# Local Bifurcation of Four Species Syn–Ecosymbiosis model

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**Abstract**
In this paper, the conditions of occurrence of the local bifurcation (such as saddle-node, transcritical and pitchfork) near each of the equilibrium points of a mathematical model consists from four-species Syn- Ecosymbiosis are established.

**Keywords:** equilibrium point, , bifurcation, sotomayor theorem

# التفرع المحلي لنظام بيئي رباعي الاجناس

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**الخلاصة:**
في هذا البحث،شروط التفرع المحلي (سدل–نود، ترانسكرتكل و بجفورك) بالقرب من كل نقطة من نقاط التوازن لنظام بيئي رباعي الاجناس وجدت.

## 1. Introduction:

Mathematical modeling is an important interdisciplinary activity which involves the study of some aspects of diverse disciplines. Biology, Epidemiodology, Physiology, Ecology, Immunology, Bio-economics, Genetics, Pharmacokinetics are some of those disciplines. This mathematical modeling has taken a lot of attentions in recent years and spread to all branches of life and drewing the attention of every one. Ecology relates to study of living beings in relation with their living styles. Research in the branch of theoretical ecology was initiated by Lotka [1] and by Volterra [2]. Since then many scientists and researchers gave a lot of time and interest to this branch of study, see for example Meyer [3], Cushing [4], Paul Colinvaux[5], Freedman [6], Kapur [7, 8].

Bifurcation analysis gives regimes in the parameter space with quantitatively different asymptotic dynamic behavior of the system. Bob W. Kooi [9] studied the numerical bifurcation analysis of dynamical systems with simple Lotka-Volterra models or more elaborated models with more biological detail. Remy and Christiane R. [10 ], studied the bifurcation analysis of a generalized gause model with prey harvesting and a generalized Holling response function of type III. Rami & Raid[11] proposed and analyzed a prey-predator model with four Syniecological system with Holling type-II functional response, they obtained a set of sufficient and necessary condition which guarantee the lacal and global stability of this system.

In this paper however, we will established the conditions of the occurrence of local bifurcation of a mathematical model proposed by Rami & Raid[11].

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**2. Mathematical model:[11]**

An ecological model of four species Syn-Ecosymbiosis, comprising of prey-predator, commensalisms and competition, model is proposed in [11] .

$$\frac{dN_1}{dT} = r_1 N_1\left(1 - \frac{N_1}{k_1}\right) - \frac{a_1 N_1}{b + N_1} N_2 + cN_1 N_3$$
$$\frac{dN_2}{dT} = e\frac{a_1 N_1}{b + N_1} N_2 - d_1 N_2 - d_2 N_2^2$$
$$\frac{dN_3}{dT} = r_2 N_3\left(1 - \frac{N_3}{k_2}\right) - \alpha_1 N_3 N_4 \qquad (2.1)$$
$$\frac{dN_4}{dT} = r_3 N_4\left(1 - \frac{N_4}{k_3}\right) - \alpha_2 N_3 N_4$$

where $0 < e < 1$ represents the conversion rate.

This model consists of a prey (for example, Anemone) whose population density at time $T$ denoted by $N_1$, the predator (for example, Butterfly fish) whose population density at time $T$ denoted by $N_2$, the host (for example, Hermit crabs) whose population density at time $T$ denoted by $N_3$, and the host's competitor species (for example, other type of Hermit crabs) whose population density at time $T$ denoted by $N_4$ .Moreover all the parameters are assumed to be positive and described as given in [11].

Now, for further simplification of the system (2), the following dimensionless variables are used in[11].

$$t = r_1 T,\ x = \frac{N_1}{k_1},\ y = \frac{N_2}{k_1},\ z = \frac{cN_3}{r_1},\ w = \frac{\alpha_1 N_4}{r_1},\ u_1 = \frac{a_1}{r_1}$$

$$u_2 = \frac{b}{k_1},\ u_3 = \frac{d_1}{r_1},\ u_4 = \frac{d_2 k_1}{r_1},\ u_5 = \frac{r_2}{r_1},$$

$$u_6 = \frac{r_1}{ck_2},\ u_7 = \frac{r_3}{r_1},\ u_8 = \frac{r_1}{\alpha_1 k_3},\ u_9 = \frac{\alpha_2}{c}$$

Thus, system (2) can be turned into the following dimensionless form:

$$\frac{dx}{dt} = x\left[(1-x) - \frac{u_1 y}{u_2 + x} + z\right] = xf_1(x,y,z,w)$$
$$\frac{dy}{dt} = y\left[\frac{eu_1 x}{u_2 + x} - u_3 - u_4 y\right] = yf_2(x,y,z,w)$$
$$\frac{dz}{dt} = z\left[u_5(1 - u_6 z) - w\right] = zf_3(x,y,z,w) \qquad (2.2)$$
$$\frac{dw}{dt} = w\left[u_7(1 - u_8 w) - u_9 z\right] = wf_4(x,y,z,w)$$

with $x(0) \ge 0, y(0) \ge 0, z(0) \ge 0$ and $w(0) \ge 0$. It is observed that the number of parameters have been reduced from fourteen in the system (2.1) to ten in the system (2.2). Obviously the interaction functions of the system (2.2) are continuous and have continuous partial derivatives on the following positive four dimensional space:

$R_+^4 = \left\{ (x,y,z,w) \in R^4 : x(0) \geq 0, y(0) \geq 0, z(0) \geq 0, w(0) \geq 0 \right\}$. Therefore these functions are Lipschitzian on $R_+^4$, and hence the solution of the system (2.2) exists and is unique. Further, in the following theorem, the boundedness of the solution of the system (2.2) in $R_+^4$ is established by [11].

**Theorem 1:** All the solutions of system (2.2) which initiate in $R_+^4$ are uniformly bounded.

**3. Existence and stability analysis of system (2.2):[11]**

The four-species Syn-Ecosymiois model given by system (2.2) has at most twelve equilibrium points, which are mentioned with their existence conditions in [11] as in the following:

The equilibrium points $E_0 = (0,0,0,0)$, which known as the washout point, and the single species points $E_1 = (1,0,0,0)$, $E_2 = \left(0,0,\frac{1}{u_6},0\right)$, $E_3 = \left(0,0,0,\frac{1}{u_8}\right)$ are always exists.

The first planar equilibrium point $E_4 = (\hat{x},\hat{y},0,0)$ exists uniquely in $Int.R_+^2$ (interior of $R_+^2$)of $xy-$plane if in addition to the condition $\hat{x} < 1$ at least one of the following conditions are satisfied:

$$u_2 > \frac{1}{2} \tag{2.3a}$$

$$eu_1^2 + u_2^2 u_4 < u_1 u_3 + 2u_2 u_4 \tag{2.3b}$$

The second planar equilibrium point

$$E_5 = (0,0,\tilde{z},\tilde{w}) \text{ where } \tilde{w} = \frac{u_5(u_6u_7 - u_9)}{u_5u_6u_7u_8 - u_9} \text{ and } \tilde{z} = \frac{u_7(u_5u_8 - 1)}{u_5u_6u_7u_8 - u_9} \tag{2.4a}$$

exists uniquely in the $Int.R_+^2$of $zw-$plane provided that one set of the following conditions is satisfied:

$$u_5u_8 > 1 \text{ and } u_6u_7 > u_9 \tag{2.4b}$$

$$u_5u_8 < 1 \text{ and } u_6u_7 < u_9 \tag{2.4c}$$

The third planar equilibrium point $E_6 = (\bar{x},0,\bar{z},0) = \left(\frac{u_6+1}{u_6},0,\frac{1}{u_6},0\right)$always exists in $Int.R_+^2$ of $xz-$plane .

The fourth planar equilibrium point $E_7 = (\bar{\bar{x}},0,0,\bar{\bar{w}}) = \left(1,0,0,\frac{1}{u_8}\right)$ always exists in $Int.R_+^2$ of $xw-$plane.

Now, the first three species equilibrium point

$$E_8 = (\breve{x},\breve{y},\breve{z},0) \text{ where } \breve{y} = \frac{u_6[(u_2+\breve{x})(1-\breve{x})] + (u_2+\breve{x})}{u_1u_6} \text{ and } \breve{z} = \frac{1}{u_6} \tag{2.5a}$$

And $\breve{x}$ is positive constant exists uniquely in $Int.R_+^3$ of $xyz-space$ if the following conditions are satisfied:

$$2u_2u_6 > u_6 + 1 \tag{2.5b}$$

$$u_6\left(eu_1^2 + u_2^2u_4\right) < u_6(u_1u_3 + 2u_2u_4) + 2u_2u_4 \tag{2.5c}$$

$$u_6 + 1 > u_6\breve{x} \tag{2.5d}$$

The second three species equilibrium point

$$E_9 = (\hat{x},\hat{y},0,\hat{w}) \text{ where } \hat{y} = \frac{(1-\hat{x})(u_2+\hat{x})}{u_1},\ \hat{w} = \frac{1}{u_8}, \text{ and } 0 < \hat{x} < 1 \tag{2.6a}$$

exists uniquely in $Int.R_+^3$ of $xyw-space$ if the following conditions are hold :

$u_2 > \frac{1}{2}$ (2.6b)

$eu_1^2 + u_2^2 u_4 < u_1 u_3 + 2u_2 u_4$ (2.6c)

The third three species equilibrium point $E_{10} = (x^\bullet, 0, z^\bullet, w^\bullet)$ where

$x^\bullet = \frac{u_5 u_6 u_7 u_8 - u_9 + u_7(u_5 u_8 - 1)}{u_5 u_6 u_7 u_8 - u_9}$, $z^\bullet = \tilde{z}$, $w^\bullet = \tilde{w}$ (2.7)

exists uniquely in the $Int.R_+^3$ of $xzw-space$ if condition (2.4a) or (2.4b) is satisfied.

Finally the positive (coexistence) equilibrium point $E_{11} = (x^*, y^*, z^*, w^*)$ where $z^* = \tilde{z}$, $w^* = \tilde{w}$ (2.8a) , and

$y = \frac{(u_2 + x)[s_2(1-x) + u_7 s_1]}{u_1 s_2}$ (2.8b)

exists uniquely in $Int.R_+^4$ if and only if the following condition is satisfied.

$0 < x^* < \frac{s_2 + u_7 s_1}{s_2}$ (2.8c)

where $s_1 = u_5 u_6 - 1$ and, $s_2 = u_5 u_6 u_7 u_8 - u_9$

**4- The stability analysis:[11]**

In the following the stability analysis of all feasible equilibrium points of system (2.2), which is down by [11], is summarized in the following in order to study the bifurcation that depends on this results .

Note that, the symbols $\lambda_{ix}, \lambda_{iy}, \lambda_{iz}$ and $\lambda_{iw}$ represent the eigenvalues of the Jacobian matrix $J(E_i); i = 1,2,\ldots,11$ that describe the dynamics in the $x-direction$, $y-direction$, $z-direction$ and $w-direction$ respectively,

A- The Jacobian matrix $J(E_0)$ of system (2.2) at the trivial equilibrium point $E_0 = (0,0,0,0)$ has the eigenvalues: $\lambda_{0x} = 1 > 0$ , $\lambda_{0y} = -u_3 < 0$, $\lambda_{0z} = u_5 > 0$ and $\lambda_{0w} = u_7 > 0$, so $E_0$ is a saddle point.

B-The eigenvalues of the Jacobian matrix $J(E_1)$ of system (2.2) at the first single species equilibrium point $E_1 = (1,0,0,0)$ are:

$\lambda_{1x} = 1 > 0$ , $\lambda_{1y} = \frac{eu_1}{u_2 + 1} - u_3$, $\lambda_{1z} = u_5 > 0$ and $\lambda_{1w} = u_7 > 0$, accordingly $E_1$ is a saddle point.

C-The eigenvalues of the Jacobian matrix $J(E_2)$ of system (2.2) at the second single species equilibrium point $E_2 = \left(0, 0, \frac{1}{u_6}, 0\right)$ are:

$\lambda_{2x} = 1 + \frac{1}{u_6} > 0$ , $\lambda_{2y} = -u_3 < 0$ $\lambda_{2z} = -u_5 < 0$ and $\lambda_{2w} = u_7 - \frac{u_9}{u_6}$, thus $E_2$ is a saddle point.

D-The Jacobian matrix $J(E_3)$ of system (2.2) at the third single species equilibrium point $E_3 = \left(0, 0, 0, \frac{1}{u_8}\right)$ has the following eigenvalues:

$\lambda_{3x} = 1 > 0$ , $\lambda_{3y} = -u_3 < 0$ , $\lambda_{3z} = u_5 - \frac{1}{u_8}$ and $\lambda_{3w} = -u_7 < 0$, then $E_3$ is a saddle point.

E-The Jacobian matrix $J(E_4)$ of system (2.2) at the first two species equilibrium point $E_4 = (\hat{x}, \hat{y}, 0, 0)$ has the following eigenvalues:

$\lambda_{4x} = -\frac{A_1}{2} + \frac{1}{2}\sqrt{A_1^2 - 4A_2}$, *and* $\lambda_{4y} = -\frac{A_1}{2} - \frac{1}{2}\sqrt{A_1^2 - 4A_2}$

$\lambda_{4z} = u_5 > 0$ , and $\lambda_{4w} = u_7 > 0$

where

$$A_1 = \hat{x} - \frac{u_1 \hat{x} \hat{y}}{(u_2 + \hat{x})^2} + u_4 \hat{y}, \text{ and } A_2 = u_4 \hat{x} \hat{y}\left(1 - \frac{u_1 \hat{y}}{(u_2 + \hat{x})^2}\right) + \frac{e u_1^2 u_2 \hat{x} \hat{y}}{(u_2 + \hat{x})^3}$$

Thus $E_4$ is unstable.

F-The Jacobian matrix of system (2.2) at the second two species equilibrium point $E_5 = (0, 0, \tilde{z}, \tilde{w}) = \left(0, 0, \frac{u_7(u_5 u_8 - 1)}{u_5 u_6 u_7 u_8 - u_9}, \frac{u_5(u_6 u_7 - u_9)}{u_5 u_6 u_7 u_8 - u_9}\right)$ has one positive eigenvalues given by:

$\lambda_{5x} = 1 + \tilde{z} > 0$ , $\lambda_{5y} = -u_3 < 0$. Thus $E_5$ is saddle unstable.

G-The eigenvalues of the Jacobian matrix $J(E_6)$ of system (2.2) at the third two species equilibrium point $E_6 = (\bar{x}, 0, \bar{z}, 0) = \left(\frac{u_6 + 1}{u_6}, 0, \frac{1}{u_6}, 0\right)$ are:

$\lambda_{6x} = -\bar{x} < 0$, $\lambda_{6y} = \frac{e u_1 + e u_1 u_6 - u_2 u_3 u_6 - u_3 - u_3 u_6}{u_2 u_6 + u_6 + 1}$,

$\lambda_{6z} = -u_5 < 0$ and $\lambda_{6w} = \frac{u_6 u_7 - u_9}{u_6}$

Therefore, if the following conditions hold

$$e u_1 (1 + u_6) < u_3 (u_2 u_6 + u_6 + 1) \qquad (2.9a)$$

$$u_6 u_7 < u_9 \qquad (2.9b)$$

Then $E_6$ is locally asymptotically stable. However, it is a saddle point otherwise.

H-The eigenvalues of Jacobian matrix $J(E_7)$ of system (2.2) at the forth two species equilibrium point $E_7 = \left(1, 0, 0, \frac{1}{u_8}\right)$ are:

$\lambda_{7x} = -1 < 0$ , $\lambda_{7y} = \frac{e u_1 - u_3 (u_2 + 1)}{u_2 + 1}$, $\lambda_{7z} = \frac{u_5 u_8 - 1}{u_8}$ and $\lambda_{7w} = -u_7 < 0$.

Therefore, if the following conditions hold

$$e u_1 < u_3 (u_2 + 1) \qquad (2.10a)$$

$$u_5 u_8 < 1 \qquad (2.10b)$$

Then $E_7$ is locally asymptotically stable. However, it is a saddle point otherwise.

I- The Jacobian matrix $J(E_8)$ of system (2.2) at the first three species equilibrium point $E_8 = (\breve{x}, \breve{y}, \breve{z}, 0) = \left(\breve{x}, \breve{y}, \frac{1}{u_6}, 0\right)$ has the following eigenvalues:

$\lambda_{8x} = -\frac{\breve{A}_1}{2} + \frac{1}{2}\sqrt{\breve{A}_1^2 - 4\breve{A}_2}$, *and* $\lambda_{8y} = -\frac{\breve{A}_1}{2} - \frac{1}{2}\sqrt{\breve{A}_1^2 - 4\breve{A}_2}$

$\lambda_{8z} = -u_5 < 0$ and $\lambda_{8w} = \dfrac{u_6 u_7 - u_9}{u_6}$ .

Where, $\breve{A}_1 = -\breve{x}\left(-1 + \dfrac{u_1 \breve{y}}{(u_2 + \breve{x})^2}\right) + u_4 \breve{y}$, and $\breve{A}_2 = u_4\, \breve{x}\, \breve{y}\left(1 - \dfrac{u_1 \breve{y}}{(u_2 + \breve{x})^2}\right) + \dfrac{e u_1^2 u_2\, \breve{x}\, \breve{y}}{(u_2 + \breve{x})^3}$

Therefore if the following conditions are satisfied

$$\frac{u_1 \breve{y}}{(u_2 + \breve{x})^2} < 1 \qquad (2.11a)$$

$$u_6 u_7 < u_9 \qquad (2.11b)$$

then, $E_8$ is locally asymptotically stable in the $R_+^4$. However, it is a saddle point otherwise.

J- The Jacobin matrix $J(E_9)$ of system (2.2) at the second three species equilibrium point $E_9 = (\hat{x}, \hat{y}, 0, \hat{w}) = \left(\hat{x}, \hat{y}, 0, \dfrac{1}{u_9}\right)$ has the following eigenvalues:

$\lambda_{9x} = -\dfrac{B_1}{2} + \dfrac{1}{2}\sqrt{B_1^2 - 4B_2}$, $\lambda_{9y} = -\dfrac{B_1}{2} - \dfrac{1}{2}\sqrt{B_1^2 - 4B_2}$

$\lambda_{9z} = \dfrac{u_5 u_8 - 1}{u_8}$ and $\lambda_{9w} = -u_7 < 0$

where

$B_1 = -\hat{x}\left(-1 + \dfrac{u_1 \hat{y}}{(u_2 + \hat{x})^2}\right) + u_4 \hat{y}$ , and $B_2 = u_4\, \hat{x}\, \hat{y}\left(1 - \dfrac{u_1 \hat{y}}{(u_2 + \hat{x})^2}\right) + \dfrac{e u_1^2 u_2\, \hat{x}\, \hat{y}}{(u_2 + \hat{x})^3}$

Therefore if the following conditions are satisfied:

$$\frac{u_1 \hat{y}}{(u_2 + \hat{x})^2} < 1 \qquad (2.12a)$$

$$u_5 u_8 < 1 \qquad (2.12b)$$

So, $E_9$ is locally asymptotically stable in the $R_+^4$. However, it is a saddle point otherwise.

K-The Jacobian matrix $J(E_{10})$ of system (2.2) at the third three species equilibrium point $E_{10} = (x^\bullet, 0, z^\bullet, w^\bullet)$ has the following eigenvalues:

$\lambda_{10z} = -\dfrac{B_1^\bullet}{2} + \dfrac{1}{2}\sqrt{B_1^{\bullet 2} - 4B_2^\bullet}$, *and* $\lambda_{10w} = -\dfrac{B_1^\bullet}{2} - \dfrac{1}{2}\sqrt{B_1^{\bullet 2} - 4B_2^\bullet}$

$\lambda_{10x} = -x^\bullet < 0$, and $\lambda_{10y} = \dfrac{e u_1 x^\bullet - u_3(u_2 + x^\bullet)}{u_2 + x^\bullet}$

where

$B_1^\bullet = u_5 u_6 z^\bullet + u_7 u_8 w^\bullet$, and $B_2^\bullet = (u_5 u_6 u_7 u_8 - u_9) z^\bullet w^\bullet$

Thus if the following conditions are satisfied

$$u_5 u_6 u_7 u_8 > u_9 \qquad (2.13a)$$

$$e u_1 x^\bullet < u_3(u_2 + x^\bullet) \qquad (2.13b)$$

then, $E_{10}$ is locally asymptotically stable in the $R_+^4$. However, it is a saddle point otherwise.

L-The Jacobian matrix $J(E_{11})$ of system (2.2) at the positive equilibrium point $E_{11} = (x^*, y^*, z^*, w^*)$ has the following eigenvalues:

$\lambda_{11x} = -\frac{R_1}{2} + \frac{1}{2}\sqrt{R_1^2 - 4R_2}$, $\lambda_{11y} = -\frac{R_1}{2} - \frac{1}{2}\sqrt{R_1^2 - 4R_2}$ ,

$\lambda_{11z} = -\frac{R_3}{2} + \frac{1}{2}\sqrt{R_3^2 - 4R_4}$ , *and* $\lambda_{11w} = -\frac{R_3}{2} - \frac{1}{2}\sqrt{R_3^2 - 4R_4}$

where

$$R_1 = -x^*\left(-1 + \frac{u_1 y^*}{(u_2 + x^*)^2}\right) + u_4 y^* \text{ , } R_2 = u_4 x^* y^*\left(1 - \frac{u_1 y^*}{(u_2 + x^*)^2}\right) + \frac{e u_1^2 u_2 x^* y^*}{(u_2 + x^*)^3},$$

$R_3 = u_5 u_6 z^* + u_7 u_8 w^*$, and $R_4 = (u_5 u_6 u_7 u_8 - u_9) z^* w^*$

Thus if the following conditions are satisfied.

$$\frac{u_1 y^*}{(u_2 + x^*)^2} < 1 \qquad (2.14a)$$

$$u_5 u_6 u_7 u_8 > u_9 \qquad (2.14b)$$

Hence, $E_{11}$ is locally asymptotically stable in the. However, it is a saddle point otherwise.

**5.The local Bifurcation.**

In this section an investigation for dynamical behavior of system (2.2) under the effect of varying one parameter at each time is carried out. The occurrence of local bifurcation in the neighborhood of the equilibrium point of system (2.2) are studied in the following theorem.

**Theorem 2:** If the parameter $u_3$ passes through the value $u_3^\circ = \frac{e u_1}{1 + u_2}$, then the equilibrium point $E_1$ transforms into nonhyperpolic equilibrium point and if

$$u_1 + 2u \ (1 + u_2)^2 \neq 1 \qquad (2.15)$$

then system (2.2) possesses transcritical bifurcation, but no saddle-node bifurcation.However violate condition (2.15) gives pitch-fork bifurcation.

**Proof:** According to the Jacobian matrix of system (2.2) at $E_1$ that is given by $J(E_1)$ it is easy to verify that as $u_3 = u_3^\circ$**,** the $J(E_1, \bar{u}_3)$ has the following eigenvalues**:**

$\lambda_{1x} = 1 > 0, \quad \lambda_{1y} = 0$

$\lambda_{1z} = u_5 > 0$ *and* $\lambda_{1w} = u_7 > 0.$

Let $v^\circ = (\theta_1^\circ, \theta_2^\circ, \theta_3^\circ, \theta_4^\circ)^T$ be the eigenvector of $J(E_1, u_3^\circ)$ corresponding to the eigenvalue of $\lambda_{1y} = 0$. Then it is easy to check that $v^\circ = (-\frac{b_{12}^\circ}{b_{11}^\circ}\theta_2^\circ, \theta_2^\circ, 0, 0)^T$ ,where $b_{11}^\circ = -1 < 0$ , $b_{12}^\circ = -\frac{u_1}{1 + u_2}$, and $\theta_2^\circ$ represents any nonzero real value. Also, let $y^\circ = (h_1^\circ, h_2^\circ, h_3^\circ, h_4^\circ)^T$ represents the eigenvector of $J^T(E_1, u_3^\circ)$ that corresponding to the eigenvalue $\lambda_{1y} = 0$. Straight forward calculation shows that $y^\circ = (0, h_2^\circ, 0, 0)^T$ , where $h_2^\circ$ represents any nonzero real number.

Now, since $\frac{\partial F}{\partial u_3} = F_{u_3}(X, u_3) = [0, -y, 0, 0]^T$ , *where* $X = (x, y, z, w)^T$ *and* $F = (f_1, f_2, f_3, f_4)^T$

With $f_i$ ; $i = 1,2,3,4$ represent the right hand side of system (2.2). Then we get $\frac{\partial F}{\partial u_3} = F_{u_3}(E_1, u_3^\circ) = [0,0,0,0]^T$, and the following is obtained:

$y^{\circ T}[F_{u_3}(E_1,u_3^{\circ})]=(0,h_2^{\circ},0,0)(0,0,0,0)^T=0.$ Thus system (2.2) at $E_1$ does not experience any saddle-node bifurcation in view of sotomayor theorem [12]. Also, since

$y^{\circ T}[DF_{u_3}(E_1,u_3^{\circ})v^{\circ}]=(0,h_2^{\circ},0,0)(0,\theta_2^{\circ},0,0)^T=h_2^{\circ}\theta_2^{\circ}\neq 0.$ here, $DF_{u_3}(E_1,u_3^{\circ})=\frac{\partial}{\partial X}F_{u_3}(X,u_3)\,|_{X=E_1,u_3=u_3^{\circ}}$.

Moreover, we have $y^{\circ T}[D^2F_{u_3}(E_1,u_3^{\circ})(v^{\circ},v^{\circ})]=\frac{b_{12}^{\circ}}{b_{11}^{\circ}}\theta_2^{\circ 2}\left[-1+u_1+2u_4(1+u_2)^2\right]\neq 0.$ by condition (2.15).Here, $D^2F_{u_3}(E_1,u_3^{\circ})=DJ(X,u_3)\,|_{X=E_1,u_3=u_3^{\circ}}$. Then by sotomayor theorem, system (2.2) possesses a transcritical bifurcation but not pitch-fork bifurcation near $E_1$ where $u_3=u_3^{\circ}$ **.**

However, violate condition (2.15) gives that $y^{\circ T}[D^2F_{u_3}(E_1,u_3^{\circ})(v^{\circ},v^{\circ})]=0$, and hence further computation shows $y^{\circ T}[D^3F_{u_3}(E_1,u_3^{\circ})(v^{\circ},v^{\circ},v^{\circ})]=-\frac{\overline{b}_{12}^{3}eu_1u_2}{\overline{b}_{11}^{3}(1+u_2)^3}\overline{\theta}_2^{3}\overline{h}_2\neq 0$ .Therefore according to Sotomayor theorem, system (2.2) possesses a pitch-fork bifurcation.

**Theorem 3:** If the parameter $u_7$ passes through the value $\underline{u_7}=\frac{u_9}{u_6}$, then the equilibrium point $E_2$ transforms into nonhyperpolic equilibrium point and if

$$u_8\neq\frac{u_5(1-u_9)}{2u_9} \qquad (2.16)$$

then system (2.2) possesses transcritical bifurcation, but no saddle-node bifurcation, nor pitch-fork bifurcation can occur.

**Proof:** According to the Jacobian matrix of system (2.2) at $E_2$ that is given by $J(E_2)$ it is easy to verify that as $u_7=\underline{u_7}$ **,** the $J(E_2,\underline{u_7})$ has the following eigenvalues**:**

$$\lambda_{2x}=1+\frac{1}{u_6}>0,\ \lambda_{2y}=-u_3<0,\lambda_{2z}=-u_5<0\ \ and\ \ \lambda_{2w}=0.$$

Let $\underline{v}=(\underline{\theta_1},\underline{\theta_2},\underline{\theta_3},\underline{\theta_4})^T$ be the eigenvector of $J(E_2,\underline{u_7})$ corresponding to the eigenvalue of $\lambda_{2w}=0.$ Then it is easy to check that $\underline{v}=(0,0,-\frac{\underline{b_{34}}}{\underline{b_{33}}}\underline{\theta_4},\underline{\theta_4})^T$ ,where $\underline{b_{33}}=-u_5<0$ ,$\underline{b_{34}}=-\frac{1}{u_6}<0$, and $\underline{\theta_4}$ represents any nonzero real value. Also, let $\underline{y}=(\underline{h_1},\underline{h_2},\underline{h_3},\underline{h_4})^T$ represents the eigenvector of $J^T(E_2,\underline{u_7})$ that corresponding to the eigenvalue $\lambda_{2w}=0.$ Straight forward calculation shows that $\underline{y}=(0,0,0,\underline{h_4})^T$, where $\underline{h_4}$ represents any nonzero real number.

Now, since $\frac{\partial F}{\partial u_7}=F_{u_7}(X,u_7)=[0,0,0,w(1-u_8w)]^T$, $where\ X=(x,y,z,w)^T\ and\ F=(f_1,f_2,f_3,f_4)^T$

With $f_i$ ; $i=1,2,3,4$ represent the right hand side of system (2.2). Then we get $\frac{\partial F}{\partial u_7}=F_{u_7}(E_2,\underline{u_7})=[0,0,0,0]^T$ and the following is obtained:

$\underline{y}^T[F_{u_7}(E_2,\underline{u_7})]=(0,0,0,\underline{h_4})(0,0,0,0)^T=0.$ Thus system (2.2) at $E_2$ does not experience any saddle-node bifurcation in view of sotomayor theorem. Also, since

$\underline{y}^T[DF_{u_7}(E_2,\underline{u_7})\underline{v}]=(0,0,0,\underline{h_4})(0,0,0,\underline{\theta_4})^T=\underline{h_4}\underline{\theta_4}\neq 0.$

here, $DF_{u_7}(E_2,\underline{u_7})=\frac{\partial}{\partial X}F_{u_7}(X,u_7)\,|_{X=E_2,u_7=\underline{u_7}}$.

Moreover, we have

$\underline{y}^T[D^2F_{u_7}(E_2,\underline{u_7})(\underline{v},\underline{v})]=\underline{\theta_4}^2\underline{h_4}\left[u_5(u_9-1)+2u_9u_8\right]\neq 0.$ by condition (2.16).

Here, $D^2F_{u_7}(E_7,\underline{u_7})=DJ(X,u_7)\,|_{X=E_2,u_7=\underline{u_7}}$. Then by sotomayor theorem, system (2.2) possesses a transcritical bifurcation but not pitch-fork bifurcation near $E_2$ where $u_7=\underline{u_7}$ **.**

However, violate condition (2.16) gives that $\underline{y}^T[D^2F_{u_7}(E_2,\underline{u_7})(\underline{v},\underline{v})]=0$, and hence further computation shows $\underline{y}^T[D^3F_{u_7}(E_2,\underline{u_7})(\underline{v},\underline{v},\underline{v})]=(0,0,0,\underline{h_4})(0,0,0,0)^T=0$. Therefore according to Sotomayor theorem, system (2.2) possesses a pitch-fork bifurcation.

**Theorem 4:** If the parameter $u_5$ passes through the value $\overline{\overline{u_5}}=\frac{1}{u_8}$, then the equilibrium point $E_3$ transforms into nonhyperpolic equilibrium point and system (2.2) not possesses any saddle-node bifurcation ,transcritical bifurcation, but no bifurcation, and no pitch-fork bifurcation can occur.

**Proof:** According to the Jacobian matrix of system (2.2) at $E_3$ that is given by $J(E_3)$ it is easy to verify that as $u_5=\overline{\overline{u_5}}$ **,** the $J(E_3,\overline{\overline{u_5}})$ has the following eigenvalues**:**

$\lambda_{3x}=1>0,\quad \lambda_{3y}=-u_3<0,\ \lambda_{3z}=0\ \ and\ \ \lambda_{3w}=-u_7<0.$

Let $\overline{\overline{v}}=(\overline{\overline{\theta_1}},\overline{\overline{\theta_2}},\overline{\overline{\theta_3}},\overline{\overline{\theta_4}})^T$ be the eigenvector of $J(E_3,\overline{\overline{u_5}})$ corresponding to the eigenvalue of $\lambda_{3z}=0$. Then it is easy to check that $\overline{\overline{v}}=(0,0,\overline{\overline{\theta_3}},-\frac{\overline{\overline{b_{43}}}}{\overline{\overline{b_{44}}}}\overline{\overline{\theta_3}})^T$ ,where $\overline{\overline{b_{43}}}=-\frac{u_9}{u_8}<0$ ,$\overline{\overline{b_{44}}}=-u_7<0$, and $\overline{\overline{\theta_3}}$ represents any nonzero real value. Also, let $\overline{\overline{y}}=(\overline{\overline{h_1}},\overline{\overline{h_2}},\overline{\overline{h_3}},\overline{\overline{h_4}})^T$ represents the eigenvector of $J^T(E_3,\overline{\overline{u_5}})$ that corresponding to the eigenvalue $\lambda_{3z}=0$. Straight forward calculation shows that $\overline{\overline{y}}=(0,0,\overline{\overline{h_3}},0)^T$, where $\overline{\overline{h_3}}$ represents any nonzero real number.

Now, since

$$\frac{\partial F}{\partial u_5}=F_{u_5}(X,u_5)=[0,0,(1-u_6z)z,0]^T,\ where\ X=(x,y,z,w)^T\ \ and\ F=(f_1,f_2,f_3,f_4)^T$$

With $f_i\ ;i=1,2,3,4$ represent the right hand side of system (2.2). Then we get $\frac{\partial F}{\partial u_5}=F_{u_5}(X,\overline{\overline{u_5}})=[0,0,0,0]^T$ and the following is obtained:

$$\overline{\overline{y}}^T[F_{u_5}(E_3,\overline{\overline{u_5}})]=(0,0,\overline{\overline{h_3}},0)(0,0,0,0)^T=0.$$

Thus system (2.2) at $E_3$ does not experience any saddle-node bifurcation in view of sotomayor theorem . Also, since $\overline{\overline{y}}^T[DF_{u_5}(E_3,\overline{\overline{u_5}})\overline{\overline{v}}]=(0,0,\overline{\overline{h_3}},0)(0,0,0,0)^T=0.$

here, $DF_{u_5}(E_3,\overline{\overline{U_5}})=\frac{\partial}{\partial X}F_{u_5}(X,u_5)\,|_{X=E_3,u_5=\overline{\overline{U_5}}}$. Thus system (2.2) at $E_3$ does not experience any transcritical bifurcation and pitch-fork bifurcation occurs at $E_3$ where $u_5=\overline{\overline{u_5}}$ .

**Theorem 5:** Assume that $\hat{x}<1$ and at least one of conditions (2.3a) and (2.3b) are hold and the parameter $u_1$ passes through the value $\hat{u}_1=\frac{\hat{y}(u_2+\hat{x})}{2(eu_2\hat{x}\hat{y})}+\frac{\sqrt{(u_2+\hat{x})^2\hat{y}^2-4(eu_2\hat{x}\hat{y})(u_2+\hat{x})^3}}{2(eu_2\hat{x}\hat{y})}$ , then the equilibrium point $E_4$ transforms into nonhyperpolic equilibrium point and if the condition

$\frac{\hat{u}_1\hat{y}}{(u_2+\hat{x})^2}>1$ (2.17a), and

$e\neq\frac{\hat{b}_{21}}{\hat{b}_{11}}$ (2.17b) ,

where $\hat{b}_{11}=\hat{x}\left(-1+\frac{\hat{u}_1\hat{y}}{(u_2+\hat{x})^2}\right)$ and $\hat{b}_{21}=\left(\frac{e\hat{u}_1u_2\hat{y}}{(u_2+\hat{x})^2}\right)$ are hold then system (2.2) possesses a saddle-node bifurcation, violate condition(2.17b) and if the condition

$$\hat{\theta}_1^2\hat{h}_1(u_2+\hat{x})\hat{b}_{21}\begin{bmatrix}2(u_2+\hat{x})^2(\hat{b}_{12}^2+\hat{b}_{11}^2u_2)+\\ \hat{b}_{12}\hat{u}_1(\hat{b}_{11}+\hat{b}_{21}\hat{b}_{11}u_2+\hat{b}_{11}(u_2+\hat{x}))\end{bmatrix}\neq\ \hat{b}_{21}\hat{b}_{12}^2\hat{u}_1\hat{\theta}_1\hat{h}_1\hat{y}\left[\hat{\theta}_1u_2+\hat{\theta}_1\hat{x}\hat{y}+2u_2\right]\quad(2.17c).$$

where $\hat{b}_{12}=-\frac{\hat{u}_1\hat{x}}{u_2+\hat{x}}$, holds then system (2.2) possesses a transcritical bifurcation ,finally, if condition (2.17c) reverses and the condition

$M_1\neq M_2$, $(2.17d)$, *where*

$$M_1=\frac{\hat{h}_1\hat{u}_1\hat{\theta}_1^2}{(u_2+\hat{x})^3}\left[2\hat{y}\hat{\theta}_1+\frac{\hat{b}_{11}}{\hat{b}_{12}}\left[2\hat{\theta}_1+\hat{\theta}_1(u_2+\hat{x})+\frac{6eu_2\hat{y}}{(u_2+\hat{x})}+2eu_2\right]\right]\text{ and,}$$

$$M_2=\frac{\hat{h}_1\hat{b}_{11}\hat{\theta}_1^3\hat{u}_1}{\hat{b}_{12}(u_2+\hat{x})^3}\left[(u_2+\hat{x})+2\frac{\hat{b}_{11}eu_2}{\hat{b}_{21}}\right]$$

holds, then system (2.2) experience a pitch-fork bifurcation at $E_4$ where $u_1=\hat{u}_1$. .

**Proof:** According to the Jacobian matrix of system (2.2) at $E_4$ that is given by $J(E_4)$ it is easy to verify that as $u_1=\hat{u}_1$, the $J(E_4,\hat{u}_1)$ has the following eigenvalues**:**

$\lambda_{4x}\cdot\lambda_{4y}=0, so\ either\ \lambda_{4x}=0\ or\ \lambda_{4y}=0$

$\lambda_{4z}=u_5>0\ \ and\ \lambda_{4w}=u_7>0.$

We will take $\lambda_{4x}=0\ at\ u_1=\hat{u}_1$ .

Let $\hat{v}=(\hat{\theta}_1,\hat{\theta}_2,\hat{\theta}_3,\hat{\theta}_4)^T$ be the eigenvector of $J(E_4,\hat{u}_1)$ corresponding to the eigenvalue of $\lambda_{4x}=0$ Then it is easy to check that $\hat{v}=(\hat{\theta}_1,-\frac{\hat{b}_{11}}{\hat{b}_{21}}\hat{\theta}_1,0,0)^T$ ,where $\hat{\theta}_1$ represents any nonzero real value.

Also, let $\hat{y}=(\hat{h}_1,\hat{h}_2,\hat{h}_3,\hat{h}_4)^T$ represents the eigenvector of $J^T(E_4,\hat{u}_1)$ that corresponding to the eigenvalue $\lambda_{4x}=0$ Straight forward calculation shows that

$\hat{y}=(\hat{h}_1,-\frac{\hat{b}_{11}}{\hat{b}_{21}}\hat{h}_1,-\frac{\hat{b}_{13}}{u_5}\hat{h}_1,0)^T$ , where $\hat{h}_1$ represents any nonzero real number.

Now, since $\frac{\partial F}{\partial u_1}=F_{u_1}(X,u_1)=[-\frac{yx}{u_2+x},\frac{exy}{u_2+x},0,0]^T$ , *where* $X=(x,y,z,w)^T$ *and* $F=(f_1,f_2,f_3,f_4)^T$

With $f_i$ ; $i=1,2,3,4$ represent the right hand side of system (2.2). Then we get $\frac{\partial F}{\partial u_1}=F_{u_1}(E_4,\hat{u}_1)=[-\frac{\hat{x}\hat{y}}{u_2+\hat{x}},\frac{e\hat{x}\hat{y}}{u_2+\hat{x}},0,0]^T$ and the following is obtained:

$$\hat{y}^T[F_{u_1}(E_4,\hat{u}_1)]=(\hat{h}_1,-\frac{\hat{b}_{11}}{\hat{b}_{21}}\hat{h}_1,-\frac{\hat{b}_{13}}{u_5}\hat{h}_1,0)(-\frac{\hat{x}\hat{y}}{u_2+\hat{x}},\frac{e\hat{x}\hat{y}}{u_2+\hat{x}},0,0)^T=\frac{\hat{x}\hat{y}}{u_2+\hat{x}}\hat{h}_1\left[-1-\frac{\hat{b}_{11}}{\hat{b}_{21}}e\right]\neq 0.$$

Also, since $\hat{y}^T[DF_{u_1}(E_4,\hat{u}_1)\hat{v}] = \frac{\hat{b}_{12}\hat{b}_{21}(-u_2-1)+\hat{b}_{11}(u_2+\hat{x})\hat{\theta}_1\hat{h}_1\hat{x}(\hat{b}_{21}+e\hat{b}_{11})}{\hat{b}_{12}\hat{b}_{21}(u_2+\hat{x})^2} + \frac{u_2\hat{y}\hat{\theta}_1\hat{h}_1}{\hat{b}_{12}\hat{b}_{21}(u_2+\hat{x})^2} \neq 0,$

by condition(2.17b), here, $DF_{u1}(E_4,\hat{u}) = \frac{\partial}{\partial X}F_{u_1}(X,u_1)\,|_{X=E_4,u_1=\hat{u}_1}$.

Then system (2.2) possesses a saddle-node bifurcation in view of sotomayor theorem.

Now, violate condition (2.17b) gives that $\hat{y}^T[DF_{u_1}(E_4,\hat{u}_1)\hat{v}] = 0$. Moreover, we have

$$\hat{y}^T[D^2F_{u_1}(E_4,\hat{u}_1)(\hat{v},\hat{v})] = -\hat{\theta}_1^2\hat{h}_1(u_2+\hat{x})\hat{b}_{21}\begin{bmatrix} 2(u_2+\hat{x})^2(\hat{b}_{12}^2+\hat{b}_{11}^2u_2)+ \\ \hat{b}_{12}\hat{u}_1(\hat{b}_{11}+\hat{b}_{21}\hat{b}_{11}u_2+\hat{b}_{11}(u_2+\hat{x})) \end{bmatrix} + \hat{b}_{21}\hat{b}_{12}^2\hat{u}_1\hat{\theta}_1\hat{h}_1\hat{y}\left[\hat{\theta}_1u_2+\hat{\theta}_1\hat{x}\hat{y}+2u_2\right]$$

$$\neq 0$$

by condition (2.17c). Here, $D^2F_{u1}(E_1,\hat{u}_1) = DJ(X,u_1)\,|_{X=E_4,u_1=\hat{u}_1}$. Then by sotomayor theorem, system (2.2) possesses a transcritical bifurcation but not pitch-fork bifurcation near $E_4$ where $u_1=\hat{u}_1$ **.** However, violate condition (2.17c) gives that $\hat{y}^T[D^2F_{u_1}(E_4,\hat{u}_1)(\hat{v},\hat{v})] = 0$, and hence further computation shows $\hat{y}^T[D^3F_{u_1}(E_4,\hat{u}_1)(\hat{v},\hat{v},\hat{v})] = -M_1+M_2 \neq 0$ by condition (2.17d).

Therefore according to Sotomayor theorem, system (2.2) possesses a pitch-fork bifurcation.

but no transcritical nor pitch-fork bifurcation occurs in view of sotomayor theorem near $E_4$ where $u_1=\hat{u}_1$. **.**

**Theorem 6:** Assume that condition (2.4b) or (2.4c) holds and the parameter $u_9$ passes through the value $\tilde{u}_9 = u_5u_6u_7u_8$, then the equilibrium point $E_5$ transforms into nonhyperpolic equilibrium point and if

$$\frac{u_5u_6\tilde{z}}{\tilde{w}} \neq 1 \qquad (2.18)$$

then system (2.2) possesses a saddle-node bifurcation but no transcritical bifurcation, , nor pitch-fork bifurcation can occur.

**Proof:** According to the Jacobian matrix of system (2.2) at $E_5$ that is given by $J(E_5)$ it is easy to verify that as $u_9=\tilde{u}_9$ **,** $J(E_5,\tilde{u}_9)$ has the following eigenvalues**:**

$$\lambda_{5x} = 1+\tilde{z} > 0, \qquad \lambda_{5y} = -u_3 < 0$$
$$\lambda_{5z} + \lambda_{5w} = u_5u_6\tilde{z} + u_7u_8\tilde{w} > 0, and$$
$$\lambda_{5z}\cdot\lambda_{5w} = 0.$$

Then either $\lambda_{5z}=0$ $or$ $\lambda_{5w}=0$ .We will assume that $\lambda_{5z}=0$ .

Let $\tilde{v}=(\tilde{\theta}_1,\tilde{\theta}_2,\tilde{\theta}_3,\tilde{\theta}_4)^T$ be the eigenvector of $J(E_5,\tilde{u}_9)$ corresponding to the eigenvalue of $\lambda_{5z}=0$. Then it is easy to check that $\tilde{v}=(0,0,\tilde{\theta}_3,-\frac{\tilde{b}_{33}}{\tilde{b}_{34}}\tilde{\theta}_3)^T$ ,where $\tilde{b}_{33}=-u_5u_6\tilde{z}<0$ ,$\tilde{b}_{34}=-\tilde{z}<0$, and $\tilde{\theta}_3$ represents any nonzero real value. Also, let $\tilde{y}=(\tilde{h}_1,\tilde{h}_2,\tilde{h}_3,\tilde{h}_4)^T$ represents the eigenvector of $J^T(E_5,\tilde{u}_9)$ that corresponding to the eigenvalue $\lambda_{5z}=0$ Straight forward calculation shows that $\tilde{y}=(0,0,\tilde{h}_3,-\frac{\tilde{b}_{33}}{\tilde{b}_{43}}\tilde{h}_3)^T$ , where $\tilde{h}_3$ represents any nonzero real number.

Now, since $\frac{\partial F}{\partial u_9} = F_{u_9}(X,u_9) = [0,0,0,-wz]^T$ , $where\ X=(x,y,z,w)^T\ and\ F=(f_1,f_2,f_3,f_4)^T$

With $f_i$ ;$i=1,2,3,4$ represent the right hand side of system (2.2). Then we get $\frac{\partial F}{\partial u_9} = F_{u_9}(E_5,\tilde{u}_9) = [0,0,0,-\tilde{w}\tilde{z}]^T$ and the following is obtained:

$\tilde{y}^T[F_{u_9}(E_5,\tilde{u}_9)] = (0,0,\tilde{h}_3,-\frac{\tilde{b}_{33}}{\tilde{b}_{43}}\tilde{h}_3)[0,0,0,-\tilde{w}\tilde{z}]^T = \frac{\tilde{z}^2\tilde{h}_3}{u_7u_8} \neq 0.$

Also, $\tilde{y}^T[DF_{u_9}(E_5,\tilde{u}_9)\tilde{v}] = (0,0,\tilde{h}_3,-\frac{\tilde{b}_{33}}{\tilde{b}_{43}}\tilde{h}_3)[0,0,0,-\tilde{w}\tilde{\theta}_3+\frac{\tilde{b}_{33}\tilde{\theta}_3}{\tilde{b}_{34}}\tilde{z}]^T = \frac{\tilde{z}\tilde{h}_3\tilde{\theta}_3}{u_7u_8}\left[1-\frac{u_5u_6}{\tilde{w}}\tilde{z}\right] \neq 0.$

By condition (2.18) .Thus system (2.2) at $E_5$ possesses a saddle-node but does not experience any transcritical or pitch-for bifurcation in view of sotomayor theorem.

**Theorem 7:** If the parameter $u_9$ passes through the value $\overline{u_9} = u_6u_7$, then the equilibrium point transforms into nonhyperpolic equilibrium point and system (2.2) does not experience any saddle-node ,transcritical and pitch-fork bifurcation at $E_6$ where $u_9 = \overline{u_9}$.

**Proof:** According to the Jacobian matrix of system (2.2) at $E_6$ that is given by $J(E_6)$ it is easy to verify that as $u_9 = \overline{u_9}$., the $J(E_6,\overline{u}_9)$ has the following eigenvalues**:**

$$\lambda_{6x} = -\bar{x} > 0,\quad \lambda_{6y} = \frac{eu_1(1+u_6)-u_3(1+u_6(1+u_2))}{1+u_6(1+u_2)} < 0 \quad provided\ that\ condition(2.9a)\ holds$$

$$\lambda_{6z} = -u_5 < 0 \quad and \quad \lambda_{6w} = 0.$$

Let $\bar{v} = (\overline{\theta_1},\overline{\theta_2},\overline{\theta_3},\overline{\theta_4})^T$ be the eigenvector of $J(E_6,\overline{u_9})$ corresponding to the eigenvalue of $\lambda_{6w} = 0$. Then it is easy to check that $\bar{v} = (-\frac{\overline{b_{13}}\bar{b}_{34}}{\overline{b_{11}}\bar{b}_{33}}\bar{\theta}_4, 0, -\frac{\bar{b}_{34}}{\bar{b}_{33}}\bar{\theta}_4, \bar{\theta}_4)^T$ ,where $\overline{b_{11}} = -\bar{x} < 0$ ,$\overline{b_{13}} = \bar{x} > 0$, $\bar{b}_{33} = -u_5 < 0$ , $\bar{b}_{34} = -\frac{1}{u_6}$ and $\overline{\theta 4}$ represents any nonzero real value. Also, let $\bar{y} = (\overline{h_1},\overline{h_2},\overline{h_3},\overline{h_4})^T$ represents the eigenvector of $J^T(E_6,\overline{u_9})$ that corresponding to the eigenvalue $\lambda_{6w} = 0$. Straight forward calculation shows that $\bar{y} = (0,,0,0,\overline{h_4})^T$ , where $\overline{h_4}$ represents any nonzero real number.

Now, since $\frac{\partial F}{\partial u_9} = F_{u_9}(X,u_9) = [0,0,0,-wz]^T$, $where X = (x,y,z,w)^T$ $and$ $F = (f_1, f_2, f_3, f_4)^T$

With $f_i$ ; $i = 1,2,3,4$ represent the right hand side of system (2.2). Then we get $\frac{\partial F}{\partial u_9} = F_{u_9}(E_6,\bar{u}_9) = [0,0,0,0]^T$, and the following is obtained:

$\bar{y}^T[F_{u_9}(E_6,\overline{u_6})] = (0,0,0,\bar{h}_4)(0,0,0,0)^T = 0.$ Thus system (2.2) at $E_6$ does not experience any saddle-node bifurcation in view of sotomayor theorem. Also, since

$\bar{y}^T[DF_{u_3}(E_1,\overline{u_3})\bar{v}] = (0,0,0,\bar{h}_4)(0,0,0,0)^T = 0.$ Here, $DF_{u_9}(E_6,\overline{u_9}) = \frac{\partial}{\partial X}F_{u_9}(X,u_9)\,|_{X=E_6,u_9=\bar{u}_9}$. Thus again by sotomayor theorem, system (2.2) does not possesses any transcritical bifurcation and pitch-fork bifurcation near $E_6$ where $u_9 = \overline{u_9}$.

**Theorem 8:** If the parameter $u_5$ passes through the value $u_5' = \frac{1}{u_8}$ , then the equilibrium point $E_7$ transforms into nonhyperpolic equilibrium point and if

$$u_9 \neq u_5u_6u_7u_8 \qquad (2.19)$$

then system (2.2) possesses transcritical bifurcation, but no saddle-node bifurcation, nor pitch-fork bifurcation can occur.

**Proof:** According to the Jacobian matrix of system (2.2) at $E_7$ that is given by $J(E_7)$ it is easy to verify that as $u_5 = u_5'$, the $J(E7,u_5')$ has the following eigenvalues**:**

$$\lambda_{7x} = -1 < 0, \qquad \lambda_{7y} = \frac{eu_1 - u_3(u_2+1)}{u_2+1} < 0\ if\ condition(2.10a)\ holds,$$

$$\lambda_{7z} = 0 \quad and \quad \lambda_{7w} = -u_7 < 0.$$

Let $v' = (\theta'_1, \theta'_2, \theta'_3, \theta'_4)^T$ be the eigenvector of $J(E_7, u'_5)$ corresponding to the eigenvalue of $\lambda_{7z} = 0$. Then it is easy to check that $v' = (-\frac{b'_{13}}{b'_{11}}\theta'_3, 0, \theta'_3, -\frac{b'_{43}}{b'_{44}}\theta'_3)^T$ ,where $b'_{13} = 1 > 0,\ b'_{43} = -\frac{u_9}{u_8} < 0,\ b'_{44} = -u_7 < 0,$ and $\theta'_3$ represents any nonzero real value. Also, let $y' = (h'_1, h'_2, h'_3, h'_4)^T$ represents the eigenvector of $J^T(E_7, u'_5)$ that corresponding to the eigenvalue $\lambda_{7z} = 0$. Straight forward calculation shows that

$y' = (0, 0, h'_3, 0)^T$ , where $h'_3$ represents any nonzero real number.

Now, since $\frac{\partial F}{\partial u_5} = F_{u_5}(X, u_5) = [0, 0, z(1 - u_6 z), 0]^T$ , $where\ X = (x, y, z, w)^T\ and\ F = (f_1, f_2, f_3, f_4)^T$

With $f_i$ ; $i = 1,2,3,4$ represent the right hand side of system (2.2). Then we get $\frac{\partial F}{\partial u_5} = F_{u_5}(E_7, u'_5) = [0, 0, 0, 0]^T$ and the following is obtained:

$y'^T[F_{u_5}(E_7, u'_5)] = (0, 0, h'_3, \underline{0})(0, 0, 0, 0)^T = 0.$ Thus system (2.2) at $E_2$ does not experience any saddle-node bifurcation in view of sotomayor theorem. Also, since

$y'^T[DF_{u_5}(E_7, u'_5)v'] = (0, 0, h'_3, \underline{0})(0, 0, \theta'_3, 0)^T = h'_3\theta'_3 \neq 0.$ here,

$DF_{u_5}(E_7, u'_5) = \frac{\partial}{\partial X} F_{u_5}(X, u_5)\ |_{X = E_7, u_5 = u'_5}.$

Moreover, we have $y'^T[D^2F_{u_5}(E_7, u'_5)(v', v')] = h'_3\theta'^2_3\left[\frac{u_9 - u_5u_6u_7u_8}{u_7u_8}\right] \neq 0,$

by condition (2.19). Here, $D^2F_{u_5}(E_7, u'_5) = DJ(X, u_5)\ |_{X = E_7, u_5 = u'_5}.$ Then by sotomayor theorem, system (2.2) possesses a transcritical bifurcation but not pitch-fork bifurcation near $E_2$ where $u_7 = \underline{u_7}$ .However, violate condition (2.19) gives that $y'^T[D^2F_{u_5}(E_7, u'_5)(v', v')] = 0.$ , and hence further computation shows

$$y'^T[D^3F_{u_5}(E_7, u'_5)(v', v', v')] = (0, 0, h'_3, 0)(\frac{b'^3_{13}}{b'^3_{11}}\theta'^3_3(u_2 - 1)u_1, -2\frac{b'^3_{13}}{b'^3_{11}}\theta'^3_3\frac{eu_1u_2}{(1 + u_2)^3}, 0, 0)^T = 0.$$

Therefore according to Sotomayor theorem, there is no pitch-fork bifurcation.

**Theorem 9:** Assume that conditions (2.5a),(2.5b) and (2.5c) hold and the parameter $u_9$ passes through the value $\breve{u}_9 = u_6u_7$, then the equilibrium point $E_8$ transforms into nonhyperpolic equilibrium point and

$$\text{if } u_5 \neq \frac{1}{u_8} \qquad (2.20)$$

then system (2.2) possesses transcritical bifurcation, but no saddle-node bifurcation, nor pitch-fork bifurcation can occur.

**Proof:** According to the Jacobian matrix of system (2.2) at $E_8$ that is given by $J(E_8)$ it is easy to verify that as $u_9 = \breve{u}_9$ **,** the $J(E_8, \breve{u}_9)$ has the following eigenvalues**:**

$$\lambda_{8x} = -\frac{\breve{A}_1}{2} + \frac{1}{2}\sqrt{\breve{A}_1^2 - 4\breve{A}_2}\ , and\ \ \lambda_{8y} = -\frac{\breve{A}_1}{2} - \frac{1}{2}\sqrt{\breve{A}_1^2 - 4\breve{A}_2}\ \ where$$

$$\breve{A}_1 = -\breve{x}(-1 + \frac{u_1\breve{y}}{(u_2 + \breve{x})^2}) + u_4\breve{y},\ and\ \breve{A}_2 = u_4\breve{x}\breve{y}(1 - \frac{u_1\breve{y}}{(u_2 + \breve{x})^2}) + \frac{eu_1^2u_2\breve{x}\breve{y}}{(u_2 + \breve{x})^3},$$

$$\lambda_{8z} = -u_5 < 0\ \ and\ \lambda_{8w} = 0.$$

Let $\breve{v}=(\breve{\theta}_1,\breve{\theta}_2,\breve{\theta}_3,\breve{\theta}_4)^T$ be the eigenvector of $J(E_8,\breve{u}_9)$ corresponding to the eigenvalue of $\lambda_{8w}=0.$ Then it is easy to check that $\breve{v}=(-\frac{\breve{b}_{22}\breve{b}_{13}\breve{b}_{34}}{\breve{b}_{33}(\breve{b}_{12}\breve{b}_{21}-\breve{b}_{11}\breve{b}_{22})}\breve{\theta}_4,\frac{\breve{b}_{21}\breve{b}_{13}\breve{b}_{34}}{\breve{b}_{33}(\breve{b}_{12}\breve{b}_{21}-\breve{b}_{11}\breve{b}_{22})}\breve{\theta}_4,-\frac{\breve{b}_{34}}{\breve{b}_{33}}\breve{\theta}_4,\breve{\theta}_4)^T$ ,where

$$\breve{b}_{11}=-\breve{x}+\frac{u_1\breve{x}\breve{y}}{(u_2+\breve{x})^2},\ \breve{b}_{12}=-\frac{u_1\breve{x}}{u_2+\breve{x}}<0,\breve{b}_{13}=\breve{x}>0,\breve{b}_{21}=\frac{eu_1u_2\breve{y}}{(u_2+\breve{x})^2}>0\ ,\breve{b}_{22}=-u_4\breve{y}<0,$$

$$b_{33}=-u_5<0,\breve{b}_{34}=-\frac{1}{u_6}<0$$

and $\breve{\theta}_4$ represents any nonzero real value. Also, let $\breve{y}=(\breve{h}_1,\breve{h}_2,\breve{h}_3,\breve{h}_4)^T$ represents the eigenvector of $J^T(E_8,\breve{u}_9)$ that corresponding to the eigenvalue $\lambda_{8w}=0.$ Straight forward calculation shows that $\breve{y}=(0,0,0,\breve{h}_4)^T$ , where $\breve{h}_4$ represents any nonzero real number.

Now, since $\frac{\partial F}{\partial u_9}=F_{u_9}(X,u_9)=[0,0,0,-wz]^T\ ,where\ X=(x,y,z,w)^T\ and\ F=(f_1,f_2,f_3,f_4)^T$

With $f_i\ ;i=1,2,3,4$ represent the right hand side of system (2.2). Then we get $\frac{\partial F}{\partial u_9}=F_{u_9}(E_8,\breve{u}_9)=[0,0,0,0]^T$ and the following is obtained:

$\breve{y}^T[F_{u_9}(E_8,\breve{u}_9)]=(0,0,0,\breve{h}_4)(0,0,0,0)^T=0.$ Thus system (2.2) at $E_2$ does not experience any saddle-node bifurcation in view of sotomayor theorem . Also, since

$$\breve{y}^T[DF_{u_9}(E_8,\breve{u}_9)\breve{v}]=(0,0,0,\breve{h}_4)(0,0,0,-\breve{z}\breve{\theta}_4)^T=-\breve{h}_4\breve{\theta}_4\breve{z}\neq 0.$$

here, $DF_{u_9}(E_8,\breve{u}_9)=\frac{\partial}{\partial X}F_{u_9}(X,u_9)\ |_{X=E_8,u_5=\breve{u}_5}.$

Moreover, we have $\breve{y}^T[D^2F_{u_9}(E_8,\breve{u}_9)(\breve{v},\breve{v})]=\left[\frac{u_6u_7(1-u_5u_8)}{u_6u_9}\right]\breve{\theta}_4^2\breve{h}_4\neq 0,$ by condition (2.20).

Here, $D^2F_{u_5}(E_8,\breve{u}_5)=DJ(X,u_9)\ |_{X=E_8,u_5=\breve{u}_5}.$ Then by sotomayor theorem, system (2.2) possesses a transcritical bifurcation but not pitch-fork bifurcation near $E_8$ where $u_9=\breve{u}_9$ **.**However, violate condition (2.20) gives that

$\breve{y}^T[D^2F_{u_9}(E_8,\breve{u}_9)(\breve{v},\breve{v})]=0.$ , and hence further computation shows

$$\breve{y}^T[D^3F_{u_9}(E_8,\breve{u}_9)(\breve{v},\breve{v},\breve{v})]=(0,0,0,\breve{h}_4)(k_1,k_2,0,0)^T=0$$

$$where\ k_1=-u_1\breve{y}\breve{\theta}_1^3+\frac{u_2(u_1-\breve{x})}{(u_2+\breve{x})^3}\breve{\theta}_1^2\breve{\theta}_2+\frac{u_1\breve{\theta}_1^2\breve{\theta}_2(u_2-\breve{x})}{(u_2+\breve{x})^3}+(u_1u_2-u_1\breve{x})\breve{\theta}_1^2\breve{\theta}_2\ \ and$$

$$k_2=\frac{2eu_1u_2\breve{\theta}_1^2}{(u_2+\breve{x})^3}(\frac{2\breve{y}\breve{\theta}_1}{(u_2+\breve{x})^2}-\breve{\theta}_2-\frac{\breve{\theta}_2}{(u_2+\breve{x})}$$

Therefore according to Sotomayor theorem, there is no pitch-fork bifurcation.

**Theorem 10:**Assume that conditions (2.6b)-(2.6c) are hold and the parameter $u_5$ passes through the value , then the equilibrium point $E_9$ transforms into nonhyperpolic equilibrium point and if

$$u_9\neq u_6u_7 \qquad (2.21)$$

then system (2.2) possesses transcritical bifurcation, but no saddle-node bifurcation, nor pitch-fork bifurcation can occur.

**Proof:** According to the Jacobian matrix of system (2.2) at $E_9$ that is given by $J(E_9)$ it is easy to verify that as $u_5 = \hat{u}_5$, the $J(E_9, \hat{u}_5)$ has the following eigenvalues**:**

$$\lambda_{9x} = -\frac{B_1}{2} + \frac{1}{2}\sqrt{B_1^2 - 4B_2}\,, \lambda_{9y} = -\frac{B_1}{2} - \frac{1}{2}\sqrt{B_1^2 - 4B_2} \;\; where$$

$$B_1 = -\breve{x}(-1 + \frac{u_1\hat{y}}{(u_2+\hat{x})^2}) + u_4\hat{y},\; B_2 = u_4\hat{x}\hat{y}(1 - \frac{u_1\hat{y}}{(u_2+\hat{x})^2}) + \frac{eu_1^2u_2\hat{x}\hat{y}}{(u_2+\hat{x})^3},$$

$$\lambda_{9z} = 0 \;\; and \;\; \lambda_{9w} = -U_7 < 0.$$

Let $\hat{V} = (\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{\theta}_4)^T$ be the eigenvector of $J(E_9, \hat{u}_5)$ corresponding to the eigenvalue of $\lambda_{9z} = 0.$ Then it is easy to check that $\breve{v} = (\frac{\hat{b}_{22}\hat{b}_{13}}{(\hat{b}_{12}\hat{b}_{21} - \hat{b}_{11}\hat{b}_{22})}\hat{\theta}_3, -\frac{\hat{b}_{21}\hat{b}_{13}}{(\hat{b}_{12}\hat{b}_{21} - \hat{b}_{11}\hat{b}_{22})}\hat{\theta}_3, \hat{\theta}_3, -\frac{\hat{b}_{43}}{\hat{b}_{44}}\hat{\theta}_3)^T$ ,

where

$$\hat{b}_{11} = -\hat{x} + \frac{u_1\hat{x}\hat{y}}{(u_2+\hat{x})^2},\; \hat{b}_{12} = -\frac{u_1\hat{x}}{u_2+\hat{x}} < 0,\; \hat{b}_{13} = \hat{x} > 0,\; \hat{b}_{21} = \frac{eu_1u_2\hat{y}}{(u_2+\hat{x})^2} > 0,\; \hat{b}_{22} = -u_4\hat{y} < 0,$$

$$\hat{b}_{43} = -\frac{u_9}{u_8} < 0,\; \hat{b}_{44} = -u_7 < 0$$

and $\hat{\theta}_3$ represents any nonzero real value. Also, let $\hat{y} = (\hat{h}_1, \hat{h}_2, \hat{h}_3, \hat{h}_4)^T$ represents the eigenvector of $J^T(E_9, \hat{u}_5)$ that corresponding to the eigenvalue $\lambda_{9z} = 0.$ Straight forward calculation shows that $\hat{y} = (0, 0, \hat{h}_3, 0)^T$ , where $\hat{h}_3$ represents any nonzero real number.

Now, since $\frac{\partial F}{\partial u_5} = F_{u_5}(X, u_5) = [0, 0, z(1 - u_6 z), 0]^T, where \; X = (x, y, z, w)^T \; and \; F = (f_1, f_2, f_3, f_4)^T$

With $f_i$ ; $i = 1,2,3,4$ represent the right hand side of system (2.2). Then we get $\frac{\partial F}{\partial u_5} = F_{u_5}(E_9, \hat{u}_5) = [0, 0, 0, 0]^T$ and the following is obtained:

$$\hat{y}^T[F_{u_5}(E_9, \hat{u}_5)] = (0, 0, \hat{h}_3, 0)(0, 0, 0, 0)^T = 0.$$

Thus system (2.2) at $E_9$ does not experience any saddle-node bifurcation in view of sotomayor theorem. Also, since $\hat{y}^T[DF_{u_5}(E_9, \hat{u}_5)\hat{v}] = (0, 0, \hat{h}_3, 0)(0, 0, \hat{\theta}_3, 0)^T = \hat{h}_3\hat{\theta}_3 \neq 0.$

here, $DF_{u_5}(E_9, \hat{u}_5) = \frac{\partial}{\partial X}F_{u_5}(X, u_5)\,|_{X = E_9, u_5 = \hat{u}_5}$. Moreover, we have

$$\hat{y}^T[D^2F_{u_5}(E_9, \hat{u}_5)(\hat{v}, \hat{v})] = \left[\frac{u_9 - u_6u_7}{u_7u_8}\right]\hat{\theta}_3^2\hat{h}_4 \neq 0,$$ by condition (2.21).

Here, $D^2F_{u_5}(E_9, \hat{u}_5) = DJ(X, u_5)\,|_{X = E_9, u_5 = \hat{u}_5}$. Then by sotomayor theorem, system (2.2) possesses a transcritical bifurcation but not pitch-fork bifurcation near $E_9$ where $u_5 = \hat{u}_5$ **.**However, violate condition (2.21) gives that $\hat{y}^T[D^2F_{u_5}(E_9, \hat{u}_5)(\hat{v}, \hat{v})] = 0$, and hence further computation shows $\hat{y}^T[D^3F_{u_5}(E_9, \hat{u}_5)(\hat{v}, \hat{v}, \hat{v})] = (0, 0, \hat{h}_3, 0)(S_1, S_2, 0, 0)^T = 0.$

$$where \quad S_1 = -u_1\breve{y}\breve{\theta}_1^3 + \frac{u_2(u_1-\breve{x})}{(u_2+\breve{x})^3}\breve{\theta}_1^2\breve{\theta}_2 + \frac{u_1\breve{\theta}_1^2\breve{\theta}_2(u_2-\breve{x})}{(u_2+\breve{x})^3} + (u_1u_2 - u_1\breve{x})\breve{\theta}_1^2\breve{\theta}_2, and$$

$$S_2 = \frac{2eu_1u_2\breve{\theta}_1^2}{(u_2+\breve{x})^3}\left(\frac{2\breve{y}\breve{\theta}_1}{(u_2+\breve{x})^2} - \breve{\theta}_2 - \frac{\breve{\theta}_2}{(u_2+\breve{x})}\right.$$

Therefore according to Sotomayor theorem, there is no pitch-fork bifurcation.

**Theorem 11:** Assume that conditions (2.4b) or (2.4c) holds and the parameter $u_3$ passes through the value $u^\bullet_3 = \frac{eu_1x^\bullet}{u_2+x^\bullet}$ where $x^\bullet$ given in (2.7), then the equilibrium point $E_{10}$ transforms into nonhyperpolic equilibrium point and system (2.2) possesses transcritical bifurcation, but no saddle-node bifurcation, nor pitch-fork bifurcation can occur at $E_{10}$ where $u_3 = u^\bullet_3$ .

**Proof:** According to the Jacobian matrix of system (2.2) at $E_{10}$ that is given by $J(E_{10})$ it is easy to verify that as $u_3 = u^\bullet_3$, the $J(E_{10}, u^\bullet_3)$ has the following eigenvalues:

$$\lambda_{10x} = -x^\bullet < 0 \;, \; \lambda_{10y} = 0, \; \lambda_{10z} = -\frac{B^\bullet_1}{2} + \frac{1}{2}\sqrt{B^{\bullet 2}_1 - 4B^\bullet_2}, and$$

$$\lambda_{10w} = -\frac{B^\bullet_1}{2} - \frac{1}{2}\sqrt{B^{\bullet 2}_1 - 4B^\bullet_2}$$

$$where \; B^\bullet_1 = u_5u_6z^\bullet + u_7u_8w^\bullet, and \; B^\bullet_2 = (u_5u_6u_7u_8 - u_9)z^\bullet w^\bullet.$$

Let $v^\bullet = (\theta^\bullet_1, \theta^\bullet_2, \theta^\bullet_3, \theta^\bullet_4)^T$ be the eigenvector of $J(E_{10}, u^\bullet_3)$ corresponding to the eigenvalue of $\lambda_{10y} = 0.$ Then it is easy to check that $v^\bullet = (-\frac{b^\bullet_{12}}{b^\bullet_{11}}\theta^\bullet_2, \theta^\bullet_2, 0, 0)^T$ ,where

$$b^\bullet_{11} = -\hat{x} < 0, \; b^\bullet_{12} = -\frac{u_1x^\bullet}{u_2+x^\bullet} < 0,$$

and $\theta^\bullet_2$ represents any nonzero real value. Also, let $y^\bullet = (h^\bullet_1, h^\bullet_2, h^\bullet_3, h^\bullet_4)^T$ represents the eigenvector of $J^T(E_{10}, u^\bullet_3)$ that corresponding to the eigenvalue $\lambda_{10y} = 0.$ Straight forward calculation shows that

$y^\bullet = (0, h^\bullet_2, 0, 0)^T$ , where $h^\bullet_2$ represents any nonzero real number.

Now, since $\frac{\partial F}{\partial u_3} = F_{u_3}(X, u_3) = [0, 0, -y, 0]^T, where\, X = (x, y, z, w)^T \; and\, F = (f_1, f_2, f_3, f_4)^T$

With $f_i$ ; $i = 1,2,3,4$ represent the right hand side of system (2.2). Then we get $\frac{\partial F}{\partial u_3} = F_{u_3}(E_{10}, u^\bullet_3) = [0,0,0,0]^T$ and the following is obtained:

$$y^{\bullet T}[F_{u_3}(E_{10}, u^\bullet_3)] = (0, 0, h^\bullet_2, 0)(0,0,0,0)^T = 0.$$

Thus system (2.2) at $E_{10}$ does not experience any saddle-node bifurcation in view of sotomayor theorem. Also, since $y^{\bullet T}[DF_{u_3}(E_{10}, u^\bullet_3)v^\bullet] = (0, 0, h^\bullet_2, 0)(0, -\theta^\bullet_2, 0, 0)^T = -h^\bullet_2\theta^\bullet_2 \neq 0.$

here, $DF_{u_3}(E_{10}, u^\bullet_3) = \frac{\partial}{\partial X}F_{u_3}(X, u_3)\,|_{X=E_{10}, u_3=u^\bullet_3}.$ Moreover, we have

$$y^{\bullet T}[D^2F_{u_3}(E_{10},u^{\bullet}{}_3)(v^{\bullet},v^{\bullet})] = (0,0,h^{\bullet}{}_2,0)(-\frac{b_{12}^{\bullet}}{b_{11}^{\bullet}}(\frac{b_{12}^{\bullet}}{b_{11}^{\bullet}}+2\frac{u_1x^{\bullet}}{(u_2+x^{\bullet})^2}\theta_2^{\bullet 2}),-\frac{b_{12}^{\bullet}}{b_{11}^{\bullet}}\frac{eu_1u_2}{(u_2+x^{\bullet})^2}\theta_2^{\bullet}-2u_2\theta_2^{\bullet 2},0,0)^T$$

$$=\left[-\frac{b_{12}^{\bullet}}{b_{11}^{\bullet}}\frac{eu_1u_2}{(u_2+x^{\bullet})^2}\theta_2^{\bullet}-2u_2\theta_2^{\bullet 2}\right]h_2^{\bullet}\neq 0.$$

Here, $D^2F_{u_3}(E_{10},u_3^{\bullet}) = DJ(X,u_3)\mid_{X=E_{10},u_3=u_3^{\bullet}}$. Then by sotomayor theorem, system (2.2) possesses a transcritical bifurcation but not pitch-fork bifurcation near $E_{10}$ where $u_3 = u^{\bullet}{}_3$ **.**

**Theorem 12:** Assume that $E_{11} = (x^*, y^*, z^*, w^*)$ exist and the parameter $u_9$ passes through the value $u_9^* = u_5u_6u_7u_8$ then the equilibrium point $E_{11}$ transforms into nonhyperpolic equilibrium point and if

$$w^* \neq u_5u_6z^* \qquad (2.22)$$

where $z^*$ and $w^*$ are given in (2.7a) and (2.7b), then system (2.2) possesses a saddle-node bifurcation, but not transcritical bifurcation nor pitch-fork bifurcation can occur at $E_{11}$ where $u_9 = u_9^*$ .

**Proof:** According to the Jacobian matrix of system (2.2) at $E_{11}$ that is given by $J(E_{11})$ it is easy to verify that as $u_9 = u_9^*$ **,** the $J(E_{11},u_9^*)$ has the following eigenvalues**:**

$$\lambda_{11x} = -\frac{R_1}{2}+\frac{1}{2}\sqrt{R_1^2-4R_2}, \lambda_{11y} = -\frac{R_1}{2}-\frac{1}{2}\sqrt{R_1^2-4R_2},$$

$$\lambda_{11z} = -\frac{R_3}{2}+\frac{1}{2}\sqrt{R_3^2-4R_4^2} = -\frac{R_3}{2}+\frac{R_3}{2} = 0, \lambda_{11w} = -\frac{R_3}{2}-\frac{1}{2}\sqrt{R_3^2-4R_4^2} = -R_3$$

$$where\ R_1 = -x^*(-1+\frac{u_1y^*}{(u_2+x^*)^2})+u_4y^*,\ R_2 = u_4x^*y^*(1-\frac{u_1y^*}{(u_2+x^*)^2})+\frac{eu_1^2u_2x^*y^*}{(u_2+x^*)^3},$$

$$R_3 = u_5u_6z^*+u_7u_8w^*, and R_4 = (u_5u_6u_7u_8-u_9^*)z^*w^* = 0$$

Let $v^* = (\theta_1^*,\theta_2^*,\theta_3^*,\theta_4^*)^T$ be the eigenvector of $J(E_{11},u_9^*)$ corresponding to the eigenvalue of $\lambda_{11z} = 0$. Then it is easy to check that

$$\breve{v} = (-\frac{b^*_{22}b^*_{13}}{(-b^*_{12}b^*_{21}+b^*_{11}b^*_{22})}\theta^*_3,-\frac{b^*_{21}b^*_{13}}{(b^*_{12}b^*_{21}-b^*_{11}b^*_{22})}\theta_3^*,\theta^*_3,-\frac{b^*_{33}}{b^*_{34}}\theta^*_3)^T$$

$$where,\ b^*_{11} = -x^*+\frac{u_1x^*y^*}{(u_2+x^*)^2},\ b^*_{12} = -\frac{u_1x^*}{u_2+x^*}<0, b^*_{13} = x^*>0, b^*_{21} = \frac{eu_1u_2y^*}{(u_2+x^*)^2}>0,$$

$$b^*_{22} = -u_4y^*<0, b^*_{33} = -u_5u_6z^*<0, b^*_{34} = -z^*<0$$

and $\theta_3^*$ represents any nonzero real value. Also, let $y^* = (h_1^*,h_2^*,h_3^*,h_4^*)^T$ represents the eigenvector of $J^T(E_{11},u_9^*)$ that corresponding to the eigenvalue $\lambda_{11z} = 0$. Straight forward calculation shows that

$$y^* = (-\frac{b^*_{43}b^*_{34}-b^*_{33}b^*_{44}}{b^*_{13}b^*_{44}}h^*_3,-\frac{b^*_{11}(b^*_{33}b^*_{44}-b^*_{43}b^*_{34})}{b^*_{21}b^*_{13}b^*_{44}}h_3^*,h^*_3,-\frac{b^*_{34}}{b^*_{44}}h^*_3)^T,$$

where $b^*_{43} = -u_9^*w^*<0, b^*_{44} = -u_7u_8w^*<0, and$ $h_3^*$ represents any nonzero real number.

Now, since $\frac{\partial F}{\partial u_9} = F_{u_9}(X,u_9^*) = [0,0,0,-wz]^T, where\ X = (x,y,z,w)^T\ and\ F = (f_1,f_2,f_3,f_4)^T$

With $f_i \; ; i = 1,2,3,4$ represent the right hand side of system (2.2). Then we get $\frac{\partial F}{\partial u_9} = F_{u_9}(E_{11}, u_9^*) = [0,0,0,-w^* z^*]^T$ and the following is obtained:

$y^{*T}[F_{u_9}(E_{11}, u_9^*)] = \frac{u_5 u_6 z^{*2}}{u_7 u_8} h_3^* \neq 0.$ Also, since $y^{*T}[DF_{u_9}(E_{11}, u_9^*)v^*] = \left[-w^* + u_5 u_6 z^*\right]\theta_3^* \neq 0,$ by condition (2.22) .Here, $DF_{u_9}(E_{11}, u_9^*) = \frac{\partial}{\partial X} F_{u_9}(X, u_9) \,|_{X = E_{11}, u_9 = u_9^*}.$ Then by sotomayor theorem, system (2.2) possesses a saddle-node bifurcation but not transcritical bifurcation nor pitch-fork bifurcation can occur at $E_{11}$ where $u_9 = u_9^*$ .

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