Iraqi Journal of Science
ISSN: 0067-2904
GIF: 0.851

On Jordan Generalized Reverse Derivations on Γ-rings

Salah Mehdi Salih, Marwa Riyadh Salih*
Department of Mathematics, College of Education, AL-Mustansirya University, Baghdad, Iraq.

Abstract

In this paper, we study the concepts of generalized reverse derivation, Jordan generalized reverse derivation and Jordan generalized triple reverse derivation on Γ-ring M. The aim of this paper is to prove that every Jordan generalized reverse derivation of Γ-ring M is generalized reverse derivation of M.

Keywords: Γ-ring, prime Γ-ring, semiprime Γ-ring, derivation, generalized higher derivation of Γ-ring, reverse derivation of R

حول تعميم مشتقات جوردان المعكوسة لحلقات– Γ

صلاح مهدي صالح، مروة رياض صالح*
قسم الرياضيات،كلية التربية، الجامعة المستنصرية ،بغداد، العراق

الخلاصة:

في هذا البحث،سندرس مفهوم تمعمم المشتقات المعكوسة و تمعمم مشتقات جوردان المعكوسة و تعميم مشتقات جوردان المعكوسة الثلاثية على حلقة من النمط – Γ .الهدف من البحث هواثبات ان تعميم مشتقات جوردان المعكوسة لحلقة من النمط – Γ هو تعميم المشتقات المعكوسة لحلقة من النمط – Γ .

1. Introduction

The concepts of a Γ-ring was first introduced by N.Nobusause [1] in 1964, this Γ-ring is generalized by W.E.Barnes in a broad sense that served now – a day to call a Γ-ring.
Let M and Γ be two additive abelian groups. Suppose that there is a mapping from $M \times\Gamma\times M\rightarrow M$(the image of$(a,\alpha,b)$ being denoted by $a\alpha b$, $a, b\in M$ and $\alpha\in\Gamma$) satisfying for all $a, b, c \in M$ and $\alpha,\beta \in\Gamma$

i) $(a +b) \alpha c = a\alpha c + b\alpha c$
$a(\alpha + \beta) c = a\alpha c + a\beta c$
$a\alpha(b + c) = a\alpha b + a\alpha c$

ii) $(a\alpha b)\beta c = a\alpha(b\beta c)$

Then M is called a Γ-ring. [2]

Throughout this paper M denotes a Γ-ring with center Z (M) [3], recall that a -Γring M is called prime If $a\Gamma M\Gamma b= (0)$ implies a=0 or b=0 [4], and it is called semiprime if $a\Gamma M\Gamma a= (0)$ implies a=0[6], a primΓ-ring is obviously semiprime and a Γ-ring M is called 2-torisiofree if 2a=0 implies a=0for every $a\in M$ [5],an additive mapping d from M into itself is called a derivations if $d(a\alpha b)=d(a)\alpha b + a\alpha d(b)$,for all $a, b\in M ,\alpha\in\Gamma$, [7] and d is said to be Jordan derivation of a Γ-ring M if $d(a\alpha a)= d(a)\alpha a + a\alpha d(a)$,for all $a, b \in M ,\alpha \in \Gamma$, [7].A mapping f from M into itself is called generalized derivation of M if there exists derivation of M such that

*Email: noor.nr286@gmail.com

f ($a\alpha b$)=f(a)αb +aαd(b), for all a, b $\in$ M, $\alpha \in \Gamma$,[8]. And f is said to be Jordan generalized derivation of Γ-ring M if there exists Jordan generalized derivation of M such that f ($a\alpha a$) =f (a) αa +aαd (a) for all a $\in$ M and $\alpha \in \Gamma$, [8].

Bresar and Vukman, [9] have introduced the notion of a reverse derivation as an additive mapping d from a ring R in to itself satisfying d ($x\alpha y$) = d(y) αx +yαd(x) for all x, y $\in$ R.

M. Samman, [10] presented study between the derivation and reverse derivation in semiprime rings R. Also it is shown that non-commutative prime rings do not admit a non-trivial skew commuting derivation.

We defined in [11] the concepts of reverse derivation of Γ-ring M d($x\alpha y$) = d(y) αx + yαd(x) for all x,y $\in$M,$\alpha\in\Gamma$

2. Generalized reverse derivation of Γ-ring:

In this section we introduce and study the concepts of generalized reverse derivation, Jordan generalized reverse derivation and Jordan generalized triple reverse derivation of Γ-ring.

Definition 2.1

Let M be a Γ-ring and f: M $\rightarrow$ M be an additive mapping then f is called **generalized reverse derivation on M** if there exists a reverse derivation d: M $\rightarrow$ M such that

f($x\alpha y$) = f(y)αx + yαd(x) ……(1)

f is said to be **a Jordan generalized reverse derivation of M** if there exists a Jordan reverse derivation such that f($x\alpha x$) = f(x)αx + xαd(x) …(2) for every x $\in$M and $\alpha\in\Gamma$

f is said to be a Jordan generalized triple reverse derivation of M if there exists Jordan triple higher reverse derivation of M such that:

f($x\alpha y\beta x$) = f(x)$\beta x\alpha y$ + xβd(y)αx + x$\beta y\alpha$d(x) …..(3)

Example 2.2.

Let f be a generalized reverse derivation on a ring R then there exists a reverse derivation d of R such that f (xy) = f(y) x + y d(x)

We take M = $M_{1\times 2}$(R) and Γ = then M is Γ-ring.

We define D be an additive mappings of M such that D (a b) = (d (a) d (b)) then D is reverse derivation of M.

Let F be additive mappings of M defined by F (a b) = (f(a) f(b)) Then F is a generalized reverse derivation of M.

It is clear that every generalized reverse derivation of a Γ-ring M is Jordan generalized reverse derivation of M, But the converse is not true.

Lemma 2.3.

Let M be a Γ-ring and let f be a Jordan generalized reverse derivation of M then for all x, y, z $\in$ M and$\alpha,\beta \in \Gamma$, the following statements hold:

i) f($x\alpha y + y\alpha x$) = f(y)α x + yαd(x) + f(x)αy + xαd(y)

ii) f($x\alpha y\beta x$+ $x\beta y\alpha x$) = f(x)$\beta x\alpha y$ +xβd(y)αx + x$\beta y\alpha$d(x) + f(x)$\alpha x\beta y$ + xαd(y)βx +x$\alpha y\beta$d(x)

iii)f($x\alpha y\alpha x$)=f(x)$\alpha x\alpha y$+ xαd(y)αx+ x$\alpha y\alpha$d(x)

iv)f ($x\alpha y\alpha z$ +$z\alpha y\alpha x$) = f(z)$\alpha x\alpha y$ + zαd(y)αx +z$\alpha y\alpha$d(x)+ f(x)$\alpha z\alpha y$ + xαd(y)αz + x$\alpha y\alpha$d(z)

v)f($x\alpha y\beta z$) = f(z)$\beta x\alpha y$ + zβd(y)αx + z$\beta y\alpha$d(x)

vi)f ($x\alpha y\beta z$ + $z\alpha y\beta x$) = f(z)$\beta x\alpha y$ + zβd(y)αx + z$\beta y\alpha$d(x)+ f(x)$\beta z\alpha y$ + xβd(y)αz + x$\beta y\alpha$d(z)

Proof:

i) Replace (x + y) for x and y in definition 2.1 (1) we get:

f((x+y)α(x+y))=f(x+y)α(x+y)+ (x +y)αd(x+y)

=f(x) αx +f(y) αx + f(x) αy + f(y) αy +xαd(x) + yαd(x) + xαd(y) + yαd(y) …… (1)

On the other hand:

f((x + y)α(x + y)) = f($x\alpha x$ +$x\alpha y$ + $y\alpha x$ + $y\alpha y$)

=f($x\alpha x$ + $y\alpha y$) + f($x\alpha y$ + $y\alpha x$)

$= f(x)\, \alpha x + x\alpha d(x) + f(y)\alpha y + y\alpha d(y) + f(x\alpha y + y\alpha x)$ ..… (2)

Compare (1) and (2) we get:

$f(x\alpha y + y\alpha x) = f(y)\alpha x + y\alpha d(x) + f(x)\alpha y + x\alpha d(y)$

ii) Replacing $x\beta y + y\beta x$ for y in 2.3 (i) we get:

$f(x\alpha(x\beta y+y\beta x)+ (x\beta y + y\beta x)\alpha x)$

$= f(x\alpha(x\beta y) +x\, \alpha((y\beta x) + (x\beta y)\, \alpha x +(y\beta x)\, \alpha x)$

$= f((x\alpha x)\, \beta y + (x\alpha y)\beta x + (x\beta y)\alpha x + (y\beta x)\alpha x)$

$=f(y)\, \beta x\alpha x+y\beta d(x)\, \alpha x+y\beta x\alpha d(x) +f(x)\beta x\alpha y+x\beta d(y)\alpha x+x\beta y\alpha d(x)+f(x)\alpha x\beta y+x\alpha d(y)\beta x$
$+x\alpha y\beta d(x) +f(x)\, \alpha y\beta x+x\alpha d(x)\, \beta y + x\alpha x\beta d(y)$ …… (1)

On the other hand

$f(x\alpha(x\beta y + y\beta x) + (x\beta y + y\beta x)\alpha x)$

$= f(x\alpha x\beta y + x\alpha y\beta x + x\beta y\alpha x + y\beta x\alpha x)$

$= f(x\alpha x\beta y + y\beta x\alpha x) + f(x\alpha y\beta x + x\beta y\alpha x)$

$=f(y)\, \beta x\alpha x + y\beta d(x)\, \alpha x + y\beta x\alpha d(x) + f(x)\alpha y\beta x + x\alpha d(x)\beta y + x\alpha x\beta d(y) +f(x\alpha y\beta x + x\beta y\alpha x)$ …..(2)

Compare (1) and (2) we get the require result.

iii) Replacing α for β in 2.3 (ii) we have:

$f(x\alpha y\alpha x + x\alpha y\alpha x) = 2(f(x\alpha y\alpha x))$

$=2(f(x)\, \alpha x\alpha y+x\alpha d(y)\, \alpha x+ x\alpha y\alpha d(x))$

Since M is 2-torsion free then we get:

$f(x\alpha y\alpha x) = f(x)\alpha y\alpha x +x\alpha d(y)\alpha x + x\alpha y\alpha d(x)$

iv) Replace (x+z) for x in 2.3(iii) we have:

$f((x+z)\alpha y\alpha(x+z))$

$=f(x+z)\alpha(x+z)\alpha y+(x+z)\alpha d(y)\alpha(x+z)+(x+z)\alpha y\alpha d(x+z)$

$=f(x)\alpha x\alpha y + f(z)\alpha x\alpha y + f(x)\alpha z\alpha y + f(z)\alpha z\alpha y+x\alpha d(y)\alpha x+z\alpha d(y)\alpha x+x\alpha d(y)\alpha z+z\alpha d(y)\alpha z +x\alpha y\alpha d(x)$
$+ x\alpha y\alpha d(z) + z\alpha y\alpha d(x) + x\alpha y\alpha d(z) + z\alpha y\alpha d(z)$ …..(1)

On the other hand

$f((x+z)\alpha y\alpha(x+z))$

$= f(x\alpha y\alpha x + x\alpha y\alpha z + z\alpha y\alpha x + z\alpha y\alpha z)$

$=f(x\alpha y\alpha x + z\alpha y\alpha z) + f(x\alpha y\alpha z + z\alpha y\alpha x)$

$=f(x)\alpha x\alpha y + x\alpha d(y)\alpha x + x\alpha y\alpha d(x) + f(z)\alpha z\alpha y + z\alpha d(y)\alpha z + z\alpha y\alpha d(z)+f(x\alpha y\alpha z + z\alpha y\alpha x)$ ….(2)

Compare (1) and (2) we get:

$f(x\alpha y\alpha z + z\alpha y\alpha x) = f(z)\alpha x\alpha y + z\alpha d(y)\alpha x + z\alpha y\alpha d(x) +f(x)\alpha z\alpha y + x\alpha d(y)\alpha z + x\alpha y\alpha d(z)$

v) Replace (x+z) for x in definition 2.1(3) we have:

$f((x+z)\alpha y\beta(x+z))$

$=f(x+z)\, \beta(x+z)\alpha y+(x+z)\beta d(y)\alpha(x+z)+(x+z)\beta y\alpha d(x+z)$

$=f(x)\beta x\alpha y +f(z)\beta x\alpha y + f(x)\beta z\alpha y + f(z)\beta z\alpha y + x\beta d(y)\alpha x+z\beta d(y)\alpha x + x\beta d(y)\alpha z + z\beta d(y)\alpha z$
$+x\beta y\alpha d(x)+x\beta y\alpha d(z)+z\beta y\alpha d(x) + z\beta y\alpha d(z)$ ….(1)

On the other hand

$f((x+z)\alpha y\beta(x+z))=f(x\alpha y\beta x+x\alpha y\beta z+z\alpha y\beta x +z\alpha y\beta z)$

$= f(x\alpha y\beta x + z\alpha y\beta x +z\alpha y\beta z) + f(x\alpha y\beta z)$

$=f((x)\beta x\alpha y + x\beta d(y)\alpha x + x\beta y\alpha d(x) + f(x)\beta z\alpha y + x\beta d(y)\alpha z +x\beta y\alpha d(z) + f(z)\beta z\alpha y + z\beta d(y)\alpha z +$
$z\beta y\alpha d(z) + f(x\alpha y\beta z)$ ….(2)

Compare (1) and (2) we get

$f(x\alpha y\beta z) = f(x)\beta z\alpha y + x\beta d(y)\alpha z + x\beta y\alpha d(z)$

vi) Replacing x+z for x in definition 2.1(3) we have:

$f((x+z)\alpha y\beta(x+z))=f(x+z)\beta(x+z)\alpha y + (x+z)\beta d(y)\alpha(x+z) + (x+z)\beta y\alpha d(x+z)$

$=f(x)\beta x\alpha y + f(z)\beta x\alpha y + f(x)\beta z\alpha y + f(z)\beta z\alpha y + x\beta d(y)\alpha x + z\beta d(y)\alpha x + x\beta d(y)\alpha z + z\beta d(y)\alpha z + x\beta y\alpha d(x) + z\beta y\alpha d(x) + x\beta y\alpha d(z) + z\beta y\alpha d(z)$ ….(1)

On the other hand:

$f((x+z)\alpha y\beta(x+z))$

$=(x\alpha y\beta x+x\alpha y\beta z+z\alpha y\beta x+z\alpha y\beta z)$

$= f(x\alpha y\beta x + z\alpha y\beta z) + f(x\alpha y\beta z + z\alpha y\beta x)$

$=f(x)\beta x\alpha y+x\beta d(y)\alpha x+x\beta y\alpha d(x)+ (x\alpha y\beta z +z\alpha y\beta x)$ …. (2)

Compare (1) and (2) we get the require result.

Definition 2.4

Let f be a Jordan generalized reverse derivation of a Γ-ring M, then for all $x,y\in M$ and $\alpha\in\Gamma$ we define:

$\delta(x,y)_\alpha = f(x\alpha y) - f(y)\alpha x - y\alpha d(x)$

In the following lemma we introduce some properties of $\delta(x,y)_\alpha$

Lemma 2.5

If f is a Jordan generalized reverse derivation of Γ-ring M then for all $x,y,z \in M$ and $\alpha,\beta \in\Gamma$

i) $\delta(x,y)_\alpha = -\delta(y,x)_\alpha$

ii) $\delta(x+y,z)_\alpha = (x,z)_\alpha + (y,z)_\alpha$

iii) $\delta(x,y+z)_\alpha = \delta(x,y)_\alpha + \delta(x,z)_\alpha$

iv) $\delta(x,y)_{\alpha+\beta} = \delta(x,y)_\alpha + \delta(x,y)_\beta$

Proof:

i) By lemma 2.3 (i) and since f is additive mapping of M we get:

$f(x\alpha y + y\alpha x) = f(y)\alpha x + y\alpha d(x) + f(x)\alpha y + x\alpha d(y)$

$f(x\alpha y)+f(y\alpha x)=(f(y)\alpha x+y\alpha d(x))+ (f(x)\alpha y +x\alpha d(y))$

$f(x\alpha y)-f(y)\alpha x - y\alpha d(x) = -f(y\alpha x) + f(x)\alpha y +x\alpha d(y)$

$f(x\alpha y)-f(y)\alpha x-y\alpha d(x) = -(f(y\alpha x) - f(x)\alpha y - x\alpha d(y))$

$\delta(x,y)_\alpha = -\delta(y,x)_\alpha$

ii) $\delta(x+y,z)_\alpha=f((x+y)\alpha z)-(f(z)\alpha(x+y) +z\alpha d(x+y))$

$= f(x\alpha z + y\alpha z) - (f(z)\alpha x + f(z)\alpha y + z\alpha d(x) + z\alpha d(y))$

Since f is additive mapping of the Γ-ring

$=f(x\alpha z)-f(z)\alpha x - z\alpha d(x) + f(y\alpha z) - f(z)\alpha y -z\alpha d(y)$

$= \delta(x,z)_\alpha + \delta(y,z)_\alpha$

iii) $\delta(x,y+z)_\alpha=f(x\alpha(y+z))-(f(y+z)\alpha x +(y+z)\alpha d(x))$

$= f(x\alpha y)-f(y)\alpha x-y\alpha d(x) + f(x\alpha z) - f(z)\alpha x - z\alpha d(x)$

$= \delta(x,y)_\alpha + \delta(x,z)_\alpha$

iv) $\delta(x,y)_{\alpha+\beta}= f(x(\alpha+\beta)y)-(f(y)(\alpha+\beta)x+ y(\alpha+\beta)d(x))$

$= f(x\alpha y+x\beta y)-(f(y)\alpha x + f(y)\beta x + y\alpha d(x) + y\beta d(x))$

Since f is additive mapping of aΓ-ring

$=f(x\alpha y) - f(y)\alpha x - y\alpha d(x) + f(x\beta y) - f(y)\beta x - y\beta d(x)$

$= \delta(x,y)_\alpha + \delta(x,y)_\beta$

Remark 2.6.

Note that f is generalized reverse derivation of a Γ-ring M if and only if $\delta(x,y)\alpha = 0$ for all $x, y \in M,\alpha \in\Gamma$.

3. The main result

In this section we present the main results of this paper.

Theorem 3.1

.Let f be a Jordan generalized reverse derivation of M then $\delta(x,y)\alpha = 0$ for all x, y $\in$ M, $\alpha \in \Gamma$.

Proof:

By lemma 2.3 (i) we get:

$f(x\alpha y + y\alpha x) = f(y)\alpha x + y\alpha d(x) + f(x)\alpha y + x\alpha d(y)$ …..(1)

On the other hand:

Since f is additive mapping of the Γ-ring M we have:

$f(x\alpha y + y\alpha x) = f(x\alpha y) + f(y\alpha x)$

$= f(x\alpha y) + f(x)\alpha y + x\alpha d(y)$ ….(2)

Compare (1) and (2) we get

$f(x\alpha y) = f(y)\alpha x + y\alpha d(x)$

$f(x\alpha y) - f(y)\alpha x - y\alpha d(x) = 0$

By definition 2.5 we get:

$\delta(x,y)\alpha = 0$

Corollary 3.2

Every Jordan generalized reverse derivation of Γ-ring M is generalized reverse derivation of M.

Proof:

By Theorem 3.1 we get $\delta(x,y)\alpha = 0$ and by Remark 2.6 we get the require result

Proposition 3.3

Every Jordan generalized reverse derivation of a 2-torision free of a Γ-ring M where $x\alpha y\beta z = x\beta y\alpha z$ is Jordan generalized triple reverse derivation of M.

Proof:

Let f be a Jordan generalized reverse derivation of M

Replace y by $(x\beta y + y\beta x)$ in lemma 2.3 (i) we get

$f(x\,\alpha(x\beta y + y\beta x) + (x\beta y + y\beta x)\alpha\, x)$

$= f((x\alpha(x\beta y) + x\alpha(y\beta x) + (x\beta y\alpha)\, x + (y\beta x)\,\alpha x)$

$= f((x\alpha x)\,\beta y + (x\alpha y)\,\beta x + (x\beta y)\,\alpha x + (y\beta x)\,\alpha x)$

$= f(y)\,\beta x\alpha x + y\beta d(x\alpha x) + f(x)\,\beta(x\alpha y) + x\beta d(x\alpha y) + f(x)\,\alpha(x\beta y) + x\alpha d(x\beta y) + f(x)\alpha(y\beta x) + x\alpha d(y\beta x)$

$= f(y)\,\beta x\alpha x + y\beta d(x)\,\alpha x + y\beta x\alpha d(x) + f(x)\,\beta x\alpha y + x\beta d(y)\,\alpha x + x\beta y\alpha d(x) + f(x)\,\alpha x\beta y + x\alpha d(y)\,\beta x$
$+ x\alpha y\beta d(x) + f(x)\,\alpha y\beta x + x\alpha d(x)\,\beta y + x\alpha x\beta d(y)$ … (1)

On the other hand:

$f(x\alpha(x\beta y + y\beta x) + (x\beta y + y\beta x)\alpha x)$

$= f(x\alpha x\beta y + x\alpha y\beta x + x\beta y\alpha x + y\beta x\alpha x)$

$= f(x\alpha xy + y\beta x\alpha x) + (x\alpha y\beta x + x\beta y\alpha x)$ …. (2)

Compare (1) and (2) and since $x\alpha y\beta z = x\beta y\alpha z$ we get

$f(x\alpha y\beta x + x\alpha y\beta x) = 2(f(x\alpha y\beta x)) = 2(f(x)\beta x\alpha y + x\beta d(y)\alpha x + x\beta y\alpha d(x))$

Since M is a 2-torision free then we have

$f(x\alpha y\beta x) = f(x)\beta x\alpha y + x\beta d(y)\alpha x + x\beta y\alpha d(x)$.

References

1. Nobusawa, N. **1964**. On a generalization of the ring theory, *Osaka J. Math*. 1, PP: 81-89.
2. Barnes,W.E. **1966**. On The Γ-rings of Nobusawa, *Pacific J. Math*, 18, PP: 411-422.
3. Asci, M., and ceran, S. **2007**. The commutativity in prime gamma rings with left derivation, *International Mathematical*, 2(3), PP: 103-108.
4. Kyuno, S. **1987**. On Prime Gamma Rings, *pacific J. of Math*., 75, PP: 185-190.
5. Dutta, T. K., and Sadar, S., K. **2000**. Semmiprime Ideals and Irreducible Ideal Of Γ-semirings, *Novi Sad J. Math*, 30(1), pp: 97-108.
6. Chakraborty, S., and Paul, A.C. **2008**. On Jordan K-derivation of 2-torision free ΓN-ring, *Punjab University J-of Math.*, 40, PP: 97-101.

7. Jing, F.J. **1987**. On Derivation Of Γ-rings, *Q U F U Shi Fan Daxue Xuebeo Ziran Kexue Ban*, 13(4), PP:159-161.
8. Ceven, Y. and Ozlurk, M.A. **2004**. On Jordan Generalized Derivation in Gamma rings, *Hacettepe, J. of Mathematics and Statistics*, 33, PP: 11-14.
9. Bresar, M., and Vukman, J. **1989**. On Some additive mappings in rings with involution, *Aequation Math.,* 38, pp: 178-185.
10. Samman, M., and Alyamani, N. **2007**. Derivation and Reverse Derivation in Semiprime Rings, *International Mathematical Forum*, 2(39), pp: 1895-1902.
11. Dey, K., Paul, A.C., and Rakhimov, I.S. **2013**. semiprime gamma rings with orthogonal reverse derivations, *international journal of pure and applied mathematics*, 83(2), PP: 233-245