**I-Semiprime Submodules**

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**Abstract**  
Let $R$ be a commutative ring with identity and $I$ a fixed ideal of $R$ and $M$ be an unitary $R$-module. We say that a proper submodule $N$ of $M$ is $I$-semiprime submodule if $a \in R$, $x \in M$ with $a^2 x \in N - IN$ implies that $ax \in N$. In this paper, we investigate some properties of this class of submodules. Also, some characterizations of $I$-semiprime submodules will be given, and we show that under some assumptions $I$-semiprime submodules and semiprime submodules are coincided.

**Keywords:** Prime submodules, weakly semiprime submodules, semiprime submodules, $I$-semiprime submodules.

**المقاسات الجزئية شبه الأولية من النمط $I$**

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**الخلاصة**

$R$ حلقة ابجالية ذات عنصر محايد، وليكن $I$ مثالي من $R$، مقاسا احاديا معكفا على $I$، $M$ هو مقاس جزئي شبه أولي من النمط $I$، إذا كان $r \in R$ ينتمي إلى $I$، فإنه يؤدي إلى $r^2 x$ ينتمي إلى $N$، $N - IN$ ينتمي إلى $M$ بحيث $x$ ينتمي إلى $R$. هذا البحث قد درسونا وأعطينا بعض خواص و متغيرات هذا النوع من المقاسات الجزئية، وبها تحت شروط معينة ان المقاسات الجزئية شبه الأولية وهذا النوع من المقاسات الجزئية يكونان متكافئين.

**Introduction**

Throughout, $R$ represents an associative ring with nonzero identity and $I$ a fixed ideal of $R$ and $M$ be a unitary $R$-module. The concept of semiprime submodules was introduced and studied in [1980], where a proper submodule $N$ of $M$ is called a semiprime submodule if for each $r \in R$, $x \in M$, $k \in \mathbb{Z}_+$ with $r^k x \in N$ implies that $rx \in N$, [1]. Then, many generalizations of semiprime submodules were studied such as weakly semiprime submodules in [2], $S$-semiprime submodules in [3] and nearly semiprime submodules in [4].

In this paper, we extend the concept of semiprime submodules. Let $I$ a fixed ideal of $R$. A proper submodule $N$ of $M$ is called $I$-semiprime if whenever $a \in R$, $x \in M$ with $a^2 x \in N - IN$ implies that $ax \in N$. We generalize some basic properties of prime and semiprime to $I$-semiprime submodules and give some characterizations of $I$-semiprime submodules.

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1. Main result

Definition (1.1): (i) Let I be an ideal of R and M an R–module. A proper submodule N of M is called a I-semiprime submodule of M, if $a^2x \in N – IN$ for all $a \in R, x \in M$ implies that $x \in N$.

(ii) An ideal A is called I-semiprime ideal iff for every $a \in R$ and any ideal I, $(a^2) \subseteq A – IA$ implies $(a) \subseteq A$.

Hence, it is clear that every semiprime submodule N of M is an I-semiprime submodule of M. But the converse need not be true. For example, consider $Z$-module $M = Z_{2^4}$ and $N = \langle \overline{8} \rangle$. Then if $I = [N: M]N = [(\overline{8}); Z_{2^4}]\langle\overline{8}\rangle = \langle\overline{8}\rangle$. So N is an I-semiprime submodule of M. But N is not semiprime in M, since $2^2 \cdot (2) = \overline{8} \in N$, but $2.2 \notin N$.

Proposition (1.2):
1. Let N, K are two submodules of an R–module M. If $N \subseteq K$ and N is I-semiprime submodule of M and then N is I-semiprime submodule of K.
2. If $I_1 \subseteq I_2$. Then if N is $I_1$- semiprime implies N is $I_2$- semi prime.
3. If N is semiprime then N is I-semiprime.

Proof: 1, 2 and 3 are trivial.

The following theorem gives a useful characterization for I-semiprime submodules.

Theorem (1.3): Let N a proper submodule of an R–module M. Then N is I-semiprime submodule in M if and only if for any ideal A of R and submodule K of M such that $A^2K \subseteq N – IN$, we have $AK \subseteq N$.

Proof. Suppose that N is I-semi prime submodule of M, and $A^2K \subseteq N – IN$ for A is an ideal of R and submodule K of M. If $AK \not\subseteq N$, there exist $x \in K$ and $a \in A$ such that $ax \notin N$. Now, $a^2x \in A^2K \subseteq N – IN$. We claim that $a^2x \notin IN$, because if $a^2x \notin IN$, we get $ax \notin N$ which is a contradiction. Thus $a^2x \notin IN$. Since $a^2K \subseteq N – IN$, there exists $m \in K$ such that $a^2m \in a^2K \subseteq N – IN$. This implies $am \in N$. On the other hand $a^2x + a^2m = a^2(x + m) \notin N – IN$. This implies $a(x + m) \notin N$; that is $ax + am \in N$. But $am \in N$, so $ax \notin N$ which is a contradiction. Therefore $AK \subseteq N$.

Conversely suppose that $a^2m \notin N – IN$ for $a \in R$ and $m \in M$. Then $(a^2) \langle m \rangle \subseteq N – IN$. So by assumption, $(a) \langle m \rangle \subseteq N$. Therefore $(a) \langle m \rangle \subseteq N$. Thus N is an I-semiprime submodule of M.

Corollary (1.4): Let N a proper submodule of an R–module M. Then N is I-semiprime submodule in M if and only if for any ideal A of R such that $A^2M \subseteq N – IN$, we have $AM \subseteq N$.

Remark (1.5): If I-semiprime submodule of an R-module M, then it is not necessarily that $[N: M]$ I-semi prime ideal, for example: If $N = \langle \overline{0} \rangle$ of the Z-module $Z_{2^4}$, then N is I-semiprime. But $[N: M] = \langle \langle \overline{0} \rangle; Z_{2^4} \rangle = 4Z$ is not an I-semiprime ideal of Z where $I = [N: M]$, since $2^2 \in [N: M] – I[N: M]$, but $2 \notin \langle \langle 0 \rangle; Z_{2^4} \rangle = 4Z$.

Now, we give characterizations of I-semiprime submodule. But first, we need the following definitions.

[Recall that an R-module M is called a multiplication module if every submodule N of M has the form IM for some ideal I of R. [5]. And an R-module is called faithful if it has zero annihilator, [6].]

Theorem (1.6): Let N a proper submodule of a finitely generated faithful multiplication R–module M with $I[N: M] = [IN: M]$. If N is I-semiprime submodule in M if and only if $[N: M]$ is an I-semi prime ideal of R.

Proof. $\Rightarrow$ Suppose that N is I-semi prime submodule in M. Let $a \in R$ with $a^2 \in [N: M] – I[N: M]$. Then $a^2M \subseteq N$. If $a^2M \subseteq IN$. Then $a^2 \in [IN: M] = I[N: M]$ which is contradiction. Assume $a^2M \not\subseteq IN$. Then $a^2M \subseteq N – IN$. But N is I-semi prime submodule. So $aM \subseteq N$, thus $a \in [N: M]$. Hence: $[N: M]$ is an I-semi prime ideal of R.

$\Leftarrow$ Suppose that $[N: M]$ is I-semi prime ideal. Let $a \in R, m \in M$, such that $a^2m \in N – IN$. $a^2[Rm: M] = [a^2 \langle Rm \rangle: M] \subseteq [N: M]$ and $a^2[Rm: M] \not\subseteq I[N: M]$ otherwise $a^2 \langle Rm \rangle = a^2[Rm: M]M \subseteq I[N: M]M \subseteq IN$. Thus $a^2[Rm: M] \subseteq [N: M] – I[N: M]$. But $[N: M]$ is an I-semi prime ideal, so $a\langle Rm \rangle \subseteq [N: M]$ and implies that $a\langle Rm \rangle \subseteq [N: M]$. Hence $aRm \subseteq N$, so $am \in N$. Thus N is an I-semiprime submodule of M.

Recall that a proper submodule N of M is called I- prime submodule if $r \notin N – IN$ for all $r \in R, \ x \in M$ implies that either $r \in [N: M]$ or $x \in N$. Also a proper submodule N of M is called I-
primary submodule if \( rx \in N - IN \) for all \( r \in R, x \in M \) implies that either \( r \in \sqrt{[N:M]} \) or \( x \in N \), [8]. And recall that an ideal \( I \) is called radical if \( I = \sqrt{I} \), [9].

By using these concepts we can give the following proposition.

**Proposition (1.7):** Let \( N \) a proper submodule of an \( R \)-module \( M \). If \( N \) is \( I \)-prime then \( N \) is \( I \)-semiprime.

**Proof:** Let \( N \) be \( I \)-prime submodule of an \( R \)-module \( M \). Assume that \( a^2m \in N - IN \), where \( a \in R, m \in M \). Since \( a^2m = a(am) \in N - IN \) and \( N \) is \( I \)-prime submodule of \( M \), then either \( am \in N \) or \( a \in [N:M] \). In any case, we have \( am \in N \). Therefore \( N \) is \( I \)-semi prime submodule of \( M \).

**Proposition (1.8):** Let \( N \) a proper submodule of an \( R \)-module \( M \) such that \([N:M]\) is radical ideal. If \( N \) is \( I \)-primary submodule in \( M \), then \( N \) is \( I \)-prime (and hence \( I \)-semi prime) submodule of \( M \).

**Proof:** Let \( N \) is \( I \)-primary submodule and \([N:M]\) is radical ideal. Assume that \( a^2m \in N - IN \), where \( a \in R, m \in M \), suppose \( m \in N \). Since \( N \) is \( I \)-primary submodule of \( M \) and \( m \notin N \), then \( a \in \sqrt{[N:M]} \). But \([N:M]\) is radical, so \( a \in [N:M] \). Therefore \( N \) is \( I \)-prime (and hence \( I \)-semi prime) submodule of \( M \).

From proposition (1.8) we get the following:

**Corollary (1.9):** Let \( N \) a proper submodule of an \( R \)-module \( M \) such that \([N:M]\) is semi prime ideal of \( R \). If \( N \) is \( I \)-primary submodule in \( M \), then \( N \) is \( I \)-prime (and hence \( I \)-semi prime) submodule of \( M \).

**Proposition (1.10):** Let \( M \) be an \( R \)-module. Let \( N \) be an \( I \)-semi prime submodule of \( M \). If \((r + [N:M])^2N \notin IN \) for all \( r \in R - [N:M] \), then \( N \) is a semiprime submodule of \( M \).

**Proof:** Suppose that \((r + [N:M])^2N \notin IN \), we show that \( N \) is a semiprime. Let \( a \in R \) and \( m \in M \) such that \( a^2m \in N \). If \( a^2m \notin IN \), then \( N \) is \( I \)-semi prime gives \( am \in N \). So assume that \( a^2m \in IN \).

First suppose that \( a^2 \notin IN \), say \( a^2n \notin IN \) where \( n \in N \). Then \( a^2(m + n) \in N - IN \), so \( am + n \in N \). Hence \( am \in N \). So we can assume that \( a^2n \subseteq IN \). Next, suppose that \( (a+b)^2m \notin IN \) for some \( b \in [N:M] \). Therefore \((a+b)^2m \notin N - IN \) and so \((a+b)m \in N \). Hence \( am \in N \). So we can assume that \((a + [N:M])^2m \subseteq IN \).

Since \((a + [N:M])^2m \notin IN \) there exists \( r \in [N:M] \) and \( x \in N \) such that \((a + r)^2x \notin IN \). Then \((a + r)^2(m + x) \in N - IN \). So \((a + r)(m + x) \in N \).

Recall that a proper submodule \( N \) of \( M \) called an irreducible submodule if for each \( K \), \( L \) be two submodules of \( M \) such that \( L \cap K = N \), then either \( L = N \) or \( K = N \), [1].

**Theorem (1.12):** Let \( N \) be an irreducible submodule of an \( R \)-module \( M \). Then \( N \) is an \( I \)-prime if and only if \( I \)-semi prime submodule of \( M \).

**Proof:** Suppose that \( N \) is \( I \)-semi prime irreducible submodule in \( M \). Assume that \( N \) is not \( I \)-prime, so there exists \( a \in R; a \notin [N:M] \) and \( m \in M; m \notin N \) such that \( am \in N - IN \). Since \( a \notin [N:M] \), there exists \( x \in M \) such that \( ax \notin N \). Claim that \( L \cap K = N \) where \( K = N + (ax) \), \( L = N + (m) \). Now, let \( b \in L \cap K \), then \( b \in N \) and \( ax \), \( b \in N + (m) \), therefore there exists \( n, w \in N \) and \( r, s \in R \) such that \( bstisy \; b = w + sax \; n + rm \), then \( sax - n + w + rm \) and so \( sa^2x = an + aw + ram \). Therefore \( sa^2 \in N - IN \). But \( N \) is \( I \)-prime, then \( sax \in N \) and so \( b = sax + n \in N \). Thus, \( L \cap K \subseteq N \) and it is clear that \( N \subseteq L \cap K \). Therefore the claim \( L \cap K = N \) is true. But \( N \) an irreducible, so which is contradiction. Therefore \( N \) is an \( I \)-prime submodule of \( M \).

\( \Leftarrow \): It follows direct by (1.7).

**Theorem (1.13):** Let \( N \) a proper submodule of a faithful multiplication \( R \)-module \( M \) and \( A \) be a finitely generated faithful multiplication ideal of \( R \). Then \( N \) is \( I \)-semi prime submodule in \( AM \) if and only if \([N:A] \) is an \( I \)-semi prime in \( M \).

**Proof:** Suppose that \( N \) is \( I \)-semi prime submodule in \( AM \). Let \( a \in R \) and \( m \in M \) such that \( a^2m \notin [N:A] \). Then \( a^2Am \subseteq N - IN \). If \( a^2Am \notin IN \), so by [8, lemma 2.15]
\[a^2m \in [N:A] = I[N:A]\] which a contradiction. Since \(N\) is \(I\)-semi prime in \(AM\), then \(aM \subseteq N\) and so \(am \in [N:A]\). Hence \([N:A]\) is an \(I\)-semi prime in \(M\).

\( \iff \) Suppose that \([N:A]\) is an \(I\)-semi prime submodule in \(M\). Let \(I\) be an ideal of \(R\), and \(K\) be a submodule of \(AM\) such that \(J^2K \subseteq N - IN\). Then \(J^2[K:A] \subseteq [J^2K:A] \subseteq [N:A]\). Moreover, if \(J^2[K:A] \subseteq I[N:A]\), by [8], Lemma 2.15 \(J^2K = J^2[AK: A] = J^2[K:A]A \subseteq I[N:A]A = AN\) which is a contradiction. Hence \(J^2[K:A] \subseteq [N:A] - I[N:A]\). Since \([N:A]\) is an \(I\)-semi prime in \(M\), so \([K:A] \subseteq [N:A]\) which impliethat \([JK:A] \subseteq [N:A]\). Thus \(JK \subseteq N\) and therefore \(N\) is an \(I\)-semi prime submodule in \(AM\).

[Recall that a subset \(S\) of a ring \(R\) is called multiplicatively closed subset of \(R\) if \(1 \in S\) and \(ab \in S\) for every \(a, b \in S\). We known that a proper ideal \(P\) of \(R\) is prime if and only if \(R - P\) is a multiplicatively closed subset of \(R\), [10].]

[Now, let \(M\) be an \(R\)-module and \(S\) be a multiplicatively closed subset of \(R\) and let \(R_S\) be the set of all fractionals \(s, t \in S\) where \(x \in M\) and \(s \in S\). For \(x_1, x_2 \in M\) and \(s_1, s_2 \in S\), \(x_1/s_1 = x_2/s_2\) if and only if there exists \(t \in S\) such that \(t(s_1 - s_2) = 0\).

[So, we can make \(M_S\) in to \(R_S\)-module by setting \(x/s + y/t = (tx + sy)/st\) and \(r \cdot t\). \(x / s = rx / ts\) for every \(x, y \in M\) and \(t \in S, r \in R\). And \(M_S\) is the module of fractions.]

[Recall that if \(N\) is a submodule of an \(R\)-module \(M\) and \(S\) be a multiplicatively closed subset of \(R\) so \(N_S = \{n/s : n \in N, s \in S\}\) be a submodule of the \((R - module map).\]

The quotient and localization of prime submodules are again prime submodules. But in case of \(I\)-semi prime submodules. We give a condition under which the quotient and localization becomes true as we see in the following theorem.

**Theorem (1.14):** Let \(M\) be an \(R\)-module. Let \(N\) be an \(I\)-semi prime sub module of \(M\).

Then:
1) Suppose that \(S\) is a multiplicatively closed subset of \(R\) such that \(N_S \neq M_S\) and \((IN)_S \subseteq ISNs\). Then \(N_S\) is an \(I_S\)-semi prime submodule of an \(R_S\)-module \(M_S\).

2) If \(K \subseteq N\) is a submodule of \(M\), then \(N/K\) is an \(I\)-semi prime submodule of \(M/K\).

**Proof:** (1): For all \(a/s \in R_S\) and \(x/t \in M_S\), let \((a/s)^2. x/t = a^2 x/s^2 t \in N_S - ISNs \subseteq N_S - (IN)_S = (N - IN)_S\). Then \(a^2 x/s^2 t = n/u\) for \(n \in N - IN\) and \(u \in S\). So there exists \(v \in S\) such that \(vua^2 = a^2(vux) \in N - IN\). As \(N\) is \(I\)-semi prime submodule, then \(axu \in N\). So \(axu/vxu = a/s. x/t \in N_S\). Hence \(N_S\) is an \(I_S\)-semi prime submodule of an \(R_S\)-module \(M_S\).

(2): Let \(a \in R, m \in M\) such that \(a^2 (m + K) = a^2 m + K \in N/K - I(N/K)\). Then \(a^2 m + K \in [N - IN]/K\). So \(a^2 m - N - IN\). Since \(N\) is \(I\)-semi prime submodule of \(M\), so \(am \in N\). Therefore \(am + K \in N/K\). Hence \(N/K\) is \(I\)- nearly prime submodule of \(M/K\).

**Theorem (1.15):** Let \(M\) be an \(R\)-module. Let \(N\) and \(K\) be submodules of \(M\) such that \(K \subseteq N\).

Then \(N\) is \(I\)-semi prime submodule of \(M\) if and only if \(N/K\) is an \(I\)-semi prime submodule of \(M/K\).

**Proof:** \( \Rightarrow \) It follows by part 2 of Theorem (1.14).

\( \Leftarrow \) Let \(N/K\) is an \(I\)-semi prime submodule of \(M/K\) and assume that \(a^2 m \in N - IN\), where \(a \in R\) and \(m \in M\). If \(a^2 (m + K) \in N/K = IN/K\), then \(a^2 m \in IN\), which is a contradiction. So we have \(a^2(m + K) \in N/K - I(N/K)\). Thus \(a(a + K) \in N/K\), because \(N/K\) is an \(I\)-semi prime. So \(a^2 m \in N\). Thus \(N/K\) is an \(I\)-semi prime submodule of \(M\).

Recall that submodule \(N\) of an \(R\)-module \(M\) is called 0-semi prime if for \(r \in R, x \in M\) with \(0 \neq r^2 x \in N\) implies that \(rx \in N\), [2].

**Proposition (1.16):** Let \(M\) be an \(R\)-module and let \(N\) be a proper submodule of \(M\). Then \(N\) is \(I\)-semi prime in \(M\) if and only if \(N/IN\) is \(0\)-semi prime in \(M/IN\).

**Proof:** Suppose that \(N\) is \(I\)-semi prime in \(M\). Let \(a \in R, x \in M\) such that \(0 \neq a^2 x + IN = a^2(x + IN) \in N/IN\) in \(M/IN\). Then \(a^2 x \in N - IN\). Since \(N\) is \(I\)-semi prime submodule of \(M\), so \(ax \in N\). Therefore \(a(x + IN) \in N/IN\). Hence \(N/IN\) is \(0\)-semi prime submodule of \(M/IN\).

Conversely suppose that \((N/IN)\) is \(0\)-nearly prime in \(M/IN\). Let \(a \in R, x \in M\) such that \(a^2 x \in N - IN\). So \(0 \neq a^2(x + IN) \in N/IN\). But \(N/IN\) is \(0\)-semi prime in \(M/IN\). Thus \(a(x + IN) \in N/IN\). Hence, \(ax \in N\) and \(N\) is \(0\)-semi prime.

**Proposition (1.17):** Let \(M_1, M_2\) be two \(R\)-modules and \(M = M_1 \oplus M_2\). If \(N_1 \oplus N_2\) is an \(I\)-semi prime submodule of \(M = M_1 \oplus M_2\) such that then \(N_1\) and \(N_2\) are \(I\)-semi prime in \(M_1, M_2\) respectively.
Proof. Let \( a^2m_1 \in N_1 - IN_1 \) where \( a \in R, m_1 \in M_1 \). Then \( a^2(m_1, 0) \in (N_1 \oplus N_2) - I(N_1 \oplus N_2) \).
Since \((N_1 \oplus N_2)\) is an \( I\)-semiprime, then \( a(m_1, 0) \in N_1 \oplus N_2 \) and so \( am_1 \in N_1 \). Hence \( N_1 \) is \( I\)-semiprime in \( M_1 \).

similarly \( N_2 \) is an \( I\)-semiprime in \( M_2 \).

In what follows give some of charactrizations for \( I\)-semiprime submodules.

**Theorem (1.18):** Let \( N \) be a proper submodule of \( M \), then the following are equivalent:

1. \( N \) is an \( I\)-semiprime submodule of \( M \).
2. For \( r \in R, [N : M (r^2)] = [IN : M (r^2)] \cup [N : M (r)] \).
3. For \( r \in R, [N : M (r^2)] = [IN : M (r^2)] \) or \( [N : M (r^2)] = [N : M (r)] \).

**Proof:** (1) \( \Rightarrow \) (2): Suppose that \( N \) is an \( I\)-semiprime submodule of \( M \). Let \( r \in R, \ m_1 \in [N : M (r^2)] \). So \( r^2m \in N \). If \( r^2m \in \IN \), then \( m \in N \), because \( N \) is an \( I\)-semiprime submodule in \( M \). If \( r^2m \in \IN \), then \( m \in [N : M (r^2)] \) or \( [N : M (r^2)] = [N : M (r)] \). Since \( IN \subseteq N \), so \( IN \subseteq \IN \) or \( [N : M (r^2)] \subseteq [N : M (r^2)] \). Therefore \( [N : M (r^2)] = [IN : M (r^2)] \cup [N : M (r)] \).

(2) \( \Rightarrow \) (3): It is clear because \( [N : r] \) is a submodule of \( M \).

(3) \( \Rightarrow \) (1): Let \( r \in R \) and \( m \in M \) such that \( r^2m \in N - \IN \). Then \( m \in [N : M (r^2)] \) and \( m \notin [IN : M (r^2)] \). Then by assumption, \( m \in [N : M (r)] \). Therefore \( m \in N \). Thus \( N \) is an \( I\)-semiprime submodule of \( M \).

**Proposition (1.19):** Let \( N \) be a proper submodule of \( M \). If \( N \) is an \( I\)-semiprime submodule of \( M \), then \( \sqrt{[N : m]} = \sqrt{[IN : m]} \cup \sqrt{[N : m]} \), for all \( m \in M - N \).

**Proof:** Suppose that \( N \) is an \( I\)-semiprime submodule of \( M \). Let \( m \in M - N \) and \( r \in \sqrt{[N : m]} \). Then \( r^2m \in \IN \). Since \( N \) is an \( I\)-semiprime submodule of \( M \), \( m \in \sqrt{[N : m]} \). Hence \( \sqrt{[N : m]} \subseteq \sqrt{[IN : m]} \cup [N : m] \). Since \( IN \subseteq N \), so \( \sqrt{[IN : m]} \cup [N : m] \subseteq \sqrt{[N : m]} \). Therefore \( \sqrt{[IN : m]} \subseteq [N : m] \).

**Theorem (1.20):**
Let \( R = R_1 \times R_2 \) and \( M = M_1 \times M_2 \) with \((r_1, r_2)(m_1, m_2) = (r_1m_1, r_2m_2)\) be an \( R\)-module, where \( r_1 \in R_1, m_1 \in M_1 \). Then we have:

1. If \( N_1 \) is an \( I_1\)-semiprime submodule of \( M_1 \) such that \( \IN_1 \times M_2 \subseteq \IN_1 \times M_2 \), then \( N_1 \times M_2 \) is an \( I\)-semiprime submodule of \( M \).
2. If \( N_2 \) is an \( I_2\)-semiprime submodule of \( M_2 \) such that \( \IN_2 \times M_1 \subseteq \IN_2 \times M_1 \), then \( M_1 \times N_2 \) is an \( I\)-semiprime submodule of \( M \).

**Proof:** Because the prove of (1) and (2) are similar, so we only prove (1). Hence suppose that \( N_1 \) is an \( I_1\)-semiprime submodule of \( M_1 \) and \( \langle a, b \rangle \in R_1 \times R_2 \) and \((m_1, m_2) \in M \) with \((a, b)^2(m_1, m_2) = (a^2m_1, b^2m_2) \in N_1 \times M_2 - \IN_1 \times M_2 \) and \( N_1 \times M_2 - \IN_1 \times M_2 \subseteq N_1 \times M_2 - \IN_1 \times M_2 = (N_1 - \IN_1) \times M_2 \). We have \( a^2m_1 \in N_1 - \IN_1 \) but \( N_1 \) is an \( I_1\)-semiprime submodule of \( M_1 \). Then \( am_1 \in N_1 \). This give \((a, b)(m_1, m_2) \in N_1 \times M_2 \) . Hence \( N_1 \times M_2 \) is an \( I\)-semiprime submodule of \( M_1 \times M_2 \).

**Proposition (1.21):** Let \( R = R_1 \times R_2, M_i \) be an \( R_i \)-module \((i = 1, 2)\) with \( M = M_1 \times M_2 \). Let \( I_1 \) and \( I_2 \) be ideals of \( R_1 \) and \( R_2 \) respectively with \( I = I_1 \times I_2 \). Then all the following types are \( I\)-semiprime submodules of \( M_1 \times M_2 \).

1. \( N_1 \times M_2 \) where \( N_1 \) is an \( I_1\)-semiprime submodule of \( M_1 \) and \( I_2 M_2 = M_2 \).
2. \( M_1 \times N_2 \) where \( N_2 \) is an \( I_2\)-semiprime submodule of \( M_2 \) and \( I_1 M_1 = M_1 \).

**Proof:**
1. Suppose that \( N_1 \) is an \( I_1\)-semiprime submodule of \( M_1 \) and \( I_2 M_2 = M_2 \). Let \((a, b) \in R \) and \((m_1, m_2) \in M \) such that \((a^2, b^2)(m_1, m_2) = (a^2m_1, b^2m_2) \in N_1 \times M_2 - \IN_1 \times M_2 = N_1 \times M_2 - (I_1 \times I_2)(N_1 \times M_2) = (N_1 \times M_2 - (I_1 N_1 \times I_2 M_2)(N_1 \times M_2 - (I_1 N_1 \times I_2 M_2)) = (N_1 - \IN_1) \times M_2 \). Then \( a^2m_1 \in N_1 - \IN_1 \) and \( N_1 \) is an \( I_1\)-semi prime submodule of \( M_1 \), so \( am_1 \in N_1 \). Therefore \((a, b)(m_1, m_2) \in N_1 \times M_2 \). So \( N_1 \times M_2 \) is an \( I\)-semiprime submodule of \( M_1 \times M_2 \).
2. The proof is similar to part (1).
Remark (1.22): Let $R = R_1 \times R_2$. Let $M_i$ be an $R_i$-module $(i=1,2)$ with $M = M_1 \times M_2$. Let $I_1$ and $I_2$ be ideals of $R_1$ and $R_2$ respectively with $I = I_1 \times I_2$. Then all the following types are $I$-semiprime submodule of $M_1 \times M_2$.

1. $N_1 \times N_2$ where $N_i$ is a proper submodule of $M_i$ with $I_i N_i = N_i$ for $i = 1, 2$.
2. $N_1 \times M_2$ where $N_1$ is a prime submodule of $M_1$.
3. $M_1 \times N_2$ where $N_2$ is a prime submodule of $M_2$.

Proof. 1. Since $I_1 N_1 = N_1$ and $I_2 N_2 = N_2$. Then $I_1 N_1 \times I_2 N_2 = (I_1 \times I_2) (N_1 \times N_2) = I (N_1 \times N_2) = N_1 \times N_2$. So $N_1 \times N_2$ is a prime submodule of $M_1 \times M_2$. Hence there is nothing to prove.

2. Let $N_1$ be a prime submodule of $M_1$. Then $N_1 \times M_2$ is a prime submodule of $M_1 \times M_2$ [11] and hence $I$-prime ($I$-semiprime) submodule of $M_1 \times M_2$ by (1.6).

3. The proof is similar to the part (2).

References


