي الى R، x ينتمي الى M بحيث r^2x ينتمي الى $N - IN$ فانه يؤدي الى rx ينتمي الى المقاس الجزئي N. في هذا البحث لقد درسنا واعطينا بعض خواص ومميزات هذا النوع من المقاسات الجزئية .وبرهنا تحت شروط معينة ان المقاسات الجزئية شبه الاولية وهذا النوع من المقاسات الجزئية يكونان متكافئيين.

Introduction

Throughout, R represents an associative ring with nonzero identity and I a fixed ideal of R and M be a unitary R-module. The concept of semiprime submodules was introduced and studied in [1980], where a proper submodule N of M is called a semiprime submodule if for each $r \in R,\ x \in M$, $k \in Z_+$with $r^k x \in N$ implies that r$x \in N$,[1].Then, many generalizations of semiprime submodules were studied such as weakly semiprime submodules in [2], S-semiprime submodules in [3]and nearly semiprime submodules in [4].

In this paper, we extend the concept of semiprime submodules. Let$I$ a fixed ideal of R.A proper submodule N of M is called I-semiprime if whenever $a \in R,\ x \in M$ with $a^2x \in N - IN$ implies that $ax \in N$. We generalize some basic properties of prime and semiprime to I- semiprime submodules and give some characterizations of I- semiprime submodules.

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1. Main result

Definition (1.1):(i) Let I be an ideal of R and M an R–module. A proper submodule N of M is called a I-semiprime submodule of M, if $a^2x \in N - IN$ for all $a \in R, x \in M$ implies that $\in N$.

(ii)An ideal A is called I-semiprime ideal iff for every $a \in R$ and any ideal I, $(a^2) \subseteq A - IA$ implies (a) ⊆ A.

Now, it is clear that every semiprime submodule N of M is an I-semiprime submodule of M. But the converse need not be true. For example, consider Z-module $M = Z_{24}$ and $N = \langle\bar{8}\rangle$. Then if $I = [N:M]N = [\langle\bar{8}\rangle : Z_{24}]\langle\bar{8}\rangle = \langle\bar{8}\rangle$. So N is an I-semiprime submodule of M. But N is not semiprime in M, since $2^2 .(\bar{2}) = \bar{8} \in N$, but $2.\bar{2} \notin N$.

Propostion (1.2):

1-Let $N, K$are two submodules of an$R$–module M . If $N \leq K$ and $N$is I-semiprime submodule of M and then N is I-semiprime submodule of K.

2- If $I_1 \subseteq I_2$. Then if N is I_1 - semiprime implies N is I_2- semi prime.

3- If N is semiprime then N is I-semiprime.

Proof: 1, 2 and 3 are trivial.

The following theorem gives a useful characterization for I-semiprime submodules.

Theorem(1.3): Let N a proper submodule of an R–module M. Then N is I-semiprime submodule in M if and only if for any ideal A of R and submodule K of M such that $A^2K \subseteq N - IN$, we have AK ⊆ N.

Proof. Suppose that N is I-semi prime submodule of M, and $A^2K \subseteq N - IN$ for A is an ideal of R and submodule K of M . If A$K \nsubseteq N$,so there exist $x \in K$ and a ∈ Asuch that $ax \notin N$.Now, $a^2x \in a^2K \subseteq N - IN$. We claim that $a^2x \in IN$, because if $a^2x \notin IN$, we get a$x \in N$ which is a contradication. Thus $a^2x \in IN$. Since $a^2K \subseteq N - IN$, there exists $m \in K$ such that $a^2m \in a^2K \subseteq N - IN$. This implies a$m \in N$. On the other hand $a^2x + a^2m = a^2(x+m) \in N - IN$. This implies a$(x+m) \in N$; that is ax+$am \in N$. But a$m \in N$, so a$x \in N$ which is a contradication. Therefore AK ⊆ N.

Conversely suppoes that $a^2 m \in N - IN$ for $a \in R$ and $m \in M$. Then $(a^2)(m) \subseteq N - IN$. So by assumption, (a)$(m) \subseteq N$. Therefore a$m \in N$. Thus N is an I-semiprime submodule of M.

Corollary(1.4): Let N a proper submodule of an R–module M. Then N is I-semiprime submodule in M if and only if for any ideal A of R such that $A^2M \subseteq N - IN$, we have AM ⊆ N.

Remark (1.5): If I-semiprime submodule of an R-module M, then it is not necessarily that $[N:M]$ I-semi prime ideal, for example: If $N = \langle\bar{0}\rangle$ of the Z- module Z_4, then N is I-semiprime. But $[N:M] = [\langle\bar{0}\rangle : Z_4] = 4Z$ is not an I-semiprime ideal of Z where $I = [N:M]$, since $2^2 \in [N:M] - I[N:M]$, but $2 \notin [\langle\bar{0}\rangle : Z_4] = 4Z$.

Now, we give characterizations of I-semiprime submodule. But first, we need the following definitions.

[Recall that an R-module M is called a multiplication module if every submodule N of M has the form IM for some ideal I of R, [5]. And an R-module M is called faithful if it has zero annihiilator, [6].

Theorem (1.6): Let N a proper submodule of a finitely generated faithful multiplaction$R$–module M with $I[N:M] = [IN:M]$. IfN is I-semiprime submodule in M if and only if$[N:M]$ is an I-semi prime ideal of R.

Proof.⇒)Supposethat N is I-semi prime submodule in M.Let $a \in R$with $a^2 \in [N:M] - I[N:M]$. Then $a^2M \subseteq N$. If $a^2M \subseteq IN$. Then $a^2 \in [IN:M] = I[N:M]$ which is contradiction. Assume $a^2M \nsubseteq IN$. Then $a^2M \subseteq N - IN$. But N is I-semi prime submodule. So $aM \subseteq N$, thus $a \in [N:M]$. Hence $[N:M]$is an I-semi prime ideal of R.

⇐)Supposethat $[N:M]$ is I-semi prime ideal. Let $a \in R, m \in M,$ such that $a^2m \in N - IN$. $a^2[Rm:M] = [a^2 Rm:M] \subseteq [N:M]$ and $a^2[Rm:M] \nsubseteq I[N:M]$ otherwise $a^2 Rm$=$a^2[Rm:M]M \subseteq I[N:M]M = IN$. Thus $a^2[Rm:M] \subseteq [N:M] - I[N:M]$. But $[N:M]$ is an I-semi prime ideal, so $a[Rm:M] \subseteq [N:M]$and implies that $[aRm:M] \subseteq [N:M]$. Hence $aRm \subseteq N$, so $am \in N$. Thus$N$ is an I-semiprime submodule of M.

Recall that a proper submodule N of M is called I- prime submodule if $rx \in N - IN$ for all $r \in R$, $x \in M$ implies that either $r \in [N:M]$ or $x \in N$, [7]. Also a proper submodule N of M is called I-

primary submodule if $rx \in N - IN$ for all $r \in R,\ x \in M$ implies that either $r \in \sqrt{[N:M]}$ or $x \in N$, [8]. And recall that an ideal I is called radical if $I = \sqrt{I}$, [9].
By using these concepts we can give the following proposition.
Proposition *(1.7)*: Let N a proper submodule of an R–module M. If N is I-prime then N is I-semiprime.
Proof: Let N is I-prime submodule of an R-module M, Assume that $a^2m \in N - IN$, where $a \in R, m \in M$. Since $a^2m = a(am) \in N - IN$ and N is I-prime submodule of M, then either $am \in N$ or $a \in [N:M]$. In any case, we have $am \in N$. Therefore N is I- semi prime submodule of M.
Proposition(1.*8*): Let N a proper submodule of an R–module M such that $[N:M]$ is radical ideal . If N is I-primary submodule in M, then N is an I-prime (and hence I-semi prime) submodule of M.
Proof: Let N is I-primary submodule and $[N:M]$ is radical ideal . Assume that $a^2m \in N - IN$, where $a \in R, m \in M$, suppose $m \notin N$. Since N is I-primary submodule of M and $m \notin N$, then $a \in \sqrt{[N:M]}$. But $[N:M]$ is radical, so $a \in [N:M]$ .Therefore N is I- prime (and hence I-semi prime)submodule of M.
From proposition (1.8) we get the following:
Corollary (*1*.9): Let N a proper submodule of an R–module M such that $[N:M]$ is semi prime ideal of R. If N is I-primary submodule in M, then N is an I- prime (and hence I-semiprime) submodule of M.
Propostion (*1*.10): Let M be an R–module. Let N be an I- semiprime submodule M. If $(r + [N:M])^2N \nsubseteq IN$ for all $r \in R - [N:M]$, then N is a semiprime submodule M.
Proof: Suppose that $(r + [N:M])^2N \nsubseteq IN$, we show that N is a semiprime. Let $a \in R$ and $m \in M$ such that $a^2m \in N$. If $a^2m \notin IN$, then N,I- semiprime gives $am \in N$. So assume that $a^2m \in IN$. First suppose that $a^2N \nsubseteq IN$, say $a^2n \notin IN$ where $n \in N$.Then $a^2(m+n) \in N - IN$, so $a(m+n) \in N$. Hence $am \in N$. So we can assume that $a^2N \subseteq IN$. Next, suppose that $(a+b)^2m \notin IN$ for some $b \in [N:M]$. Therefore $(a+b)^2m \in N - IN$ and so $(a+b)m \in N$. Hence $am \in N$. So we can assume that $(a + [N:M])^2m \subseteq IN$. Since $(a + [N:M])^2m \nsubseteq IN$ there exists $r \in [N:M]$ and $x \in N$ with $(a+r)^2x \notin IN$ Then $(a+r)^2(m+x) \in N - IN$. So $(a+r)(m+x) \in N$. Hence $a\,m \in N$. So N is a semiprime submodule M.
Propostion (1.11): Let M be an R–module. Let N be an I- semiprime submodule M. If $(\mathrm{r})^2\mathrm{N} \nsubseteq \mathrm{IN}$ for some $r \in [N:M]$, then N is a semiprime submodule M.
Proof: Let a $\in$ R and m $\in$ M such that $\mathrm{a^2m} \in \mathrm{N}$. Suppose $\mathrm{a^2N} \subseteq \mathrm{IN}$. If $\mathrm{a^2m} \notin \mathrm{IN}$, then $\mathrm{a^2m} \in \mathrm{N - IN}$, and N is an I- semiprime gives am $\in$ N. Suppose that $\mathrm{r^2m} \notin \mathrm{IN}$. Therefore $\mathrm{(a+r)^2\ m} = (\mathrm{a^2 + r^2})\mathrm{m} \in \mathrm{N - IN}$ and hence (a+r)m $\in$ N. So am $\in$ N. Now, we can assume that $\mathrm{r^2m} \in \mathrm{IN}$. But $(\mathrm{r})^2\mathrm{N} \nsubseteq \mathrm{IN}$, so there exists x$\in$ N such that $\mathrm{r^2x} \notin \mathrm{IN}$. Then $\mathrm{(a+r)^2\ (m+x)} = (\mathrm{a^2 + r^2})(\mathrm{m + x}) \in \mathrm{N - IN}$ and hence (a+r)(m+x) $\in$ N. So am $\in$ N. Then N is a semiprime submodule M.
Recall that a proper submodule N of M is called an irreducible submodule if for each K , L be two submodules of M such that $\mathrm{L} \cap K = N$, then either L= N or K= N,[1].
Theorem *(1.12)*: Let N be an irreducible submodule of an R–module M.Then N is an I- prime if and only if I-semiprime submodule of M.
Proof:⇒)Supposethat N is I-semi prime irreducible submodule in M. Assume that N is not I- prime, so there exists $a \in R; a \notin [N:M]$ and $m \in M; m \notin N$ such that $am \in N - IN$. Since $a \notin [N:M]$, so there exists $x \in$ M such that $ax \notin N$. Claim that $\mathrm{L} \cap K = N$ where $K = N + (ax)$, $L = N + (m)$. Now, let b $\in \mathrm{L} \cap K$, so b $\in N + (ax)$, and b $\in N + (m)$, therefore there exists n, w $\in N$ and r, s $\in R$ such that stisfiey b= w+ sax= n +rm, then sax=n-w+ rm and so sa^2x = an-aw+ram. Therefore $sa^2x \in N - IN$. But N is I-semi prime, then sax$\in N$ and so b= sax+ n $\in N$. Thus, $\mathrm{L} \cap K \subseteq N$ *and it is clear that* $N \subseteq \mathrm{L} \cap K$. Therefore the claim $\mathrm{L} \cap K = N$ is true. But N an irreducible , so which is contradiction. Therefore N is an I- prime submodule of M.
⇐) : It follows direct by (1.7).
Theorem (*1*.13): Let N a proper submodule of a faithful multiplaction R–module M and A be a finitely generated faithful multiplaction ideal of R. Then N is I-semiprime submodule in AM if and only if $[N:A]$ is an I-semi prime in M.
Proof. ⇒)Supposethat N is I-semiprime submodule in AM. Let $a \in R$ and $m \in M$ such that $a^2m \in [N:A] - I[N:A]$. Then $a^2Am \subseteq N - IN$. If $a^2Am \nsubseteq IN$, so by [8, lemma 2.15]

$a^2m \in [IN:A] = I[N:A]$ which a contradication. Since N is I-semiprime in AM, then a$Am \subseteq N$and so a$m \in [N:A]$. Hence $[N:A]$ is an I-semi prime in M

$\Leftarrow$)Suppose that $[N:A]$ is I-semiprime submodule in M . Let J be an ideal of R, and K be a submodule of AM such that $J^2K \subseteq N - IN$. Then $J^2[K:A] \subseteq [J^2K:A] \subseteq [N:A]$. Moreover, if $J^2[K:A] \subseteq I[N:A]$, by [8], lemma 2.15] J^2K $=[J^2[AK:A]] = J^2[K:A]A \subseteq I[N:A]A$ = AN which is a contradiction. Hence $J^2[K:A] \subseteq [N:A] - I[N:A]$. Since $[N:A]$ is an I-semi prime in M, so J$[K:A] \subseteq [N:A]$ which impliesthat [JK:A] $\subseteq [N:A]$. Thus JK $\subseteq N$and therefore N is I-semiprime submodule in AM.

[Recall that a subset S of a ring R is called multiplicatively closed subset of R if $1 \in S$ and $ab \in S$ for every $a, b \in S$. We known that a proper ideal P of R is prime if and only if $R - P$ is a multiplicatively closed subset of R,[10].]

[Now, let M be an R–module and S be a multiplicatively closed subset of R and let $R_S$be the set of all fractional$r/s$where $r \in R$ and $s \in S$ and M_s be the set of all farctional x/s where $x \in M$ and $s \in S$. For x_1 , $x_2 \in M$and s_1 ,$s_2 \in S$, $x_1/s_1 = x_2/s_2$ if and only if there exists $t \in S$ such that $t\,(s_1\, x_1 – s_2\, x_2) = 0$.]

[So, we can make M_s in to R_s–module by setting $x/s + y/t = (tx + sy)/st$ and$r / t\ .\ x / s = rx / ts)$ for every$x, y \in M$and$s, t \in S$, $r \in R$. And M_s is the module of fractions.]

[Recall that if N is a submodule of an R–module M and S be a multiplicatively closed subset of R so $N_s = \{n/s : n \in N,\ s \in S\}$ be a submodule of the R_s– *module* M_s , see [10].]

The quotient and localization of prime submodules are again prime submodules. But in case of I-semiprime submodules. We give a condition under which the quotient and localization becomes true as we see in the following theorem.

Theorem(1.14): Let M be an R-module. Let N be an I-semi prime submodule of M.

Then:

1) Suppose that $S$is a multiplicatively closed subset of R such that $N_S \neq M_S$and $(IN)s \subseteq IsNs$. Then N_S is anI_S -semiprime submodule of an R_S–module M_S.

2) If $K \subseteq N$ is a submodule of M, then N/K is an I-semiprime submodule of M/K.

Proof.(1): For all $a/s \in R_S$and $x/t \in M$s, let$(a/s)^2 . x/t = a^2x/s^2t \in Ns - IsNs \subseteq N_S - (IN)s = (N - IN)s$. Then $a^2x/s^2t = n/u$ for $n \in N - IN$ and $u \in S$. So there exists $v \in S$ such that $vua^2\, x = a^2\,(vux) \in N - IN$. As N is I –semiprime submodule, then $axvu \in N. So\ axvu/stvu = a/s . x/t \in N_S$. Hence N_S is an I_S -semiprime submodule of an R_S–module M_S.

(2):Let $a \in R,\ m \in M$ such that $a^2\,(m + K) = a^2m + K \in N/K - I(N/K)$. Then $a^2m + K \in [N - IN]/K$. So $a^2m \in N - IN$. Since N is I-semiprime submodule of M, so $am \in N$.Therefore$am + K \in N/K$. HenceN/K is I- nearly prime submodule of M/K.

Theorem (1.15): Let M be an R-module. Let $N$and K be a submodules of M such that K$\subseteq IN$. Then N is I-semiprime submodule of M if and only if N/K is an I-semiprime submodule of M/K .

Proof: ⇒) It follows by part 2 of Theorem (1.14).

⇐) Let N/K is an I-semiprime submodule of M/K and assume that $a^2m \in N - IN$, where $a \in R$ and $m \in M$. If $a^2(m + K) \in I(N/K) = IN/K$, then $a^2m \in IN$, which is a contradication. So we have $a^2(m + K) \in N/K - I(N/K)$. Thus a $(m + K) \in$N/K, because N/K is an I-semiprime. So $a\,m \in N$. *Thus* $N$is an I-semiprime submodule of M.

Recall that asubmodule N of an R-module M is called 0-semiprime if for $r \in R, x \in M$ with $0 \neq r^2\,x \in N$ implies that $rx \in N$, [2].

Propostion (1.16): Let M be an R-module and let N be a proper submodule of M. Then N is I-semiprime in M if and only if N/IN is 0-semiprime in M/IN .

Proof: Suppose that N is I-semiprime in M . Let $a \in R,\ x \in M$ such that $0 \neq a^2x + IN = a^2(x + IN) \in N/IN$ in M/IN.Then $a^2x \in N - IN$. Since N is I-semiprime submodule of M, so $ax \in N$. Therefore $a(x + IN) \in N/IN$. Hence N/IN is 0- semiprime submodule of M/IN **.**

Conversely suppose that (N)/IN is 0-nearly prime in M/IN. Let $a \in R,\ x \in M$ such that $a^2x \in N - IN$. So $0 \neq a^2(x + IN) \in N/IN$. But $N/IN$is 0-semiprime in M/IN **.** Thus $a(x + IN) \in N/IN$.Hence, $ax \in N$ and$N$is 0-semiprime.

Proposition (1.17): Let M_1, M_2be two R-modules and $M = M_1 \oplus M_2$.If $N_1 \oplus N_2$ is an I-semiprime submodule of $M = M_1 \oplus M_2$such that then $N_1$and $N_2$are I-semiprime in M_1, M_2 respectively.

Proof. Let $a^2m_1 \in N_1 - IN_1$ where $a \in R, m_1 \in M_1$.Then $a^2(m_1, 0) \in (N_1 \oplus N_2) - I(N_1 \oplus N_2)$. Since $(N_1 \oplus N_2)$ is an I-semiprime, then a$(m_1, 0) \in N_1 \oplus N_2$and so $am_1 \in N_1$. Hence N_1 is I-semiprime in M_1.
similarly N_2 is an I-semiprime in M_2.
In what follows give some of charactrizations for I-semiprime submodules.
Theorem (1.18): Let N be a proper submodule of M, then the following are equivalent:
(1) $N$is an I-semiprime submodule$M$.
(2) For $r \in R, [N:_M (r^2)] = [IN:_M (r^2)] \cup [N:_M (r)]$.
(3) For $r \in R, [N:_M (r^2)] = [IN:_M (r^2)]$ or $[N:_M (r^2)] = [N:_M (r)]$.
Proof: (1) $\rightarrow$ (2): Suppose that $N$is an I-semiprime submodule of M. Let $r \in R, m \in [N:_M (r^2)]$. So $r^2m \in N$. If $r^2m \notin IN$, then$rm \in N$,because $N$is an I-semiprime submodule in M.If$r^2m \in IN$, then $m \in [IN:_M (r^2)]$. Hence $[N:_M (r^2)] \subseteq [IN:_M (r)] \cup [N:_M (r)]$. Since $IN \subseteq N$, so $[IN:_M (r^2)] \cup [N:_M (r)] \subseteq [N:_M (r^2)]$. Therefore $[N:_M (r^2)] = [IN:_M (r^2)] \cup [N:_M (r)]$.
(2) $\rightarrow$ (3): It is clear because $[N: r]$is a submodule of$M$.
(3) $\rightarrow$ (1): Let $r \in R$and $m \in M$such that $r^2m \in N - IN$. Then $m \in [N:_M (r^2)]$ and $m \notin [IN:_M (r^2)]$. Then by assumption, $m \in [N:_M (r)]$. Therefore$rm \in N$. Thus $N$is an$I$-semiprime submodule of M.
Proposition (1.19): Let N be a proper submodule of M. If N is an I-semiprime submodule M, $\sqrt{[N:m]} = \sqrt{[IN:m]}$ or $\sqrt{[N:m]} = [N:m]$, for all $m \in M - N$.
Proof: Suppose that N is an I-semiprime submodule of M. Let m $\in M - N$ and r $\in \sqrt{[N:m]} - \sqrt{[IN:m]}$. So r^{n}m $\in N - IN$ for some n$\in Z_+$. Since N is an I-semiprime submodule of M, so r $\in [N:m]$. Hence $\sqrt{[N:m]} \subseteq \sqrt{[IN:m]} \cup [N:m]$. Since $IN \subseteq N$, so $\sqrt{[IN:m]} \cup [N:m] \subseteq \sqrt{[N:m]}$ and hence $\sqrt{[N:m]} = \sqrt{[IN:m]} \cup [N:m]$. Therefore $\sqrt{[N:m]} = \sqrt{[IN:m]}$ or $\sqrt{[N:m]} = [N:m]$.
Theorem (1. 20):
Let $R = R_1 \times R_2$ *and* $M = M_1 \times M_2$ with $(r_1, r_2)(m_1, m_2) = (r_1m_1, r_2m_2)$ be an R-module, where $r_i \in R_i$, $m_i \in M_i$.Then we have:
(1) If N_1 is an I_1-semiprime submodule of M_1 such that $IN_1 \times M_2 \subseteq I(N_1 \times M_2)$,then $N_1 \times M_2$ is an *I*-semiprime submodule of *M*.
(2) If N_2 is an I_2-semiprime submodule of M_2 such that $IN_2 \times M_1 \subseteq I(N_2 \times M_1)$, then $M_1 \times N_2$ is an *I*-semiprime submodule of *M*.
Proof: Because the prove of (1) and (2) are similar, so we only prove (1). Hence suppose that N_1 is an I_1-semiprime submodule of M_1 and
let $(a, b) \in R_1 \times R_2$and $(m_1, m_2) \in M$ with$(a, b)^2(m_1, m_2) = (a^2m_1, b^2m_2) \in N_1 \times M_2 - I(N_1 \times M_2)$, and $N_1 \times M_2 - I(N_1 \times M_2) \subseteq N_1 \times M_2 - I N_1 \times M_2 = (N_1 - IN_1) \times M_2$.We have $a^2m_1 \in N_1 - IN_1$but $N_1$is I_1-semiprime submodule of M_1. Then a$m_1 \in N_1$.This give (a, b) $(m_1, m_2) \in N_1 \times M_2$. Hence $N_1 \times M_2$is an$I$-semiprime submodule of $M_1 \times M_2$.
Proposition (1.21): Let $R = R_1 \times R_2$, M_i *be an* $R_i - module$ (i=1,2) with $M = M_1 \times M_2$. Let $I_1$and I_2be ideals of $R_1$and R_2 respectively with $I = I_1 \times I_2$. Then all the following types are *I*-semiprime submodule of $M_1 \times M_2$.
1. $N_1 \times M_2$ where N_1 is an I_1- semiprime submodule of M_1 and $I_2M_2 = M_2$.
2. $M_1 \times N_2$ where N_2 is an I_2- semiprime submodule of M_2 and $I_1M_1 = M_1$.
Proof.
1. Suppose that N_1 is an I_1- semiprime submodule of M_1 and $I_2M_2 = M_2$. Let $(a, b) \in R$ *and* $(m_1, m_2) \in M$ such that $(a^2, b^2)(m_1, m_2) = (a^2m_1, b^2m_2) \in N_1 \times M_2 - I(N_1 \times M_2) = N_1 \times M_2 - (I_1 \times I_2) (N_1 \times M_2) = (N_1 \times M_2 - (I_1N_1 \times I_2M_2) = (N_1 \times M_2 - (I_1N_1 \times M_2) = (N_1 - IN_1) \times M_2$. Then $a^2m_1 \in N_1 - IN_1$ and $N_1$is I_1- semi prime submodule of M_1, so a$m_1 \in N_1$. Therefore (a, b) $(m_1, m_2) \in N_1 \times M_2$. So$N_1 \times M_2$is an *I*-semiprime submodule of$M_1 \times M_2$.
2. The proof is similar to part (1).

Remark (1.22): Let $R = R_1 \times R_2$. Let M_i *be an* R_{i-} *module* (i=1,2) with $M = M_1 \times M_2$. Let I_1 and I_2 be ideals of R_1 and R_2 respectively with $I = I_1 \times I_2$. Then all the following types are *I*-semiprime submodule of $M_1 \times M_2$.

1- $N_1 \times N_2$ where N_i is a proper submodule of M_i with $I_i N_i = N_i$ for $i = 1, 2$.

2- $N_1 \times M_2$ where N_1 is a prime submodule of M_1.

3- $M_1 \times N_2$ where N_2 is a prime submodule of M_2.

Proof. 1. Since $I_1 N_1 = N_1$ and $I_2 N_2 = N_2$. Then $I_1 N_1 \times I_2 N_2 = (I_1 \times I_2)(N_1 \times N_2) = I(N_1 \times N_2) = N_1 \times N_2$. So $N_1 \times N_2 - I(N_1 \times N_2) = \emptyset$. Thus there is nothing to prove.

2. Let N_1 be a prime submodule of M_1. Then $N_1 \times M_2$ is a prime submodule of $M_1 \times M_2$ [11] and hence *I*- prime (*I*- semiprime) submodule of $M_1 \times M_2$ by (1.6).

3. The proof is similar to the part (2).

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