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Novel Results of $\mathcal{K}$ Quasi $(\lambda - \mathcal{M})$-hyponormal Operator

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Abstract
This research aims to present some results for conceptions of $\mathcal{K} -$ quasi $(\lambda - \mathcal{M})$-hyponormal operator defined on Hilbert space$-\mathcal{H}$. Signified by the $\mathcal{KQ}$ $(\lambda - \mathcal{M})h$-operator, together with some significant characteristics of this operator and various theorems pertaining to this operator are discussed, as well as, we discussed the null space and range of these kinds of operators.

Keywords-: $\mathcal{K} -$ quasi$(\lambda - \mathcal{M})h -$Operator, Quasihyponorrmal Operator, Hilbert Space, Quasi $\mathcal{M}$ -hyponormal, $\mathcal{M}$ -hyponormal.

نتائج جديدة للمؤثر فوق السوي $(\lambda - \mathcal{M})$ شبه $\mathcal{K}$

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الخلاصة
أهداف هذا البحث تقديم نتائج اخرى لبعض المفاهيم للمؤثر فوق السوي $(\lambda - \mathcal{M})$ شبه$-$ $\mathcal{K}$ والمعرفة على فضاء هلبرت$-$ $\mathcal{H}$. ويرمز له بالمؤثر$-$ $\mathcal{KQ}$ $(\lambda - \mathcal{M})h$ مع بعض التميزات لهذا المؤثر و تمت اضافة نظريات مختلفة متعلقة بهذا المؤثر وكذلك درسنا وناقشنا فضاء النواة والمدى لهذا النوع من المؤثرات.

1. Introduction:

In the science of mathematics, specially in the field of operator theory, an extension of a normal operator is a hyponormal operator. Typically, a bounded linear operator $\mathcal{T}$ defined on a complex Hilbert space H. Halmos P.R.[1], first articulated the idea of hyponormality in [1] under a different name which is subnormal. Despite this, the later two concepts are not merely easy adaptations of one to another because many attributes do not transfer well when taking a djoints, where are the unilateral shift operator, it is a well-known example of a hyponormal operator, which is crucial, also solvability of the λ-commuting operator equation takes the following form $\mathcal{T}_1\mathcal{T}_2 = \lambda\ \mathcal{T}_2\mathcal{T}_1$, is one of the key applications of the hyponormal operator, with the equation for the $(\lambda, \mathcal{M})$-commuting has formulations $\mathcal{T}_1\mathcal{T}_2 = \lambda\ \mathcal{T}_2\mathcal{T}_1$, and $\mathcal{T}^*{}_1\mathcal{T}_2 = \mathcal{M}\ \mathcal{T}_2\mathcal{T}^*{}_1$,[2]

Putnam C.R. [3] investigated several features of the operator $J = ei\ T + ei\ T^*$, such that in the case T discrebe the hyponormal operator in 1957 and continuous this studies on hyponormal until 1961 several hyponormal operator qualities were presented to Berberain S.K. In 1962,

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Stampfli J.G. [4] demonstrated several properties of hyponormal operators, one of which was that the hyponormal operator T is normal if T^n is normal, where n is a positive integer number. It was established in 1966 by Coburn L.A.[5] that Weyl's theorem is valid for any hyponormal operator. In 1972 an important development occurred in these studies such as in another generalization of the idea of hyponormality, Devi S. [6] established the class of quasi-hyponormal operators. Moreover, in the same year, Stampfli J.G. began researching M-hyponormal operators, and in his PhD dissertation. Wadhwa B.L.[7] examined certain spectral theoretical aspects of M-hyponormal operators in addition any details regarding M-hyponormal operators were provided to Phadke S.V.and Thakare N.K.[8] in 1976. But Clancey K.[9] demonstrated three comparable hyponormal operator formula in 1979. Recently, other important studies have been appeared in the same filed, so that, in 2011 Young Min Han and Ju Hee Son, [10] given the Weyl kind theorems on algebraically quasi-M-hyponormal and focus of local spectral properties for quasi-M-hyponormal, also Mohsen, Salim D. and Atheab, Nidaa M. in 2020 explained the idea of *-quasi-hyponormal, a novel extension for the hyponormal operator, and discussed some key theorems relating to it.[11].Also Mohsen, Salim D. And Atheab, Nidaa M. [12], establish (N,k)-hyponormal operators, a novel generalization for hyponormal operators. We also investigate some of these operators' features. This paper, deals with the idea of $\mathcal{K}$ −quasi-$(\lambda - \mathcal{M})$-hyponormal operator in Hilbert space, and represented a new category of operators with examine some of the characteristics of quasi-normal operators [13]. This paper, contains four sections, such that, section one include the introduction of this work, and section two contains some basic concepts of operators, section three consist of the definition of $\mathcal{K}$ −quasi-$(\lambda - \mathcal{M})$ -hyponormal operator in Hilbert space−$\mathcal{H}$ with some remakes. Also, section four deals with main results of this type of operator such that we introduced the proves of important theorems. Finally, the conclusion of this article are highlighted in section five.

2. Basic definitions for operators:

In this section, we introduce some basic definitions about this subject, such that we recall the definitions of self-adjoint, normal, hyponormal operators, and some modifications of operators have been given.

Definition 2.1: [14]

A linear bounded operator $\mathcal{T}: \mathcal{H} \to \mathcal{H}$ which is define on Hilbert space $\mathcal{H}$ is referred to as a self-adjoint operator if the condition $\mathcal{T}^* = \mathcal{T}$, hold.

Definition 2.2: [15]

Supposing $\mathcal{T}: \mathcal{H} \to \mathcal{H}$ is a linear bounded operator which is define on a Hilbert space $\mathcal{H}$, whenever $\mathcal{T}^*\mathcal{T} = \mathcal{T}\mathcal{T}^*$, which implies, for any $x \in \mathcal{H}$, $< x, \mathcal{T}\mathcal{T}^*x > = < \mathcal{T}^*\mathcal{T}x, x >$, then $\mathcal{T}$ is regarded as normal.

Definition 2.3: [14]:

Supposing $\mathcal{T}: \mathcal{H} \to \mathcal{H}$ bounded operator, so we call $\mathcal{T}$ which is hyponormal on Hilbert space $\mathcal{H}$. If $\mathcal{T}\mathcal{T}^* \le \mathcal{T}^*\mathcal{T}$, which implies for any $x \in \mathcal{H}$, $< x, \mathcal{T}\mathcal{T}^*x > \le < \mathcal{T}^*\mathcal{T}x, x >$.

Definition 2.4: [8]

Supposing $\mathcal{T}: \mathcal{H} \to \mathcal{H}$ bounded operator, we say that to be operator type $\mathcal{M}$-hyponormal whenever existing a positive real number $\mathcal{M}$ so that $\mathcal{M}^2 (\mathcal{T} - \lambda)^*(\mathcal{T} - \lambda) \ge (\mathcal{T} - \lambda)(\mathcal{T} - \lambda)^*$ for any $\lambda \in \mathbb{C}\backslash\{0\}$.

Definition 2.5: [10]

Supposing $\mathcal{T}: \mathcal{H} \to \mathcal{H}$ bounded operator, we will called operator type quasi $\mathcal{M}$-hyponormal whenever existing a positive real number $\mathcal{M}$, with the condition $\mathcal{T}^* (\mathcal{T} - \lambda)(\mathcal{T} - \lambda)^* \mathcal{T}^* \leq (\mathcal{M}^2 (\mathcal{T} - \lambda)^*(\mathcal{T} - \lambda))\mathcal{T}$, where $\lambda \in \mathbb{C}\backslash\{0\}$.

3. On $\mathcal{K}$ quasi $(\lambda - \mathcal{M})$-hyponormal operator:

In this section, we present new modifcation of hyponormal operator and main results, starting via the defintion below, this definition close to definition appear in [14].

Definition 3.1: [16]

Assume $\mathcal{H}$ is a Hilbert space, the operator $\mathcal{T}: \mathcal{H} \to \mathcal{H}$ which is said to be $\mathcal{K}$-quasi $(\lambda - \mathcal{M})$ hyponormal operator, shortly $\mathcal{KQ}\ (\lambda - \mathcal{M})h$- operator when $\mathcal{M}$ which is a positive real number and positive integer k so has

$\mathcal{T}^{*k}(\mathcal{M}^2(\mathcal{T} - \lambda)^*(\mathcal{T} - \lambda))\mathcal{T}^k \geq \mathcal{T}^{*k}((\mathcal{T} - \lambda)(\mathcal{T} - \lambda)^*)\mathcal{T}^k - (1)$

It clearly that ,the inequality appeared in definition (3.1) is Equivalently with

$\mathcal{M}^2\mathcal{T}^{*k}\big((\mathcal{T} - \lambda)^*(\mathcal{T} - \lambda)\big)\mathcal{T}^k \geq \mathcal{T}^{*k}\big((\mathcal{T} - \lambda)(\mathcal{T} - \lambda)^*\big)\mathcal{T}^k - (2),$

where $\lambda \in \mathbb{C}\backslash\{0\}$.

Remarks 3.2:

1) It is clearly that, every self adjoint (hyponormal, $\mathcal{M}$ -hyponormal, quasi $\mathcal{M}$ -hyponormal) is $\mathcal{KQ}\ (\lambda - \mathcal{M})h$-operator.
2) $\mathcal{KQ}\ (\lambda - \mathcal{M})h$- operator becomes quasi-$\mathcal{M}$-hyponormal when $k = 1$.
3) The densely range $\mathcal{KQ}\ (\lambda - \mathcal{M})h -$ operator becomes $\mathcal{M}$-hyponormal when $k = 1$.

4. Main results:

In this section, we find some novel properties of $\mathcal{KQ}\ (\lambda - \mathcal{M})h$-operator, and proving the important theorems to explain this concept. Some charachrizations for $\mathcal{KQ}\ (\lambda - \mathcal{M})h$-operator, can be obtain in the next theorem.

Theorem 4.1:

Assume that $\mathcal{T}: \mathcal{H} \to \mathcal{H}$, is $\mathcal{KQ}\ (\lambda - \mathcal{M})h$-operator if and only if existing real number $\mathcal{M}$ which is positive satisfy $\mathcal{M}^2 \parallel (\mathcal{T} - \lambda)\mathcal{T}^k(v) \parallel \geq \parallel (\mathcal{T} - \lambda)^*\mathcal{T}^k(v) \parallel$, to each $v \in \mathcal{H}$.

Proof: $\parallel (\mathcal{T} - \lambda)^*\mathcal{T}^k(v) \parallel^2 = < (\mathcal{T} - \lambda)^*\mathcal{T}^k(v), (\mathcal{T} - \lambda)^*\mathcal{T}^k(v) >$

$= < \mathcal{T}^k(v), (\mathcal{T} - \lambda)(\mathcal{T} - \lambda)^*\mathcal{T}^k(v) >$

$= < v, \mathcal{T}^{*k}\big((\mathcal{T} - \lambda)(\mathcal{T} - \lambda)^*\big)\mathcal{T}^k(v)) >$, since $\mathcal{T}$ is $(\mathcal{KQ}\ (\lambda - \mathcal{M})h -$ operator, one can get

$\parallel (\mathcal{T} - \lambda)^*\mathcal{T}^k(v) \parallel^2 \leq \mathcal{M}^2 < v, \mathcal{T}^{*k}(\mathcal{T} - \lambda)^*(\mathcal{T} - \lambda)\mathcal{T}^k(v) >$

$\leq \mathcal{M}^2 < \mathcal{T}^k(v), (\mathcal{T} - \lambda)^*(\mathcal{T} - \lambda)\mathcal{T}^k(v) >$

$\leq \mathcal{M}^2 < (\mathcal{T} - \lambda)\mathcal{T}^k(v), (\mathcal{T} - \lambda)\mathcal{T}^k(v) >$

$\leq \mathcal{M}^2 \parallel (\mathcal{T} - \lambda)\mathcal{T}^k(v) \parallel^2$.

Conversely, we can assumption $\mathcal{T}$ satisfy,

$\parallel (\mathcal{T} - \lambda)^*\mathcal{T}^k(v) \parallel^2 \leq \mathcal{M}^2 \parallel (\mathcal{T} - \lambda)\mathcal{T}^k(v) \parallel^2$ for each $v \in \mathcal{H}$, so one can obtain that,

$< (\mathcal{T} - \lambda)^*\mathcal{T}^k(v), (\mathcal{T} - \lambda)^*\mathcal{T}^k(v) > = < \mathcal{T}^k(v), (\mathcal{T} - \lambda)(\mathcal{T} - \lambda)^*\mathcal{T}^k(v) >$

$= < v, \mathcal{T}^{*k}(v)(\mathcal{T} - \lambda)(\mathcal{T} - \lambda)^*\mathcal{T}^k(v) >$

$\leq \mathcal{M}^2 < v, \mathcal{T}^{*k}(v)(\mathcal{T} - \lambda)^*(\mathcal{T} - \lambda)\mathcal{T}^k(v) >$

for every $v \in \mathcal{H}$, so get that

$< v, \mathcal{M}^2\mathcal{T}^{*k}(v)(\mathcal{T} - \lambda)^*(\mathcal{T} - \lambda)\mathcal{T}^k(v) - \mathcal{T}^{*k}(v)(\mathcal{T} - \lambda)(\mathcal{T} - \lambda)^*\mathcal{T}^k(v) > \geq 0$

So, $\mathcal{M}^2\mathcal{T}^{*k}(v)(\mathcal{T} - \lambda)^*(\mathcal{T} - \lambda)\mathcal{T}^k(v) - \mathcal{T}^{*k}(v)(\mathcal{T} - \lambda)(\mathcal{T} - \lambda)^*\mathcal{T}^k(v) \geq 0$

Thus, $\mathcal{M}^2\mathcal{T}^{*k}(\mathcal{T}-\lambda)^*(\mathcal{T}-\lambda)\mathcal{T}^k-\mathcal{T}^{*k}(\mathcal{T}-\lambda)(\mathcal{T}-\lambda)^*\mathcal{T}^k \geq 0$.
Hence one can obtain $\mathcal{M}^2\mathcal{T}^{*k}((\mathcal{T}-\lambda)^*(\mathcal{T}-\lambda)\mathcal{T}^k) \geq \mathcal{T}^{*k}(\mathcal{T}-\lambda)(\mathcal{T}-\lambda)^*\mathcal{T}^k$.
Therefore, $\mathcal{T}$ is $\mathcal{KQ}$ $(\lambda-\mathcal{M})h$-operator.
To illustrate this concept, we can consider the next example:

Examples and Remarks 4.2:

1) According to the unilateral shift operator $\mathbb{U}:\ell_2(\mathbb{C})\to\ell_2(\mathbb{C})$, such that $\ell_2(\mathbb{C})=\{(u_1,u_2,\dots): x_i\in\mathbb{C} : \sum_{i=1}^{\infty}|u_i|^2<\infty$ for all $i=1,2,\dots\}$, which is defined as $\mathbb{U}(u_1,u_2,\dots)=(0,u_1,u,\dots)$ for any $u_i\in\mathbb{C}$, is $\mathcal{KQ}$ $(\lambda-\mathcal{M})h$-operator.
2) The bilateral shift operator $\mathbb{B}:\ell_2(\mathbb{C})\to\ell_2(\mathbb{C})$, which is defined by $\mathbb{B}(u_1,u,\dots)=(0,u_1,u_2,\dots)$ for any $u_i\in\mathbb{C}$, is not $\mathcal{KQ}$ $(\lambda-\mathcal{M})h$-operator.
3) From (2), one can consequence the fact, if $\mathcal{T}$ is $\mathcal{KQ}$ $(\lambda-\mathcal{M})h$-operator, then $\mathcal{T}^*$, need not to be $\mathcal{KQ}$ $(\lambda-\mathcal{M})h$-operator, in general.
4) Consider the operator $\mathcal{T}=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$, define on complx plane $\mathbb{C}^2$, one can evalution that to have $\mathcal{T}$ is $\mathcal{KQ}$ $(\lambda-\mathcal{M})h$-operator with $\mathcal{K}=2$, but $\mathcal{T}$ is not $\mathcal{M}$-hyponormal, and the following corollary give the conditions to obtain the fact is true.

Corollary 4.3:

If $\mathcal{T}$ is $\mathcal{KQ}$ $(\lambda-\mathcal{M})h$-operator with densly range, then to be $\mathcal{T}$ is $\mathcal{M}$ -hyponormal operator.

Proof: Supposing $\mathcal{T}$ densely range exactly one get $\overline{\mathcal{T}(\mathcal{H})}=\mathcal{H}$, and let v in $\mathcal{H}$ so by using the definition for closure there is a sequence v_n in $\mathcal{H}$ such that $\mathcal{T}^k(v_n)\to\mathcal{T}^k(v)=\omega$, as $n\to\infty$. now from hypothesis of this corollary and theorem (3.3), we have $\mathcal{M}^2\parallel(\mathcal{T}-\lambda)\mathcal{T}^k(v)\parallel \geq \parallel(\mathcal{T}-\lambda)^*\mathcal{T}^k(v)\parallel$, this implies to $\mathcal{M}^2\parallel(\mathcal{T}-\lambda)\mathcal{T}^k(v_n)\parallel\geq\parallel(\mathcal{T}-\lambda)^*\mathcal{T}^k(v_n)\parallel$, thus $\mathcal{M}^2\parallel(\mathcal{T}-\lambda)\omega\parallel=\mathcal{M}^2\parallel\lim_{n\to\infty}(\mathcal{T}-\lambda)\mathcal{T}^k(v_n)\parallel=\mathcal{M}^2\lim_{n\to\infty}\parallel(\mathcal{T}-\lambda)\mathcal{T}^k(v_n)\parallel\geq\lim_{n\to\infty}\parallel(\mathcal{T}-\lambda)^*\mathcal{T}^k(v_n)\parallel=\parallel\lim_{n\to\infty}(\mathcal{T}-\lambda)^*\mathcal{T}^k(v_n)\parallel=\parallel(\mathcal{T}-\lambda)^*\omega\parallel$, therefore $\mathcal{T}$ is $\mathcal{M}$-hyponormal.

Corollary 4.4:

If $\mathcal{T}$ is $\mathcal{KQ}$ $(\lambda-\mathcal{M})h$-operator such that $\lambda\mathcal{T}+\delta$ is $\mathcal{KQ}$ $(\lambda-\mathcal{M})h$-operator, then $\mathcal{T}$ is $\mathcal{M}$ -hyponormal operator, for each $\lambda,\delta\in\mathbb{C}$.

Proof: We know the set of spectrum of any operator which is compact subset of complex numbers $\mathbb{C}$, so we can has λ no equal to zero, also δ, that leads to $\mathcal{T}_1=\lambda\mathcal{T}+\delta$ be invertible, thus from above corollary (4.3), one can obtain $\mathcal{T}_1$ is $\mathcal{M}$-hyponormal operator, therefore, $\mathcal{T}=\frac{1}{\lambda}(\mathcal{T}_1-\delta)$.

In [10], appeared if $\mathcal{T}$ the quasi-$\mathcal{M}$ -hyponormal operator, we can get $N(\mathcal{T}-\lambda)\subseteq N(\mathcal{T}-\lambda)^*$, where $\lambda\in\mathbb{C}\backslash\{0\}$, this fact is also true when $\mathcal{T}$ is $\mathcal{KQ}$ $(\lambda-\mathcal{M})h$-operator.where $N(\mathcal{T}-\lambda)$ is null space of $\mathcal{T}-\lambda$.

Theorem 4.5:

If $\mathcal{T}$ is $\mathcal{KQ}$ $(\lambda-\mathcal{M})h$-operator, then $N(\mathcal{T}-\lambda)\subseteq N(\mathcal{T}-\lambda)^*$.

Proof: Let v be any arbitray element such as $v\in N(\mathcal{T}-\lambda)$, and from hypothesis of this theorem $\mathcal{T}$ is $\mathcal{KQ}$ $(\lambda-\mathcal{M})h-$ operator, so that from theorem 4.1, one has $\mathcal{M}^2\parallel(\mathcal{T}-\lambda)\mathcal{T}^k(v)\parallel\geq\parallel(\mathcal{T}-\lambda)^*\mathcal{T}^k(v)\parallel$, and since $\mathcal{T}v=\lambda v$ so we get $\mathcal{T}^kv=\lambda^kv$, then $\parallel$

$(\mathcal{T}-\lambda)\mathcal{T}^k(v) \parallel = \parallel \mathcal{T}^{k+1}(v) - \lambda\mathcal{T}^k(v) \parallel = \parallel \lambda^{k+1}v^k - \lambda^{k+1}v^k \parallel = 0,$ then $0 \geq \parallel (\mathcal{T}-\lambda)^*\mathcal{T}^k(v) \parallel$, thus
$\parallel (\mathcal{T}-\lambda)^*\mathcal{T}^k(v) \parallel = 0$, so $\parallel (\mathcal{T}-\lambda)^*\lambda^k v \parallel = 0$, $|\lambda^k| \parallel (\mathcal{T}-\lambda)^* v \parallel = 0$, but $|\lambda^k| \neq 0$, one can have
$\parallel (\mathcal{T}-\lambda)^* v \parallel = 0$, one can have $(\mathcal{T}-\lambda)^* v = 0$, we have $v \in N(\mathcal{T}-\lambda)^*$. Therefore,$N(\mathcal{T}-\lambda) \subseteq N(\mathcal{T}-\lambda)^*$.

Proposition 4.6:
If $\mathcal{T}: \mathcal{H} \to \mathcal{H}$ is $\mathcal{KQ}$ $(\lambda-\mathcal{M})h$-operator.define on $\mathcal{H}$, then
$\mathcal{M}^2(\mathcal{T}^{*k}((\mathcal{T}-\lambda)^*(\mathcal{T}-\lambda))\mathcal{T}^k)^n \geq (\mathcal{T}^{*k}((\mathcal{T}-\lambda)(\mathcal{T}-\lambda)^*)\mathcal{T}^k)^n$, for each natural number n

Proof: We can use the mathematical induction for proving this proposition, at first since $\mathcal{T}$ is $\mathcal{KQ}$ $(\lambda-\mathcal{M})h$-operator then $\mathcal{M}^2(\mathcal{T}^{*k}(\mathcal{M}^2(\mathcal{T}-\lambda)^*(\mathcal{T}-\lambda))\mathcal{T}^k)^n \geq (\mathcal{T}^{*k}((\mathcal{T}-\lambda)(\mathcal{T}-\lambda)^*)\mathcal{T}^k)^n$ its satisfy as $n = 1$.
$\mathcal{M}^2(\mathcal{T}^{*k}(\mathcal{M}^2(\mathcal{T}-\lambda)^*(\mathcal{T}-\lambda))\mathcal{T}^k)^1 \geq \left(\mathcal{T}^{*k}\left((\mathcal{T}-\lambda)(\mathcal{T}-\lambda)^*\right)\mathcal{T}^k\right)^1 - (a)$
Supposing the statement is true as $n = m$, so we obtain
$\mathcal{M}^2(\mathcal{T}^{*k}(\mathcal{M}^2(\mathcal{T}-\lambda)^*(\mathcal{T}-\lambda))\mathcal{T}^k)^m \geq \left(\mathcal{T}^{*k}\left((\mathcal{T}-\lambda)(\mathcal{T}-\lambda)^*\right)\mathcal{T}^k\right)^m - (b)$
Next, to show this statement as $n = m+1$, so we have
$\mathcal{M}^2(\mathcal{T}^{*k}(\mathcal{M}^2(\mathcal{T}-\lambda)^*(\mathcal{T}-\lambda))\mathcal{T}^k)^{m+1} = \mathcal{M}^2(\mathcal{T}^{*k}((\mathcal{T}-\lambda)^*(\mathcal{T}-\lambda))\mathcal{T}^k)^m . (\mathcal{T}^{*k}(\mathcal{M}^2(\mathcal{T}-\lambda)^*(\mathcal{T}-\lambda))\mathcal{T}^k)^1$
From (a) and (b) of this proposition, we must obtain
$\mathcal{M}^2(\mathcal{T}^{*k}((\mathcal{T}-\lambda)(\mathcal{T}-\lambda)^*)\mathcal{T}^k)^m . (\mathcal{T}^{*k}((\mathcal{T}-\lambda)(\mathcal{T}-\lambda)^*)\mathcal{T}^k)^1$
$= (\mathcal{T}^{*k}((\mathcal{T}-\lambda)(\mathcal{T}-\lambda)^*)\mathcal{T}^k)^{m+1}$.

Therefore, $\mathcal{M}^2(\mathcal{T}^{*k}((\mathcal{T}-\lambda)^*(\mathcal{T}-\lambda))\mathcal{T}^k)^n \geq (\mathcal{T}^{*k}((\mathcal{T}-\lambda)(\mathcal{T}-\lambda)^*)\mathcal{T}^k)^n$.
In the next theorem, we introduce the condition of the $\mathcal{KQ}$ $(\lambda-\mathcal{M})h$-operators becomes $\mathcal{KQ}$ $(\lambda-\mathcal{M})h$-operator.

Theorem 4.7:
Assume that$\mathcal{T}_1$ and $\mathcal{T}_2$ are $\mathcal{KQ}$ $(\lambda-\mathcal{M})h$ −operators defined on $\mathcal{H}$. If ${\mathcal{T}_1}^{*k}(\mathcal{T}_2-\lambda)^* = {\mathcal{T}_2}^{*k}(\mathcal{T}_1-\lambda)^* = (\mathcal{T}_1-\lambda){\mathcal{T}_2}^k = (\mathcal{T}_2-\lambda){\mathcal{T}_1}^k = (\mathcal{T}_2-\lambda)^*(\mathcal{T}_1-\lambda) = (\mathcal{T}_1-\lambda)^*(\mathcal{T}_1-\lambda) = \mathcal{T}_1\mathcal{T}_2 = \mathcal{T}_2\mathcal{T}_1 = 0$, then $\mathcal{T}_1+\mathcal{T}_2$ is $\mathcal{KQ}$ $(\lambda-\mathcal{M})h-$ operator, where $\mathcal{M}$ is real number positive. Such a way that $\mathcal{M}^2 \geq \mathcal{M}_1^2\mathcal{M}_2^2$.

Proof: From hypothesis $\mathcal{T}_1$ with $\mathcal{T}_2$ are $\mathcal{KQ}$ $(\lambda-\mathcal{M})h$-operators, then both of them satisfy (1), and by our hypothesis${\mathcal{T}_1}^{*k}(\mathcal{T}_2-\lambda)^* = {\mathcal{T}_2}^{*k}(\mathcal{T}_1-\lambda)^* = (\mathcal{T}_1-\lambda){\mathcal{T}_2}^k = (\mathcal{T}_2-\lambda){\mathcal{T}_1}^k = (\mathcal{T}_2-\lambda)^*(\mathcal{T}_1-\lambda) = (\mathcal{T}_1-\lambda)^*(\mathcal{T}_1-\lambda) = 0$, to show that $\mathcal{T}_1+\mathcal{T}_2$ is $\mathcal{KQ}$ $(\lambda-\mathcal{M})h$ −operator, we have
$\mathcal{M}^2((\mathcal{T}_1+\mathcal{T}_2)^{*k}((\mathcal{T}_1-\lambda)+(\mathcal{T}_2-\lambda))^*((\mathcal{T}_1-\lambda)+(\mathcal{T}_2-\lambda))(\mathcal{T}_1+\mathcal{T}_2)^k)$
$= \mathcal{M}^2\left({\mathcal{T}_1}^{*k} + k{\mathcal{T}_1}^{*k-1}{\mathcal{T}_2}^* + \cdots + {\mathcal{T}_2}^{*k}\right)((\mathcal{T}_1-\lambda)^* + (\mathcal{T}_2-\lambda)^*)((\mathcal{T}_1-\lambda)+(\mathcal{T}_2-\lambda))\left({\mathcal{T}_1}^k + k{\mathcal{T}_1}^{k-1}\mathcal{T}_2 + \cdots + {\mathcal{T}_2}^k\right)$
$= \mathcal{M}^2({\mathcal{T}_1}^{*k}(\mathcal{T}_1-\lambda)^* + {\mathcal{T}_1}^{*k}(\mathcal{T}_2-\lambda)^* + {\mathcal{T}_2}^{*k}(\mathcal{T}_1-\lambda)^* + {\mathcal{T}_2}^{*k}(\mathcal{T}_2-\lambda)^*)((\mathcal{T}_1-\lambda){\mathcal{T}_1}^k + (\mathcal{T}_1-\lambda){\mathcal{T}_2}^k + (\mathcal{T}_2-\lambda){\mathcal{T}_1}^k + (\mathcal{T}_2-\lambda){\mathcal{T}_2}^k)$
$= \mathcal{M}^2({\mathcal{T}_1}^{*k}(\mathcal{T}_1-\lambda)^* + {\mathcal{T}_2}^{*k}(\mathcal{T}_2-\lambda)^*)((\mathcal{T}_1-\lambda){\mathcal{T}_1}^k + (\mathcal{T}_2-\lambda){\mathcal{T}_2}^k)$
$= \mathcal{M}^2(\left({\mathcal{T}_1}^{*k}(\mathcal{T}_1-\lambda)^*\right)\left((\mathcal{T}_1-\lambda){\mathcal{T}_1}^k\right) + \mathcal{M}^2\left({\mathcal{T}_1}^{*k}(\mathcal{T}_1-\lambda)^*\right)\left((\mathcal{T}_2-\lambda){\mathcal{T}_2}^k\right) + \mathcal{M}^2\left({\mathcal{T}_2}^{*k}(\mathcal{T}_2-\lambda)^*\right)\left((\mathcal{T}_1-\lambda){\mathcal{T}_1}^k\right) + \mathcal{M}^2\left({\mathcal{T}_2}^{*k}(\mathcal{T}_2-\lambda)^*\right)\left((\mathcal{T}_2-\lambda){\mathcal{T}_2}^k\right)$.
Since $\mathcal{M}^2 \geq \mathcal{M}_1^2\mathcal{M}_2^2$, we have that,

$\mathcal{M}^2((\mathcal{T}_1{}^{*k}(\mathcal{T}_1-\lambda)^*)\left((\mathcal{T}_1-\lambda)\mathcal{T}_1{}^k\right)+\mathcal{M}^2(\mathcal{T}_1{}^{*k}(\mathcal{T}_1-\lambda)^*)\left((\mathcal{T}_2-\lambda)\mathcal{T}_2{}^k\right)+\mathcal{M}^2(\mathcal{T}_2{}^{*k}(\mathcal{T}_2-$
$\lambda)^*)\left((\mathcal{T}_1-\lambda)\mathcal{T}_1{}^k\right)+\mathcal{M}^2(\mathcal{T}_2{}^{*k}(\mathcal{T}_2-\lambda)^*)\left((\mathcal{T}_2-\lambda)\mathcal{T}_2{}^k\right).$
$=\mathcal{M}_1^2((\mathcal{T}_1{}^{*k}(\mathcal{T}_1-\lambda)^*)((\mathcal{T}_1-\lambda)\mathcal{T}_1{}^k)+\mathcal{M}_2^2(\mathcal{T}_2{}^{*k}\mathcal{T}_2{}^*)((\mathcal{T}_2-\lambda)\mathcal{T}_2{}^k)$
$=\mathcal{M}_1^2(\mathcal{T}_1{}^{*k}(\mathcal{T}_1-\lambda)^*(\mathcal{T}_1-\lambda)\mathcal{T}_1{}^k)+\mathcal{M}_2^2(\mathcal{T}_2{}^{*k}(\mathcal{T}_2-\lambda)^*(\mathcal{T}_2-\lambda)\mathcal{T}_2{}^k).$
Since $\mathcal{T}_1$ and $\mathcal{T}_2$ are $\mathcal{KQ}\ (\lambda-\mathcal{M})h$-operators then,
$\mathcal{M}_1^2(\mathcal{T}_1{}^{*k}(\mathcal{T}_1-\lambda)^*(\mathcal{T}_1-\lambda)\mathcal{T}_1{}^k)+\mathcal{M}_2^2(\mathcal{T}_2{}^{*k}(\mathcal{T}_2-\lambda)^*(\mathcal{T}_2-\lambda)\mathcal{T}_2{}^k)$
$=\mathcal{M}_1^2(\mathcal{T}_1{}^{*k}(\mathcal{T}_1-\lambda)^*(\mathcal{T}_1-\lambda)\mathcal{T}_1{}^k)+\mathcal{M}_2^2(\mathcal{T}_2{}^{*k}(\mathcal{T}_2-\lambda)^*(\mathcal{T}_2-\lambda)\mathcal{T}^k)\geq(\mathcal{T}_1{}^{*k}(\mathcal{T}_1-\lambda)(\mathcal{T}_1-$
$\lambda)^*\mathcal{T}_1{}^k)+(\mathcal{T}_2{}^{*k}(\mathcal{T}_2-\lambda)(\mathcal{T}_2-\lambda)^*\mathcal{T}_2{}^k).$
Therefore, $\mathcal{T}_1+\mathcal{T}_2$ is $\mathcal{KQ}\ (\lambda-\mathcal{M})h$-operator.

Theorem 4.8:
Assume that $\mathcal{T}_1$ and $\mathcal{T}_2$ are $\mathcal{KQ}\ (\lambda-\mathcal{M})h$-operators defined on $\mathcal{H}$, then $\mathcal{T}_1.\mathcal{T}_2$ is $\mathcal{KQ}\ (\lambda-\mathcal{M})h$-operator, whenever the the following requirements condtions are satisfied:
i) $\mathcal{T}_1$ is normal;
ii) $\mathcal{T}_1\in\{\mathcal{T}_2{}^*\}$.

Proof **:** The assumption $\mathcal{T}_1$ and $\mathcal{T}_2$ to be $\mathcal{KQ}\ (\lambda-\mathcal{M})h$-operators must have the following conditions:
$(\mathcal{T}_1-\lambda)^*(\mathcal{T}_1-\lambda)=(\mathcal{T}_1-\lambda)(\mathcal{T}_1-\lambda)^*,\ (\mathcal{T}_1-\lambda)(\mathcal{T}_2-\lambda)^*=(\mathcal{T}_2-\lambda)^*(\mathcal{T}_1-\lambda).$
Now, show that $\mathcal{T}_1.\ \mathcal{T}_2$ to be $\mathcal{KQ}\ (\lambda-\mathcal{M})h$-operator, we have
$\mathcal{M}^2((\mathcal{T}_1\mathcal{T}_2)^{*k}((\mathcal{T}_1-\lambda)(\mathcal{T}_2-\lambda))^*((\mathcal{T}_1-\lambda)(\mathcal{T}_2-\lambda))(\mathcal{T}_1\,\mathcal{T})^k)$
$=\mathcal{M}^2(\mathcal{T}_2{}^{*k}\mathcal{T}_1{}^{*k})(\mathcal{T}_1-\lambda)^*(\mathcal{T}_1-\lambda)(\mathcal{T}_2-\lambda)(\mathcal{T}_1{}^k\mathcal{T}_2{}^k)$
$\geq(\mathcal{T}_2{}^{*k}\mathcal{T}_1{}^{*k})(\mathcal{T}_2-\lambda)^*(\mathcal{T}_1-\lambda)(\mathcal{T}_1-\lambda)^*(\mathcal{T}_2-\lambda)(\mathcal{T}_1{}^k\mathcal{T}_2{}^k)$, because
$\geq(\mathcal{T}_2{}^{*k}\mathcal{T}_1{}^{*k})(\mathcal{T}_1-\lambda)((\mathcal{T}_2-\lambda)^*(\mathcal{T}_1-\lambda)^*)(\mathcal{T}_2-\lambda)(\mathcal{T}_1{}^k\mathcal{T}_2{}^k)$
$\geq(\mathcal{T}_2{}^{*k}\mathcal{T}_1{}^{*k})(\mathcal{T}_1-\lambda)(\mathcal{T}_2-\lambda)((\mathcal{T}_2-\lambda)^*(\mathcal{T}_1-\lambda)^*)(\mathcal{T}_1{}^k\mathcal{T}_2{}^k)$
$\geq((\mathcal{T}_1\mathcal{T}_2)^{*k}((\mathcal{T}_1-\lambda)(\mathcal{T}_2-\lambda))((\mathcal{T}_1-\lambda)(\mathcal{T}_2-\lambda))^*(\mathcal{T}_1\mathcal{T}_2)^k).$
Hence, $\mathcal{T}_1.\mathcal{T}_2$ is $\mathcal{KQ}\ (\lambda-\mathcal{M})h$-operator.

5. Conclusions

This article is establish new characterizations of the hyponormal operator, which is $\mathcal{KQ}\ (\lambda-\mathcal{M})h$-operator and some operations related to the $\mathcal{KQ}\ (\lambda-\mathcal{M})h$-operator. We investigate and generalized some properties of class of operators. As well as we found a strong relation between $\mathcal{M}$-hyponormal operator and the other operator which we studied. Finally, we show that the hyponormal operator may not be satisfied in $\mathcal{KQ}\ (\lambda-\mathcal{M})h$-operator, such as the property of the sum and the product of two $\mathcal{KQ}\ (\lambda-\mathcal{M})h$-operator, by proving that it is not necessarily true, one can accomplish it by including some conditions.

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