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# $s$-WH Modules

**Osama Basim Mohammed, Thaer Younis Ghawi**
*Department of Mathematics, College of Education, AL-Qadisiyah University, AL-Qadisiyah, Iraq*

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**Abstract**

This article introduces the concept of strongly-WH module which is a proper generalized of Hopfian modules. A module $M$ is called strongly-WH, briefly $s$-WH if, any $e$-small surjective $R$-endomorphism of $M$ is an automorphism. We specify and provide some properties of this concept. Furthermore, we have established connections between strongly-WH modules and various other concepts. We demonstrate that every strongly-WH module is $\delta$-weakly Hopfian. As well as, we provide cases in which the concepts of WH, $\delta$-weakly Hopfian, and strongly-WH modules are equivalent.

**Keywords:** $s$-WH modules; Hopfian modules; weakly Hopfian modules; $e$-small submodules.

# المقاسات من النمط $s$−WH

**أسامة باسم ، ثائر يونس غاوي**
قسم الرياضيات، كلية التربية، جامعة القادسية، القادسية – العراق

**الخلاصة**

تقدم هذه المقالة مفهوم المقاس القوي من النمط WH وهو تعميم فعلي للمقاسات الهوبفيينة. يقال للمقاس $M$ على الحلقة $R$ انهُ قوي من النمط WH, باختصار $s$−WH، اذا كان كل تشاكل ذاتي شامل صغير من النمط e على $M$ يكون تشاكل متقابل. نحن حددنا وأثبتنا بعض الخواص لهذه المقاسات. بالإضافة إلى ذلك، نحن أثبتنا العلاقة بين المقاس من النمط $s$−WH وبين بعض المفاهيم الأخرى المختلفة. نحن بينًا ان المقاس من النمط $s$−WH يكون هوبفيين ضعيف من النمط δ. إلى جانب ذلك، نحن بينًا الحالة التي تتكافئ فيها المفاهيم من النمط WH, هوبفيين ضعيف من النمط δ و من النمط $s$−WH.

## 1. Introduction

Throughout this paper, we consider all modules to be unitary left $R$-modules, where $R$ is an associative ring with an identity element. The $r_R(x)$ denotes the right annihilator of $x$ in $R$. And $D \leq^{\oplus} M$ denotes that a submodule $D$ is a direct summand of $M$. A non-zero submodule $E \leq M$ is said to be an essential in $M$, and its denoted by $E \trianglelefteq M$, if $N \cap E \neq 0$ for every non-zero submodule $N$ of $M$ [1]. A submodule $S$ of $M$ is called small ($e$-small), which

*Email: edu-math.post26@qu.edu.iq

is denoted by $S \ll M$ (resp., $S \ll_e M$), if for every submodule (essential submodule) $N$ of $M$ with the property $M = S + N$ implies $N = M$ [2]. A module $M$ is called hollow if every proper submodule is small in $M$ [3]. $Rad_e(M)$ and $\delta(M)$ are a generalized radical of a module $M$ defined as, $Rad_e(M) = \sum\{ S \in M | S \ll_e M \}$, and $\delta(M)$=$\sum\{N \leq M | N \ll_\delta M\}$. Since every $\delta$-small submodule is $e$-small, then $\delta(M) \leq Rad_g(M)$ see [2], and [4]. If every non-zero submodules of a module $M$ are essential then $M$ is said to be uniform [1]. If every submodule (direct summand) of a module $M$ are fully invariant then $M$ is said to be a duo (weak duo) module [5].

The study of homomorphism of modules has been extensively explored by various researchers see [6], [7], and [8]. In reference [9], they introduced the concept of Hopfian modules, which are defined as follows, a module $M$ is considered to be Hopfian if every surjective $R$-endomorphism of $M$ is an automorphism. Another generalized notion, known as generalized Hopfian (gH) modules, was presented by A. Gorbani and A. Haghany in reference [10]. A module $M$ is considered as gH if it has a small kernel for every surjective $R$-endomorphism.

In 1992, K. Varadarajan defined co-Hopfian modules as follows, a module $M$ is said to be co-Hopfian when every injective $R$-endomorphism of $M$ is an automorphism [11].

Furthermore, the concept of weakly Hopfian (WH) modules was introduced in reference [12], as a proper generalization of Hopfian modules. A module $M$ is said to be WH if every small surjective $R$-endomorphism of $M$ is an isomorphism.
Additionally, another generalization of Hopfian modules was introduced in reference [13], known as δ-weakly Hopfian. A module $M$ is considered δ-weakly Hopfian if every δ-small surjective endomorphism of $M$ is an isomorphism.

In Section 2, we introduced a new proper generalized for Hopfian called strongly weakly Hopfian for short $s$-WH, defined as, a module $M$ is said to be $s$-WH if, every $e$-small $R$-epimorphism $g \in End(M)$ is an automorphism. In the same section, we showed some important properties and examples of $s$-WH. In the end of Section 2, we investigate the behavior of $s$-WH modules Under the concept of localization. In Section 3, we have established significant relationships between $s$-WH modules and various other concepts. By exploring this connections, we have deepened our understanding of $s$-WH modules and their place within the broader context of module theory. Our research demonstrates that every $s$-WH module is $\delta$-weakly Hopfian. We give a cases that make the concepts of WH, δ-weakly Hopfian and $s$-WH modules are identical.

## 2. $s$-WH modules and some basic properties

**Definition 2.1.** A non-zero $R$-module $M$ is called strongly weakly Hopfian, for short $s$-WH if any $e$-small epimorphism $g \in End(M)$ is an automorphism. Moreover, a ring $R$ is called $s$-WH if, $R$ as an $R$-module is an $s$-WH module**.**

**Remarks and Examples 2.2.**
**(1)** Evidently, every Hopfian $R$-module is an $s$-WH.
**(2)** Every Noetherian module is $s$-WH.
**Proof.** It follows directly by ( [14], Lemma 4, p. 42), and (1). □
**(3)** Each of $\mathbb{Q}$ and $\mathbb{Z}$ are $s$-WH, because the only rings homomorphism of them is the identity map.

**(4)** From [11], the modules $\mathbb{Q}$ as $\mathbb{Z}$-module and $\mathbb{Q}$ as $\mathbb{Q}$-module are Hopfian, so they are $s$-WH, by (1).

**Example 2.3.** The $\mathbb{Z}$-module $\mathbb{Z}_n$ is $s$-WH.
**Proof.** Let $f \in End(\mathbb{Z}_n)$ be an $e$-small $\mathbb{Z}$-epimorphism. It follows that $f(\bar{x}) = \bar{x}a$ where $\gcd(a, n) = 1$ for all $a \in \mathbb{Z}^+$ and $a < n$. Evidently, $kerf = 0$, so $\mathbb{Z}_n$ is s-WH $\mathbb{Z}$-module.

The module which introduced in the next example is not an $s$-WH.

**Example 2.4.** Consider $M = \mathbb{Z}_{p^\infty}$ as $\mathbb{Z}$-module. $M$ is hollow. So, we have an $e$-small kernel for every surjective endomorphism of $M$. But, we can have a surjective endomorphism of $M$ which is not an automorphism, from the multiplication by $p$. i.e., $f(\bar{x}) = p(\bar{x})$, where $\bar{x} \in M$.

**Proposition 2.5.** Every $R$-module $M$ with $Rad_e(M) = 0$ is an $s$-WH.
**Proof.** Suppose that $M$ is an $R$-module with $Rad_e(M) = 0$ and $f \in End(M)$ be an $e$-small $R$-epimorphism. Therefore, $kerf \ll_e M$, and $kerf \subseteq Rad_e(M)$. So, $kerf = 0$. Thus, $f$ is an automorphism and $M$ is an $s$-WH. □

**Remarks 2.6.**
**(1)** As an application of Proposition 2.5, we have the $\mathbb{Z}$ as $\mathbb{Z}$-module is $s$-WH. In fact, $Rad_e(\mathbb{Z}) = 0$.
**(2)** In general, the converse of Proposition 2.5, is not true, such as the $\mathbb{Z}_{24}$ as $\mathbb{Z}$-module is $s$-WH. (See Examples 2.3). While $Rad_e(\mathbb{Z}_{24}) = 2\mathbb{Z}_{24}$, that means $Rad_e(\mathbb{Z}_{24}) \neq \bar{0}$.

**Proposition 2.7.** Let $M$ be a projective $R$-module and $\delta(M) = 0$. Then $M$ is an s-WH.
**Proof.** Assume that $M$ is a projective $R$-module with $\delta(M) = 0$. Let $f \in End(M)$ be an $e$-small $R$-epimorphism. Thus $kerf \ll_e M$, and hence $kerf \ll_\delta M$ see ( [2], p.1053). Therefore, $kerf \subseteq \delta(M)$. Hence, $kerf = 0$. Thus $f$ is an automorphism and $M$ is an $s$-WH. □

**Corollary 2.8.** If $R$ is a ring such that $\delta(R) = 0$. Then $R$ is an $s$-WH.
**Proof.** Since $R = \langle 1 \rangle$ is a free $R$-module, so it is projective. We have the result by Proposition 2.7. □

**Proposition 2.9.** If $M$ is an indecomposable $R$-module with $Rad(M) = 0$. Then $M$ is an $s$-WH.
**Proof.** Suppose that $M$ is an indecomposable $R$-module with $Rad(M) = 0$. Let $f \in End(M)$ be an $e$-small $R$-epimorphism. Therefore, $kerf$ is a proper $e$-small submodule of $M$ (since $M$ is indecomposable), [15], implies $kerf \ll M$ and $kerf \subseteq Rad(M)$, so $kerf = 0$. Thus, $f$ is an automorphism and $M$ is an s-WH. □

**Corollary 2.10.** If $M$ is a uniform $R$-module such that $Rad(M) = 0$. Then $M$ is an s-WH.
**Proof.** Assume that $M$ is a uniform module, thus $M$ is an indecomposable module by( [16], Examples 3.51(1)). So, Proposition 2.9 implies the result. □

**Proposition 2.11.** If $M$ is a weak duo $R$-module and $Rad(M) = 0$. Then $M$ is an s-WH.
**Proof.** Suppose that $f \in End(M)$ is an epimorphism and $kerf \ll_e M$. Try to show that $kerf \ll M$. Assume that $M = kerf + H$, for some $H \leq M$. Since $kerf \ll_e M$, then $M = N \oplus H$, for some semisimple submodule $N$ of $M$, by [2]. Therefore, $H \leq^{\oplus} M$, hence it is fully invariant, since $M$ is a weak duo. Therefore, $M = f(M) = f(kerf + H) = f(H) \subseteq M$. Thus

$f(H) = M$. Hence, $M = f(H) \subseteq H$ that implies $M = H$ and $kerf \ll M$. Thus, $kerf \subseteq Rad(M)$, it follows $kerf = 0$. Thus, $f$ is an automorphism and $M$ is an s-WH. □

**Corollary 2.12.** Let $M$ be a duo $R$-module with $Rad(M) = 0$. Then $M$ is an s-WH.
**Proof.** Since any duo $R$-module is weak duo $R$-module then, Proposition 2.11 implies the result. □

**Proposition 2.13.** A direct summand of a $s$-WH module is an $s$-WH.
**Proof.** Let $M$ be a $s$-WH module and $N \leq^{\oplus} M$. So, $M = N \oplus L$ for some $L \leq M$. Assume that $f \in End(N)$ is an $e$-small epimorphism. Consider $I_L: L \to L$ is an identity map over $L$. Thus $f \oplus I_L: M \to M$ with $f \oplus I_L(n + l) = f(n) + l$ for all $n \in N$ and $l \in L$ is a surjective, since $f \oplus I_L(M) = f \oplus I_L(N \oplus L) = f(N) \oplus I_L(L) = N \oplus L = M$. Then by [2], we have that $ker(f \oplus I_L) = kerf \oplus ker(I_L) = kerf \oplus 0 \ll_e N \oplus L = M$, $i.e.$, $f \oplus I_L$ is an $e$-small epimorphism. Since $M$ is an $s$-WH, so $f \oplus I_L$ is an automorphism of $M$. That is $ker(f \oplus I_L) = 0$ implies $kerf = 0$. Thus, $N$ is an s-WH, since $f$ is an automorphism of $N$. □

**Proposition 2.14.** Let $M = M_1 \oplus M_2$ such that $M_1$ and $M_2$ are fully invariant under every surjection of $M$. Then $M$ is an $s$-WH if and only if $M_i$ is an $s$-WH, for all $i = 1,2$.
**Proof.** ⟹) Follows directly by Proposition 2.13.
⟸) Let $f: M \to M$ be an $e$-small $R$-epimorphism, then $f|_{M_i}: M_i \to M_i$ is an $R$-epimorphism for all $i = 1,2$, and by assumption $M_1$, $M_2$ are fully invariant submodules. Since $kerf = ker\,(f|_{M_1} \oplus f|_{M_2}) = (kerf|_{M_1}) \oplus (kerf|_{M_2}) \ll_e M_1 \oplus M_2 = M$, then $ker\,f|_{M_1} \ll_e M_1$ and $kerf|_{M_2} \ll_e M_2$ by [2]. That means $f|_{M_1}$ and $f|_{M_2}$ are an $e$-small $R$-epimorphisms. By assumption $f|_{M_1}$ and $f|_{M_2}$ are automorphisms. Therefore, $kerf = ker\,(f|_{M_1} \oplus f|_{M_2}) = 0 \oplus 0 = 0$. Thus, $f$ is an automorphism. Hence, $M$ is an $s$-WH. □

**Corollary 2.15.** Let $M = \bigoplus_{i=1}^{n} M_i$ such that $M_i$ is fully invariant under every surjection of $M$ for all $i = 1,2, \ldots, n$. Then $M$ is an $s$-WH if and only if $M_i$ is a $s$-WH, for all $i = 1,2, \ldots, n$.

**Corollary 2.16.** If $M$ is a weak duo module and all its direct summands are under any surjection of $M$. Then $M$ is an $s$-WH if and only if any direct summand of $M$ is an s-WH.
**Proof.** By Proposition 2.14, since any direct summand of $M$ is fully invariant, as $M$ is a weak duo module. □

**Proposition 2.17.** Let $M = M_1 \oplus M_2$ be an $R$-module such that $r_R(M_1) \oplus r_R(M_2) = R$. Then $M$ is an $s$-WH if and only if $M_i$ is an $s$-WH, for all $i = 1,2$.
**Proof.** If part follows directly by Proposition 2.13.
The only if part. Assume that $f \in End(M)$ is a surjective and $kerf \ll_e M$. Since $r_R(M_1) \oplus r_R(M_2) = R$ and $Imf \leq M_1 \oplus M_2$, then by [17], there exists $X \leq M_1$ and $Y \leq M_2$ such that $Imf = X \oplus Y$. Thus, $f(M)=$ $f(M_1 \oplus M_2) = f(M_1) \oplus f(M_2) = X \oplus Y = Imf|_{M_1} \oplus Imf|_{M_2}$, so $Imf|_{M_1} \leq M_1$ and $Imf|_{M_2} \leq M_2$. Thus $f|_{M_i}$ is a surjective for all $i = 1,2$. And we have that $kerf$=$(kerf|_{M_1}) \oplus (kerf|_{M_2}) \ll_e M_1 \oplus M_2 = M$, therefore $ker\,f|_{M_1} \ll_e M_1$ and $kerf|_{M_2} \ll_e M_2$ by [2]. That means $f|_{M_1}$ and $f|_{M_2}$ are an $e$-small $R$-epimorphisms. By assumption $f|_{M_1}$ and $f|_{M_2}$ are automorphisms that implies $kerf|_{M_1} = kerf|_{M_2} = 0$. If $f(m_1 + m_2) = 0$, then $f(m_1) + f(m_2) = 0$, so $m_1 = m_2 = 0$, i.e., $kerf = 0$. Thus, $M$ is an $s$-WH. □

**Proposition 2.18.** The following are equivalent for $M$ as $R$-module.
**(1)** $M$ is $s$-WH;
**(2)** For all $e$-small submodule $N$ of $M$, $M/N \cong M$ if and only if $N = 0$.
**Proof.** $(1) \Longrightarrow)(2)$ Assume that $N = 0$. Then trivially $M/N \cong M$. Suppose that $M/N \cong M$ with $N \ll_e M$. Let $\psi: M/N \to M$ be an $R$-isomorphism. Consider a canonical $R$-epimorphism $\pi: M \to M/N$. Then $\psi\pi$ is a surjective endomorphism of $M$ with $ker(\psi\pi) = \pi^{-1}(ker\psi) = \pi^{-1}(N) = N$, that is $\psi\pi$ is an $e$-small $R$-epimorphism. Therefore, $\psi\pi$ is an automorphism. i.e., $ker(\psi\pi) = 0$ by (1). Then $N = 0$.
$(2) \Longrightarrow)(1)$ Let $f \in End(M)$ be an $e$-small $R$-epimorphism. Thus $kerf \ll_e M$. We have that $M/kerf \cong M$ by the First Isomorphism Theorem and by (2), $kerf = 0$, so $f$ is an $R$-automorphism. Hence, $M$ is an s-WH. □

**Proposition 2.19.** Let $M$ be a module, consider the following:
**(1)** $M$ is $s$-WH.
**(2)** If $M \cong M \oplus N$, then $N = 0$, for some semisimple module $N$.
Then $(1) \Longrightarrow)(2)$. And if $M$ is projective, we have $(2) \Longrightarrow)(1)$.

**Proof.** $(1) \Longrightarrow)(2)$ Assume that $M$ is an $s$-WH module, where $M \cong M \oplus N$ for some semisimple module $N$. It follows that $M = K \oplus L$ where $K \cong M$ and $L \cong N$. From [2], we deduce that $L$ is an $e$-small submodule of $M$. We have $M/L \cong K \cong M$. Thus, by Proposition 2.18, $L = 0$, hence $N = 0$.
$(2) \Longrightarrow)(1)$ Conversely, suppose that $M$ is projective and $f \in End(M)$ is an epimorphism, where $kerf \ll_e M$. Thus $f$ split, that is $M = T \oplus kerf$, for some $T \leq M$. By the First Isomorphism Theorem, we have that $T \cong M/kerf \cong M$. By [2], $M = S \oplus T$ where $S$ is a semisimple submodule of $kerf$. By the modular law, $kerf = kerf \cap M = kerf \cap (S \oplus T) = S \oplus (kerf \cap T) = S$. It follows that $M \cong M \oplus kerf$ and $kerf$ is semisimple. By (2), $kerf = 0$ and $M$ will be an $s$-WH module. □

We will offer the following condition $(E^*)$ for any $R$-module $M$:
$(E^*)$ If $f: M \to M'$ and $g: M' \to M''$ are any two $R$-endomorphisms, then $f$ and $g$ are $e$-small if and only if $gf$ is $e$-small.

**Proposition 2.20.** If $M$ is a module with the property that for any $g \in End(M)$, there exists an $n \in \mathbb{Z}^+$ such that $kerg^n \cap Img^n = 0$, then $M$ is an $s$-WH.
**Proof.** Let $g \in End(M)$ be an $e$-small epimorphism. By assumption, there is an integer $n \geq 1$ such that $kerg^n \cap Img^n = 0$. It follows that $g^n \in End(M)$ is an epimorphism, i.e., $Img^n = M$. Thus, $kerg^n \cap Img^n = kerg^n \cap M = kerg^n = 0$. But we know that $kerg \leq kerg^n$, which implies $kerg = 0$. Therefore, $g$ is an automorphism. Hence, $M$ is an $s$-WH. □

**Corollary 2.21.** Let $M$ be an $R$-module satisfies $(E^*)$ property. If $M$ has ACC on $e$-small submodules, then $M$ is an $s$-WH.

**Proposition 2.22.** Let $M$ be an $R$-module. If for any $R$-epimorphism $\varphi: M \to M$, there exist $n \geq 1$ such that $ker\varphi^n = ker\varphi^{n+i}$ for all $i \in \mathbb{Z}^+$, then $M$ is an $s$-WH.
**Proof.** Let $\varphi \in End(M)$ be any surjective. We claim that $ker\,\varphi^n \cap Im\varphi^n = 0$. Let $y \in ker\,\varphi^n \cap Im\varphi^n$. Thus $\varphi^n(y) = 0$ and $y = \varphi^n(x)$ for some $x \in M$. Hence, $\varphi^{2n}(x) = \varphi^n(y) = 0$. Hence, $x \in ker\varphi^{2n}$. But from our assumption we have that $ker\varphi^n = ker\varphi^{n+n} = ker\varphi^{2n}$. So, $x \in ker\varphi^n$. Therefore, $0 = \varphi^n(x) = y$. Hence, $ker\,\varphi^n \cap Im\varphi^n =$

0. Since $\varphi$ is a surjective, so $Im\varphi^n = M$, thus $ker\varphi^n = 0$. But $ker\varphi \subseteq ker\varphi^n$. So, $ker\varphi \ll_e M$. Therefore, $M$ is an $s$-WH. □

**Proposition 2.23.** Let $M$ be an $R$-module has $(E^*)$ property and $N$ be any non-zero $e$-small submodule of $M$, if $M/N$ is $s$-WH, then $M$ is an $s$-WH.

**Proof.** Assume $M$ is not $s$-WH, then there is an $e$-small epimorphism $f \in End(M)$ that it is not automorphism, $(kerf \neq 0)$. From 1st isomorphism theorem there is an $R$-isomorphism $\varphi: M/kerf \to M$. Consider $\pi: M \to M/kerf$ the canonical map, then $ker\pi = kerf \ll_e M$. Therefore, $\pi$ is an $e$-small epimorphism. It follows that $\pi\varphi: M/kerf \to M/kerf$ is an $e$-small epimorphism, by hypothesis, which is not isomorphism (since $ker(\pi\varphi) = \varphi^{-1}(ker\pi) = \varphi^{-1}(kerf) \neq 0_{M/kerf}$), but $M/N$ is $s$-WH, which is a contradiction. Hence, $kerf = 0$ and $M$ is an $s$-WH. □

**Theorem 2.24.** Let $M$ be a uniform quasi-projective $R$-module has $(E^*)$ property. Then $M$ is an $s$-WH if and only if $M/N$ is $s$-WH, with $N$ is an $e$-small fully invariant submodule of $M$.
**Proof.** Assume that $M$ is $s$-WH and $N$ is an $e$-small fully invariant submodule of $M$. If $f: M/N \to M/N$ is an $e$-small epimorphism. Consider the $e$-small canonical epimorphism $\pi: M \to M/N$ (as $ker\pi = N \ll_e M$), so $f\pi: M \to M/N$ is an $e$-small epimorphism, as $M$ has $(E^*)$ property. Since $M$ is quasi projective, then there exists an endomorphism $g$ of $M$ such that $\pi g = f\pi$. This equality implies that $g$ is an epimorphism, as $M$ is uniform. Since, $f\pi$ is $e$-small. Then is $\pi g$ is $e$-small that implies $g$ is $e$-small, since $M$ has $(E^*)$ property. Since $M$ is $s$-WH, then $g$ is an automorphism. For all $m \in M$, we deduce that $f(m + N) = f\pi(m) = \pi g(m) = g(m) + N$, and then $kerf = \{m + N \in M/N |\, f(m + N) = N\} = \{m + N \in M/N |\, g(m) + N = N\} = \{m + N \in M/N | g(M) \ \ \in N\} = K/N$, where $K = \{m \in M |\, g(m) \in N\}$ and $N \subseteq K\ = g^{-1}(N)$. Since $N$ is fully invariant in $M$ and $g^{-1} \in End(M)$, then $g^{-1}(N) \subseteq N$, thus $K = g^{-1}(N) = N$. Hence, $kerf = K/N = 0_{M/N}$ and $M/N$, is an $s$-WH. The converse is clear when $N = 0$. □

**Definition 2.25.** [18] Let $R$ be a ring with identity 1 a subset $S$ of a ring $R$ is called multiplicatively closed set if the following two conditions hold:
**(1)** $1 \in S$.
**(2)** For all $u$ and $v$ in $S$, the product $uv \in S$.

**Definition 2.26.** [18] Let $M$ be an $R$-module. Let $S$ be a multiplicatively closed set in $R$. Let $T$ be the set of all ordered pairs $(x, s)$ where $x \in R$ and $s \in S$. Define a relation on $T$ by $(x, s)\sim(\acute{x}, \acute{s})$ if there exists $t \in S$ such that $t(s\acute{x} - \acute{s}x) = 0$. This is an equivalence relation on $T$, and we denote the equivalence class of $(x, s)$ by $x/s$ . Let $S^{-1}M$ denote the set of equivalence classes of $T$ with respect to this relation. We can make $S^{-1}M$ into an $R$-module by setting $x/s + y/t = (tx + sy)/st$, $a(x/s) = a\,x/s$, $a \in R$. W The $R$-module $S^{-1}M$ is called a quotient module (localization of module), or a module of quotient. Note that if $0 \in S$, then $S^{-1}M = 0$.

**Definition 2.27.** [19] Let $M$ be an $R$-module and $R$ is a commutative ring. An element $r \in R$ is called prime to $L$, where $L \leq M$, if $rm \in L$ $(m \in M)$ implies that $m \in L$.
The set of all elements of $R$ that are not prime to $L$, denote by $\mathcal{L}(L)$, i.e., $\mathcal{L}(L) = \{r \in R |\, rm \in L \text{ for some } m \in M \setminus L\}$.

In the next results, we examine the behavior of the s-WH under the concept of localization.

**Proposition 2.28.** Suppose that $M$ be an $R$-module and $S$ a multiplicative closed subset of $R$ such that $\mathcal{L}(L) \cap S = \emptyset$ for any $L \leq M$. If $S^{-1}M$ is a $s$-WH as $S^{-1}R$-module, then $M$ is an $s$-WH as $R$-module.
**Proof.** Let $f: M \to M$ be an $e$-small $R$-epimorphism. Define $S^{-1}R$-endomorphism $S^{-1}f: S^{-1}M \to S^{-1}M$ by $S^{-1}f\left(\frac{m}{s}\right) = \frac{f(m)}{s}$ for all $m \in M$, $s \in S$. Then we have $Im(S^{-1}f) = S^{-1}(Imf) = S^{-1}M$, then $S^{-1}f$ is an $S^{-1}R$-epimorphism. Since $kerf \ll_e M$, thus $ker(S^{-1}f) = S^{-1}(kerf) \ll_e S^{-1}M$ from ( [20], Lemma 2.3.3). As $S^{-1}M$ is $s$-WH. Therefore, $ker(S^{-1}f) = S^{-1}(kerf) = S^{-1}(0)$, then $kerf = 0$ by ( [20], Lemma 2.3.1.) Hence, $M$ is $s$-WH. □

## 3. $s$-WH modules and related concepts

Many relations between $s$-WH modules and other types of modules are introduced in this section, such as generalized hollow, semisimple and nonsingular uniform modules. We give a case that make the concepts WH, δ-weakly Hopfian and $s$-WH modules are identical, we give two cases that make the concepts Hopfian and $s$-WH are equivalent. Also, we put a condition on co-Hopfian ring to become an $s$-WH ring.

Recall that a module $M$ is called generalized hollow if any proper submodule of $M$ is an $e$-small [21].

**Proposition 3.1.** Let $M$ be a non-zero $R$-module, if $M$ is a generalized Hollow module. Then $M$ is an $s$-WH if and only if $M$ is a Hopfian.
**Proof.** Let $M$ be an $s$-WH $R$-module. Let $f \in End(M)$ be an $R$-epimorphism, so $kerf \subset M$ (since, if $kerf = M$ then $f = 0$, a contradiction), Then $kerf \ll_e M$, as $M$ is generalized Hollow. Since $M$ is $s$-WH, so $kerf = 0$. Hence, $M$ is a Hopfian $R$-module, since $f$ is an automorphism. Conversely, follows by Remarks and Examples 2.2(1). □

**Proposition 3.2.** Every $s$-WH module is a $\delta$-weakly Hopfian.
**Proof.** Let $M$ be a $s$-WH $R$-module. If $f \in End(M)$ is a $\delta$-small $R$-epimorphism, then $kerf \ll_\delta M$, and then $kerf \ll_e M$, by ( [2], p.1052). Since $M$ is $s$-WH, then $kerf = 0$. Therefore, $M$ is a δ-weakly Hopfian $R$-module, since $f$ is an isomorphism. □

**Corollary 3.3.** Every $s$-WH module is WH.
**Proof.** Since every $\delta$-weakly Hopfian is WH from [13]. Then the result is followed by Proposition 3.2. □

Now, we will give the case that makes the concepts WH, $\delta$-weakly Hopfian and s-WH modules identical.

**Proposition 3.4.** If $M$ is a non-zero indecomposable $R$-module. Then the following are equivalent.
**(1)** $M$ is $s$-WH;
**(2)** $M$ is $\delta$-weakly Hopfian;
**(3)** $M$ is WH.
**Proof.** (1) $\Longrightarrow$)(2) By Proposition 3.2.
(2) $\Longrightarrow$)(3) By [13].
(3) $\Longrightarrow$)(1) Assume that $M$ is a WH $R$-module, let $f \in End(M)$ is an $e$-small epimorphism. If $kerf = M$, then $f = 0$, which it is a contradiction. Thus, $kerf$ is a proper $e$-small submodule of $M$, and since $M$ is indecomposable, [15], implies $kerf \ll M$, that means $f \in$

$End(M)$ is a small $R$-epimorphism. Since $M$ is a WH $R$-module, then $f$ is an automorphism. Hence, $M$ is an $s$-WH. □

**Corollary 3.5.** The following are equivalent for a non-zero uniform $R$-module $M$.
**(1)** $M$ is $s$-WH;
**(2)** $M$ is $\delta$-weakly Hopfian;
**(3)** $M$ is WH.
**Proof.** Assume that $M$ is a uniform module, thus $M$ is an indecomposable module by ( [16] ,Examples 3.51(1)). Thus, Proposition 3.4 implying the result. □

**Proposition 3.6.** Let $M$ be a uniform and torsion-free module. Then $M$ is an $s$-WH.
**Proof.** Let $f: M \to M$ be an $e$-small $R$-epimorphism. Let $0 \neq x \in M\backslash kerf$, $f(x) \neq 0$, so $-x \in M$ and $f(-x) = f(x).-1 \neq 0$, i.e., $-x \in M\backslash kerf$. For any $r \in R$, $f(xr) = f(x)r$. Since $M$ is an torsion-free $R$-module, it follows that $f(x)r \neq 0$ and then $xr \in M\backslash kerf$. Thus, $(M\backslash kerf) \cup \{0\}$ is a submodule of $M$ and so $(M\backslash kerf) \cup \{0\} \trianglelefteq M$, as $M$ is uniform. As $(M\backslash kerf) \cup \{0\} + kerf = M$ and $f$ an $e$-small $R$-epimorphism, i.e., $kerf \ll_e M$, thus $(M\backslash kerf) \cup \{0\} = M$, so $kerf = 0$. Hence, $M$ is an $s$-WH. □

**Example 3.7.** The reverse of Proposition 3.6, is not true generally. Consider the $\mathbb{Z}$-module $\mathbb{Z}_{pq}$ where $p$,$q$ are prime numbers. By Examples 2.3, $\mathbb{Z}_{pq}$ is an $s$-WH, but nor uniform neither torsion-free $\mathbb{Z}$-module.

**Theorem 3.8.** For a projective $R$-module $M$, the following are equivalent.
**(1)** $M$ is $s$-WH;
**(2)** if $f \in End(M)$ has a right inverse in $End(M)$ and $kerf$ is a semisimple, then $f$ has a left inverse in $End(M)$;
**(3)** if $f \in End(M)$ has a right inverse in $End(M)$ and $kerf \ll_e M$, then $f$ has a left inverse in $End(M)$;
**(4)** if $f \in End(M)$ has a right inverse in $End(M)$ and $(1 - gf)M \ll_e M$, then $f$ has a left inverse in $End(M)$;
**(5)** if $f \in End(M)$ is a surjective and $kerf$ is semisimple, then $f$ has a left inverse in $End(M)$.
**Proof.** It is clear that $f \in End(M)$ is a surjective if and only if $fg = 1$ for some $g \in End(M)$. Thus, $kerf = (1 - gf)M$, to see this: let $x \in kerf \implies f(x) = 0 \implies (1 - gf)(x) = x - gf(x) = x - g(0) = x \implies x \in (1 - gf)M$. Now, assume that $y \in (1 - gf)M \implies y = (1 - gf)(x)$ for some $x \in M \implies y = x - gf(x) \implies f(y) = f(x) - fgf(x) = f(x) - 1\big(f(x)\big) = f(x) - f(x) = 0 \implies y \in kerf$. So $M = kerf \oplus (gf)M = kerf \oplus Img$, since $kerf + (gf)M = (1 - gf)M + (gf)M = M$, also if $m \in kerf \cap Img \implies f(m) = 0$ and $m = g(a)$, for some $a \in M \implies 0 = f(m) = f\big(g(a)\big) = fg(a) = 1(a) = a \implies m = g(a) = g(0) = 0$.
(1) ⟹)(2) Assume that $f \in End(M)$ contain a right inverse with $kerf$ is semisimple. Thus $fg = 1$ for some $g \in End(M)$. Then $g$ is an injective, i.e., $kerg = 0$. From 1st isomorphism theorem, $M \cong M/0 = M/kerg \cong Img$. By above argument, we have $M = Img \oplus kerf \cong M \oplus kerf$, i.e., $M \cong M \oplus kerf$ and $kerf$ is semisimple, thus $kerf = 0$, by Proposition 2.19, that is $f$ is an automorphism. As $fg = 1$, then $g = f^{-1}$. Hence $gf = f^{-1}f = 1$, that mean $g$ is a left inverse of $f$ in $End(M)$.
(2) ⟹)(3) Assume that $f \in End(M)$ contain a right inverse in $End(M)$ and $kerf \ll_e M$. Since $M = kerf \oplus Img$, [2], implies $kerf$ is semisimple. From (2), $f$ has a left inverse in $End(M)$.
(3) ⟹)(4) Since $kerf = (1 - gf)M$, (3) implies (4).

$(4) \Longrightarrow)(5)$ Let $f \in End(M)$ be a surjective and $kerf$ is semisimple, then $f$ has a right inverse in $End(M)$. By above argument, we have $kerf = (1 - gf)M$ and $M = kerf \oplus Img$. By [2], $kerf = (1 - gf)M \ll_e M$, then $f$ has a left inverse in $End(M)$, by (4).
$(5) \Longrightarrow)(1)$ Assume that if $f \in End(M)$ is a surjective and $kerf \ll_e M$. Hence, $f$ has a right inverse in $End(M)$. By above argument, $M = kerf \oplus Img$. $kerf$ is semisimple from [2], so $f$ contain a left inverse in $End(M)$ by (5). That is $hf = 1$ for some $h \in End(M)$. Thus $f \in End(M)$ is an injective. Hence, it is an automorphism. Therefore, (1) holds. □

**Proposition 3.9.** Let $M$ be a semisimple module. Then $M$ is $s$-WH if and only if it is Hopfian.
**Proof.** Suppose that $M$ is an $s$-WH module. Let $f: M \to M$ be an $R$-epimorphism. As $M$ is a semisimple module, then by [22], we get $kerf \ll_e M$, i.e., $f$ is an $e$-small $R$-epimorphism and so $f$ is an automorphism. Hence, $M$ is Hopfian. Conversely, follows by Remarks 2.2(1). □

**Proposition 3.10.** Every co-Hopfian quasi-projective module is an $s$-WH.
**Proof.** Suppose that $M$ is a co-Hopfian quasi-projective module and let $\varphi: M \to M$ be an $e$-small epimorphism. Since $M$ is quasi-projective, so there is an $f \in End(M)$ such that $\varphi f = I_M$. As $I_M$ is a monomorphism, then so is $f$. As $M$ is a co-Hopfian module, thus $f$ is an epimorphism. Since $0 = kerI_M = ker(\varphi f) = f^{-1}(ker\varphi)$, then $0 = f(0) = f(f^{-1}(ker\varphi)) = ker\varphi$, that means $\varphi$ is an automorphism. Hence, $M$ is an $s$-WH. □

**Example 3.11.** The reverse of Proposition 3.10, need not be true in general. Examples 2.6(1) shows that the $\mathbb{Z}$-module $\mathbb{Z}$ is $s$-WH. But we know that $\mathbb{Z}$-module $\mathbb{Z}$ is quasi-projective not co-Hopfian see [11].

**Corollary 3.12.** Every projective co-Hopfian module is an $s$-WH.
**Proof.** Clear by Proposition 3.10. □

**Corollary 3.13.** Every co-Hopfian ring is a $s$-WH ring.
**Proof.** Suppose that $R$ is a co-Hopfian ring. As $R = \langle 1 \rangle$ is a free $R$-module, so it is projective. Then the result is followed by Corollary 3.12. □

**Proposition 3.14.** If $M$ is a nonsingular uniform module, then $M$ is an $s$-WH.
**Proof.** Let $M$ be a nonsingular uniform module. Suppose that $\varphi \in End(M)$ is an $e$-small epimorphism, i.e., $ker\varphi \ll_e M$. Assume $ker\varphi \neq 0$. We have $Ker\varphi \trianglelefteq M$ because $M$ is uniform. Thus $M/ker\varphi$ is singular by ( [1], Proposition 1.21). From the First Isomorphism Theorem $M/ker\varphi \cong M$. This is a contradiction because $M/ker\varphi$ is singular and nonsingular. Hence, $ker(\varphi) = 0$, so $\varphi$ is an automorphism. Therefore, $M$ is an $s$-WH. □

**Remarks 3.15.**
**(1)** We note that Proposition 3.14, is another proof for Example 2.6(1), of $\mathbb{Z}$-module $\mathbb{Z}$ being $s$-WH, in fact $\mathbb{Z}$ as $\mathbb{Z}$-module is nonsingular and uniform.
**(2)** The reverse of Proposition 3.14, need not be true in general. Examples 2.3 shows that the $\mathbb{Z}$-module $\mathbb{Z}_6$ is $s$-WH. But we know that $\mathbb{Z}_6$ nor nonsingular neither uniform as $\mathbb{Z}$-module.

## 4. Conclusions

We defined a new concept of modules called $s$-WH which is a proper generalized of Hopfian. It is shown and investigate some different properties and examples of this class.

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