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I-Nearly Prime Submodules

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Abstract:

Let R be a commutative ring with identity, and I a fixed ideal of R and M be an unitary R-module. In this paper we introduce and study the concept of I-nearly prime submodules as genrealizations of nearly prime and we investigate some properties of this class of submodules. Also, some characterizations of I-nearly prime submodules will be given.

Keywords: Prime submodules, nearly prime submodules, I-prime submodules.

المقاسات الجزئية الاولية تقريبا من النمط−I

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الخلاصة

لتكن R حلقة ابدالية ذات عنصرمحايد، وليكن I مثالي من R، M مقاسا احاديا معرفا على الحلقة R.
قدمنا ودرسنا في هذا البحث المفهوم :المقاسات الجزئية الاولية تقريبا من النمط−I كاعمام للمقاسات الجزئية الاولية تقريبا . لقد درسنا واعطينا بعض خواص ومميزات هذا النوع من المقاسات الجزئية.

Introduction

Throughout, R represents an associative ring with nonzero identity and I a fixed ideal of R. A proper submodule N of M is called a prime submodule if whenever $r\in R$ and $x \in M$ with $rx \in N$ implies that $r\in [N:M]$ or $x \in N$, [1]. One of generalization of this concept was studied as nearly prime, [2]. A proper submodule N of M is called nearly prime submodule if whenever $r \in R, x \in M$ and $rx \in N$ implies that either $x \in N + J(M)$ or $r \in [N + J(M): M]$. Previuos work [3] and [4] introduced the notions I- prime and I- primary submodules. A proper submodule N of M is called I- prime submodule of M if rx ∈ N − IN for all $r\in R,\ x \in M$ implies that either $r \in [N: M]$ or $x \in N$. A proper submodule N of M is called I- primary submodule of M if rx ∈ N − IN for all $r\in R,\ x \in M$ implies that either $r \in \sqrt{[N: M]}$ or $x \in N$. In this paper, we define and study I- nearly prime submodules which are generalizations of prime submodules and nearly prime submodules to I- nearly prime submodules. A proper submodule N of M is called I-nearly prime submodule if rx ∈ N − IN for all $r\in R,\ x \in M$ implies that either $r \in [N + J(M): M]$ or $x \in N + J(M)$, where $J(M)$ is the intersection of all maximal submodule of M. Here is a brief summary of the paper. In Theorem 1.4 we show that N is I-nearly prime submodule in M if and only if for any ideal J of R and submodule K of M such that $JK \subsetneq N - IN$, we have $K \subsetneq N + J(M)$ or $J \subsetneq [N + J(M): M]$. In Proposition 1.6 we show that if N is an I- nearly prime in M and $[N: M]N \nsubseteq IN$, then N is a nearly prime in M. In Theorem 1.8 we show that if N is a submodule of an R –module M then the following statements are

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equivalent: (1) N is an I-nearly prime in M. (2) For $r \in R \backslash [N + J(M): M], [N: r] = [N + J(M)] \cup [IN: r]$. (3) For $r \in R \backslash [N + J(M): M]$, $[N: r] = N + J(M)$ or $[N: r] = [IN: r]$.

1. Main result

Definition 1.1: A proper submodule N of an R-moadule M is called I-nearly prime submodule if and only if whenever $r \in R$, $x \in M$ and $rx \in N - IN$ implies that either $x \in N + J(M)$ or $r \in [N + J(M): M]$, where I is an ideal of R and $J(M)$ is the Jacobson radical of M.

For examples:1- Consider the ring of integers Z and the Z-module Z12. Take
I = 4Z as an ideal of Z and N = $(\bar{4})$ be a submodule of Z12 generated by
4. Then N is an I-nearly prime submodule of Z12 since N $-$ IN = $(\bar{4}) - 4Z.(\bar{4})$ =
$(\bar{4}) - (\bar{4}) = \emptyset$. On the other side, N is not a prime submodule since $\bar{4} = 2.\bar{2} \in$ N but not $\bar{2} \in$ N nor 2$\in$ [N:Z12].

2- Let N= $(\bar{0})$ is an I-nearly prime submodule of Z6 as a Z-module since if I = (0) is taken as an ideal of Z, then N-IN= $(\bar{0})$ – (0). $(\bar{0}) = \emptyset$. On the other side, N is not a nearly prime submodule, see [2].

3- Consider the ring of integers Z and the Z-module Z40 and N = $(\bar{8})$. Take
I = [N:M]= 8Z as an ideal of Z, then [N:M]N=N . Then N is an I-nearly prime submodule of Z12 since N $-$ IN = $(\bar{8}) - 8Z.(\bar{8}) = \emptyset$. On the other side, if I = (0) is taken as an ideal of Z, then [N:M]N=(0), then N is not an I-nearly prime submodule since $\bar{8} = 4.\bar{2} \in$ N but not $\bar{2} \in$ N nor 4$\in$ [N:Z12] .

Proposition (1. 2):

1- If N is an I- nearly prime in M and K is a submodule of M with $J(M) \subseteq J(K)$, then N is an I-nearly prime submodule of K.

2- If $I_1 \subseteq I_2$. Then N is an I_1 - nearly prime implies N is I_2- nearly prime.

Proof. (1): Suppose that $am \in N - IN$ where $a \in R$ and $m \in K$. Since N is an I-nearly prime submodule of M, so either $m \in N + J(M)$ or $a \in [N + J(M): M]$. But $J(M) \subseteq J(K)$, so either $m \in N + J(K)$ or $a \in [N + J(K): K]$ .Therefore N is an I- nearly prime submodule of K.

(2): Let $m \in M$ and $m \in M$ with $am \in N - I_2N$. Since $I_1 \subseteq I_2$, $N - I_2N \subseteq N - I_1N$,then $am \in N - I_1N$. But N is an I_1- nearly prime. So $a \in [N + J(M): M]$ or $m \in N + J(M)$. Thus N is an I_2-nearly prime.

Proposition (1.3): Let N be a submodule of an $R -$ module M.

1- If N is an I-nearly prime and $J(M) \subseteq N$, then N is an I-prime (and I-primary) .

2- If N is a maximal an I- nearly prime submodule of a local R-module M, then N is an I-prime (and I-primary) in M.

3- If N is an I- nearly prime submodule of a semisimple R-module M, then N is an I-prime (and I-primary) in M.

Proof. (1).The proof is trivial.

(2). Suppose that $am \in N - IN$ where $a \in R, m \in M$. Since N is an I-nearly prime submodule of M, so either $m \in N + J(M)$ or $a \in [N + J(M): M]$`. But M be a local and N is a maximal, so $J(M) = N$, [5]. So either $m \in N$ or $a \in [N: M]$ .Therefore N is an I-prime (and I-primary) in M.

(3). Suppose that $am \in N - IN$ where $m \in M$ and $a \in R$. Because N is an I-nearly prime submodule of M, so either $a \in [N + J(M): M]$ or $m \in N + J(M)$. But M be a semisimple an R–module, so $J(M) = 0$, [6]. So either $m \in N$ or $a \in [N: M]$. Hence N is an I-prime (and I-primary) in M.

The following theorem gives a useful characterization for an I-nearly prime submodules.

Theorem (1.4): Let N be a proper submodule of an R–module M. Then N is an I-nearly prime submodule in M if and only if for any ideal J of R and submodule K of M such that $JK \subsetneq N - IN$, we have $K \subsetneq N + J(M)$ or $J \subsetneq [N + J(M): M]$.`

Proof. Suppose that N is an I-nearly prime in M. Let $JK \subsetneq N - IN$ for some ideal J of R and submodule K of M . If $J \nsubseteq [N + J(M): M]$ and $K \nsubseteq N + J(M)$, so there exists $r \in J \backslash [N + J(M): M]$ and $x \in K \backslash [N + J(M)]$ such that $rx \in N - IN$.

By assuming that N is an I-nearly prime submodule in M, either $x \in N + J(M)$ or $r \in [N + J(M): M]$ which is a contradiction. Hence $J \subsetneq [N + J(M): M]$ or $K \subsetneq N + J(M)$.

Conversely supposse that $rm \in N - IN$ where $r \in R, m \in M$. So $(r)(m) = (rm)$ $\subsetneq N - IN$. So, either $(r) \subsetneq [N + J(M): M]$ or $(m) \subsetneq N + J(M)$. Therefore $r \in [N + J(M): M]$ or $m \in N + J(M)$. Thus N is an I-nearly prime submodule of M.

Let R be a ring . A subset S of R is called multiplicatively closed subset if $1 \in S$ and $ab \in S$, $\forall\, a, b \in S$, [7].

Let R_s be the set of all fractional r/s where $r \in R$ and $s \in S$ and M_s be the set of all farctional x/s where $x \in M$ and $s \in S$. For x_1 , $x_2 \in M$ and s_1 , $s_2 \in S$, $x_1/s_1 = x_2/s_2$ if and only if there exists $t \in S$ such that $t\,(s_1\, x_1 - s_2\, x_2) = 0$.

So, we can make M_s into R_s–module by setting $x/s + y/t = (tx + sy)/st$ and $r/t\ .\ x/s = rx/ts)$ for every $x, y \in M$ and $s, t \in S$, $r \in R$. And M_s is the module of fractions. If N is a submodule of M, so $N_S = \{n/s; n \in N, s \in S\}$ is a submodule of $M_{S,}$ [7].

The quotient and localization of prime submodules are again prime submodules. But in case of I-nearly prime submodules, we give a condition under which the quotient and localization become true.

Proposition (1.5): Suppose that N is an I-nearly prime in M.

1) If $N_S \neq M_S$ and $(IN)s \subseteq IsNs$. Then N_S is an I_S -nearly prime submodule of an R_S–module M_S.

2) If $K \subseteq N$ and $N/K + J(M/K) = N + J(M)/K$, then N/K is an I-nearly prime in M/K .

Proof (1): For all $r/s \in R_S$ and $x/t \in M$s, let $r/s.x/t = rx/st \in Ns - IsNs \subseteq N_S - (IN)s = (N - IN)s$. Then $rx/st = m/u$ for $m \in N - IN$ and $u \in S$. So for some $v \in S, vurx = vstm \in N - IN$. As N is an I-nearly prime submodule, so either $vur \in [N + J(M){:}\, M]$ or $x \in N + J(M)$. So $ruv/suv = r/s \in [N + J(M){:}\, M]_S = [N_S + J(M_S){:}\, M_S]$ or $\frac{x}{t} \in [N + J(M)]_S = N_S + J(M_S)$ by [8]. Hence N_S is an I_S -nearly prime in M_S.

(2): Suppose that $m \in M$ and $a \in R$ with $a(m + K) = am + K \in N/K - I(N/K)$.

Then $am + K \in [N - IN]/K$. So $am \in N - IN$. Since N is an I-nearly prime submodule of M, so either $m \in N + J(M)$ or $a \in [N + J(M){:}\, M]$.Therefore $m + K \in N/K + J(M/K)$ or $a \in [N/K + J(M/K){:}\, M/K]$. Therefore N/K is an I- nearly prime in M/K.

Proposition (1.6): If N is an I- nearly prime in M and $[N{:}\,M]N \nsubseteq IN$, then N is a nearly prime in M.

Proof: We show that N is a nearly prime. Suppose that $am \in N$ where $m \in M, a \in R$.

If $am \notin IN$, then N, I- nearly prime gives $m \in N + J(M)$ or $a \in [N + J(M){:}\, M]$. So assume that $am \in IN$. First suppose that $aN \nsubseteq IN$, say $an \notin IN$ where $n \in N$.Then $a(m + n) \in N - IN$, so $a \in [N + J(M){:}\, M]$ or $(m + n) \in N + J(M)$. Hence $a \in [N + J(M){:}\, M]$ or $m \in N + J(M)$. Now, if $m[N{:}\,M] \nsubseteq IN$.So $\exists\, b \in [N{:}\,M]$ such that $mb \notin IN$. So *(a+b)m*$\in N$. Therefore $m \in N + J(M)$ or $(a + b) \in [N + J(M){:}\, M]$. Then $m \in N + J(M)$ or $a \in [N + J(M){:}\, M]$. Suppose that $m[N{:}\,M] \subseteq IN$. Since $[N{:}\,M]N \nsubseteq IN$, there exists $r \in [N{:}\,M]$, $x \in N$ with $rx \notin IN$. Then $(a + r)(m + x) \in N - IN$. Then $(a + r) \in [N + J(M){:}\, M]$ or $(m + x) \in N + J(M)$. Hence $a \in [N + J(M){:}\, M]$ or $m \in N + J(M)$. So N be a nearly prime in M.

Corollary (1.7): If N is an 0- nearly prime in M and $[N{:}\,M]N \neq 0$. Then N is a nearly prime in M.

In what follows we give some charactrizations for an I-nearly prime .

Theorem (1.8): Suppose that N is a submodule of an R –module M. Then the following statements are equivalent:

(1) N is an I-nearly prime in M.

(2) For $r \in R\backslash[N + J(M){:}\, M]$, $[N{:}\,r] = [N + J(M)] \cup [IN{:}\,r]$.

(3) For $r \in R\backslash[N + J(M){:}\, M]$, $[N{:}\,r] = N + J(M)$ or $[N{:}\,r] = [IN{:}\,r]$.

Proof: (1) $\rightarrow$ (2): Suppose that N is an I-nearly prime submodule of M such that $r \notin [N + J(M){:}\, M]$. Let $m \in [N{:}\,r]$. So $rm \in N$. If $rm \notin IN$, then $m \in N + J(M)$.

Because N is an I-nearly prime submodule in M. If $rm \in IN$, so $m \in [IN{:}\,r]$. Hence $[N{:}\,r] \subsetneq [N + J(M)] \cup [IN{:}\,r]$. Now since $IN \subsetneq N$, the other inclusion is hold.

(2) $\rightarrow$ (3): Because $[N{:}\,r]$ is a submodule of M, so it is clear.

(3) $\rightarrow$ (1): Suppose that $rm \in N - IN$ where $r \in R, m \in M$. If $r \notin [N + J(M){:}\, M]$, so either $[N{:}\,r] = N + J(M)$ or $[N{:}\,r] = [IN{:}\,r]$. Since $rm \notin IN$, so $m \notin [IN{:}\,r]$. But $rm \in N$, so $m \in [N{:}\,r]$. Then $[N{:}\,r] = N + J(M)$. Therefore $m \in N + J(M)$. Thus N is an I-nearly prime submodule of M.

Proposition (1.9): Suppose that M_1be an R_1-module and M_2 be an R_2-module. Then we have :

(1) If N_1 is an I_1- nearly prime submodule of M_1 such that $IN_1 \times M_2 \subseteq I(N_1 \times M_2)$ and $J(M_1) \times M_2 \subseteq J(M_1 \times M_2)$, then $N_1 \times M_2$ is an I- nearly prime in $M_1 \times M_2$.

(2): If N_2 is an I_2- nearly prime in M_2 such that $IN_2 \times M_1 \subseteq I(N_2 \times M_1)$ and $J(M_2) \times M_1 \subseteq J(M_2 \times M_1)$, then $M_1 \times N_2$ is an I- nearly prime in $M_1 \times M_2$.

Proof: (1): Suppose that $(a,b) \in R_1 \times R_2$ and $(m_1,m_2) \in M$ with $(a,b)(m_1,m_2) = (am_1,\ bm_2) \in N_1 \times M_2 - I(N_1 \times M_2)$, and $N_1 \times M_2 - I(N_1 \times M_2) \subseteq N_1 \times M_2 - IN_1 \times M_2 = (N_1 - IN_1) \times M_2$. We have $am_1 \in N_1 - IN_1$ but N_1 is an I_1- nearly prime in M_1. So $a \in [N_1 + J(M_1): M_1]$ or $m_1 \in N_1 + J(M_1)$. So $(a,b) \in [N_1 + J(M_1): M_1] \times R_2 = \left[(N_1 + J(M_1)) \times M_2 :_{R_1 \times R_1} M_1 \times M_2\right] = \left[N_1 \times M_2 + J(M_1) \times M_2 :_{R_1 \times R_1} M_1 \times M_2\right] \subseteq \left[N_1 \times M_2 + J(M_1 \times M_2) :_{R_1 \times R_1} M_1 \times M_2\right]$ or $(m_1,m_2) \in [N_1 + J(M_1)] \times M_2 = N_1 \times M_2 + J(M_1) \times M_2 \subseteq N_1 \times M_2 + J(M_1 \times M_2)$. Hence $N_1 \times M_2$ is an I- nearly prime submodule of $M_1 \times M_2$.
The proof of (2) is similar to proof (1).

Proposition (1.10): Let I_1 and I_2 be ideals of R_1 and R_2, respectively, with $I = I_1 \times I_2$.
1. $N_1 \times N_2$ is an I-nearly prime in $M_1 \times M_2$ where $I_iN_i = N_i$ for $i = 1, 2$.
2. If N_1 is a prime in M_1, then $N_1 \times M_2$ is an I-nearly prime in $M_1 \times M_2$.
3. If N_1 is an I_1 - nearly prime in M_1 with $I_2M_2 = M_2$, then $N_1 \times M_2$ is an I-nearly prime in $M_1 \times M_2$.
4. If N_2 is a prime in M_2, then $M_1 \times N_2$ is an I-nearly prime in $M_1 \times M_2$.
5. If N_2 is an I_2 - nearly prime in M_2 with $I_1M_1 = M_1$, then $M_1 \times N_2$.

Proof (1): Since $I_1N_1 = N_1$ and $I_2N_2 = N_2$. Then $I_1N_1 \times I_2N_2 = (I_1 \times I_2)(N_1 \times N_2) = I(N_1 \times N_2) = N_1 \times N_2$. So $N_1 \times N_2 - I(N_1 \times N_2) = \emptyset$. Thus there is nothing to prove.
2. Let N_1 is a prime in M_1. Then $N_1 \times M_2$ is a prime in $M_1 \times M_2$, [9] and hence *I*- nearly prime in $M_1 \times M_2$.
3. Let N_1 is an I_1 - nearly prime in M_1 and $I_2M_2 = M_2$. Suppose that $(r_1,r_2) \in R$ and $(m_1,m_2) \in M$ with $(r_1,r_2)(m_1,m_2) = (r_1m_1, r_2m_2) \in N_1 \times M_2 - I(N_1 \times M_2) = N_1 \times M_2 - (I_1 \times I_2)(N_1 \times M_2) = (N_1 \times M_2 - (I_1N_1 \times I_2M_2) = (N_1 \times M_2 - (I_1N_1 \times M_2) = (N_1 - IN_1) \times M_2$. Then $r_1m_1 \in N_1 - IN_1$ and N_1 is an I_1- nearly prime submodule of M_1, so $r_1 \in [N_1 + J(M_1): M_1]$ or $m_1 \in N_1 + J(M_1)$.
Therefore $(r_1,r_2) \in [N_1 + J(M_1): M_1] \times R_2 = \left[(N_1 + J(M_1)) \times M_2 :_{R_1 \times R_1} M_1 \times M_2\right] = \left[N_1 \times M_2 + J(M_1) \times M_2 :_{R_1 \times R_1} M_1 \times M_2\right] \subseteq \left[N_1 \times M_2 + J(M_1 \times M_2) :_{R_1 \times R_1} M_1 \times M_2\right]$ or $(m_1,m_2) \in [N_1 + J(M_1)] \times M_2 = N_1 \times M_2 + J(M_1) \times M_2 \subseteq N_1 \times M_2 + J(M_1 \times M_2)$. So $N_1 \times M_2$ is an *I*- nearly prime in $M_1 \times M_2$.
The proofs of (4) and (5) are similar to parts (2), (3), respectively.

Proposition(1.11): Let M be an R-module and let N be a proper submodule of M such that $N/IN + J(M/IN) = N + J(M)/IN$. Then N is an I-nearly prime in M if and only if N/IN is 0-nearly prime in M/IN.

Proof: Let N be an I-nearly prime in M. Suppose that $0 \neq ax + IN = a(x + IN) \in N/IN$ in M/IN where $a \in R, x \in M$. Then $ax \in N - IN$. Since N is an I-nearly prime submodule of M, so either $x \in N + J(M)$ or $a \in [N + J(M): M] = [[N + J(M)]/IN : M/IN]$. Therefore $x + IN \in [N + J(M)]/IN = N/IN + J(M/IN)$ or $a \in [N/IN + J(M/IN): M/IN]$. Hence N/IN is 0- nearly prime in M/IN **.**
Conversely, let N/IN is an 0-nearly prime in M/IN. Assume that $a \in R,\ x \in M$ with $ax \in N - IN$. So $0 \neq a(x + IN) = ax + IN \in N/IN$. But N/IN is an 0-nearly prime in M/IN **.** Thus $x + IN \in N/IN + J(M/IN) = [N + J(M)]/IN$ or $a \in [N/IN + J(M/IN): M/IN] = [[N + J(M)]/IN : M/IN]$ and so $x \in N + J(M)$ or $a \in [N + J(M): M]$. Hence N is an 0-nearly prime.

Theorem (1.12): If M is an R-module and I is an ideal of R, then the following statements are equivalent.
1- IM is an I-nearly prime submodule M;
2- For $x \in [M \backslash (IM + J(M))]$; $[IM : x] = [I(IM): x] \cup [IM + J(M): M]$;
3- For $x \in [M \backslash (IM + J(M))]$, $[IM: x] = [I(IM): x]$ or $[IM: x] = [IM + J(M): M]$.
4- If $JK \subsetneq IM - I(IM)$, then $J \subsetneq [IM + J(M): M]$ or $K \subsetneq IM + J(M)$ for each an ideal J of R and submodule K of M.

Proof : (1) → (2): Suppose that $x \in M - IM, r \in [IM: x]$. So $rx \in IM$. If $rx \notin I(IM)$, but IM is an I -nearly prime and $x \notin IM + J(M)$, so $r \in [IM + J(M): M]$. If $rx \in I(IM)$, so $r \in [I(IM): x]$. Thus, $[IM : x] \subsetneq [IM + J(M): M] \cup [IIM : x]$. On the other hand $I(IM) \subsetneq IM$, so $[I(IM): x] \cup [IM + J(M): M] \subsetneq [IM: x]$.
(2) → (3): It follows directly by the fact that if an ideal is a union of two ideals, then it is equal to one of them.

(3)→ (4): Suppose that $JK \subsetneq IM$. Let $J \nsubseteq [IM + J(M): M]$ and $K \nsubseteq IM + J(M)$. Ausssme that $x \in K$. If $x \notin IM + J(M)$. So $Jx \subsetneq IM$ and hence $J \subsetneq [IM: x]$. But $J \nsubseteq [IM + J(M): M]$, so $J \subsetneq [IM: x] = [I(IM): x]$. Thus, $xJ \subsetneq I(IM)$, so $KJ \subsetneq I(IM)$. Suppose that $x \in IM$. Let $m \in K - IM$. Then $(x + m) \in K - IM$. So $(x + m)J \subsetneq I(IM)$. Let $r \in J$. Then $x = (x + m)r - mr \in I(IM)$. So $xJ \subsetneq I(IM)$. Thus $JK \subsetneq I(IM)$. (4) → (1): By theorem (1. 4).

Proposition (1.13):

1- Let N_1 and N_2 are two submodules of the R-0modules M_1,M_2 ,respectively. If $N_1 \oplus N_2$ is an I-nearly prime and small submodule of $M = M_1 \oplus M_2$ such that $J(M_1 \oplus M_2) \subseteq [J(M_1) \oplus M_2]$ and $J(M_1 \oplus M_2) \subseteq [M_1 \oplus J(M_2)]$, then $N_1$and N_2 are I- nearly prime in M_1, M_2 respectively.

2- Let N be a small submodule of an R-module M_1 and M_2 be any two modules with $J(M_1) \oplus M_2$ is small in M. If N is an I-nearly prime, then $N \oplus M_2$ is an I-nearly primes submodule of $M_1 \oplus M_2$.

Proof. (1). Let $am_1 \in N_1 - IN_1$ where $a \in R, m_1 \in M_1$.Then $a(m_1, 0) \in (N_1 \oplus N_2) - I(N_1 \oplus N_2)$. Since $(N_1 \oplus N_2)$ is an I-nearly prime and small, then either $(m_1, 0) \in (N_1 \oplus N_2) + J(M) = J(M_1 \oplus M_2) = J(M_1) \oplus J(M_2)$, [10] and so $m_1 \in J(M_1) \subseteq N_1 + J(M_1)$ or a $\in [(N_1 \oplus N_2) + J(M_1 \oplus M_2): M_1 \oplus M_2]$= $[J(M_1 \oplus M_2): M_1 \oplus M_2] \subseteq [J(M_1) \oplus M_2 : M_1 \oplus M_2]$ and so a $\in [J(M_1): M_1] \subseteq [N_1 + J(M_1): M_1]$. It follows that either $m_1 \in N_1 + J(M_1)$ or a $\in [N_1 + J(M_1): M_1]$. Hence N_1 is an I-nearly prime in M_1.

By a similar proof, N_2 is an I-nearly prime in M_2.

(2). Let $a\,(m_1, m_2) \in (N \oplus M_2) - I(N \oplus M_2)$, where $a \in R, (m_1, m_2) \in M$. Then $am_1 \in N - IN$. Since N is an I-nearly prime and small in M_1, then either $m_1 \in N + J(M_1) = J(M_1)$ or a $\in [N + J(M_1): M_1]$, [10]. So that

If $m_1 \in N + J(M_1) = J(M_1)$, then $(m_1, m_2) \in J(M_1) \oplus M_2 \subseteq J(M_1 \oplus M_2) \subseteq N \oplus M_2 + J(M_1 \oplus M_2)$.

If a $\in [N + J(M_1): M_1]$ and since N is small in M_1, then a $\in [J(M_1) \oplus M_2 : M_1 \oplus M_2]$. But $J(M_1) \oplus M_2$ is small in M_2, so $[J(M_1) \oplus M_2 : M_1 \oplus M_2] \subseteq [J(M_1 \oplus M_2) : M_1 \oplus M_2] \subseteq [N \oplus M_2 + J(M_1 \oplus M_2) : M_1 \oplus M_2]$, so $N \oplus M_2$ is an I-nearly prime in $M_1 \oplus M_2$.

Corollary (1.14):

1-If $N_1 \oplus N_2$ is an I-nearly prime of a hollow module $M_1 \oplus M_2$ with $J(M_1 \oplus M_2) \subsetneq [J(M_1) \oplus M_2]$ and $J(M_1 \oplus M_2) \subsetneq [M_1 \oplus J(M_2)]$, then N_1 and N_2 are I-nearly primes in M_1 , M_2 respectively.

2- If is N an I-nearly prime submodule of a hollow R-module M_1 , M_2 is any module such that $M_1 \oplus M_2$ be a hollow R-module, then $N \oplus M_2$ is an I-nearly prime submodule of $M_1 \oplus M_2$.

Proof: (1): Since $M_1 \oplus M_2$ is a hollow , so all submodules are small , [11]. Therefore the result follows (1.13,1**).**

(2): Since M_1 and M_2 are hollow modules, so every submodule of them is small, [11]. Therefore the result follows (1.13,2).

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