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s-WH Modules

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Abstract

This article introduces the concept of strongly-WH module which is a proper generalized of Hopfian modules. A module M is called strongly-WH, briefly s-WH if, any e-small surjective R-endomorphism of M is an automorphism. We specify and provide some properties of this concept. Furthermore, we have established connections between strongly-WH modules and various other concepts. We demonstrate that every strongly-WH module is δ-weakly Hopfian. As well as, we provide cases in which the concepts of WH, δ-weakly Hopfian, and strongly-WH modules are equivalent.

Keywords: s-WH modules; Hopfian modules; weakly Hopfian modules; e-small submodules.

المقاسات من النمط s–WH

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الخلاصة

تقدم هذه المقالة مفهوم المقاس القوي من النمط WH وهو تعميم فعلي للمقاسات الهوبفيينة. يقال للمقاس M على الحلقة R انهُ قوي من النمط WH, باختصار s–WH، اذا كان كل تشاكل ذاتي شامل صغير من النمط e على M يكون تشاكل متقابل. نحن حددنا وأثبتنا بعض الخواص لهذه المقاسات. بالإضافة إلى ذلك، نحن أثبتنا العلاقة بين المقاس من النمط s–WH وبين بعض المفاهيم الأخرى المختلفة. نحن بينًا ان المقاس من النمط s–WH يكون هوبفيين ضعيف من النمط δ. إلى جانب ذلك، نحن بينًا الحالة التي تتكافئ فيها المفاهيم من النمط WH, هوبفيين ضعيف من النمط δ و من النمط s–WH.

1. Introduction

Throughout this paper, we consider all modules to be unitary left R-modules, where R is an associative ring with an identity element. The $r_R(x)$ denotes the right annihilator of x in R. And $D \leq^{\oplus} M$ denotes that a submodule D is a direct summand of M. A non-zero submodule $E \leq M$ is said to be an essential in M, and its denoted by $E \trianglelefteq M$, if $N \cap E \neq 0$ for every non-

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zero submodule N of M [1]. A submodule S of M is called small (e-small), which is denoted by $S \ll M$ (resp., $S \ll_e M$), if for every submodule (essential submodule) N of M with the property $M = S + N$ implies $N = M$ [2]. A module M is called hollow if every proper submodule is small in M [3]. $Rad_e(M)$ and $\delta(M)$ are a generalized radical of a module M defined as, $Rad_e(M) = \sum\{ S \in M | S \ll_e M \}$, and $\delta(M)$=$\sum\{N \leq M | N \ll_\delta M\}$. Since every δ-small submodule is e-small, then $\delta(M) \leq Rad_g(M)$ see [2], and [4]. If every non-zero submodules of a module M are essential then M is said to be uniform [1]. If every submodule (direct summand) of a module M are fully invariant then M is said to be a duo (weak duo) module [5].

The study of homomorphism of modules has been extensively explored by various researchers see [6], [7], and [8]. In reference [9], they introduced the concept of Hopfian modules, which are defined as follows, a module M is considered to be Hopfian if every surjective R-endomorphism of M is an automorphism. Another generalized notion, known as generalized Hopfian (gH) modules, was presented by A. Gorbani and A. Haghany in reference [10]. A module M is considered as gH if it has a small kernel for every surjective R-endomorphism.

In 1992, K. Varadarajan defined co-Hopfian modules as follows, a module M is said to be co-Hopfian when every injective R-endomorphism of M is an automorphism [11].

Furthermore, the concept of weakly Hopfian (WH) modules was introduced in reference [12], as a proper generalization of Hopfian modules. A module M is said to be WH if every small surjective R-endomorphism of M is an isomorphism.
Additionally, another generalization of Hopfian modules was introduced in reference [13], known as δ-weakly Hopfian. A module M is considered δ-weakly Hopfian if every δ-small surjective endomorphism of M is an isomorphism.

In Section 2, we introduced a new proper generalized for Hopfian called strongly weakly Hopfian for short s-WH, defined as, a module M is said to be s-WH if, every e-small R-epimorphism $g \in End(M)$ is an automorphism. In the same section, we showed some important properties and examples of s-WH. In the end of Section 2, we investigate the behavior of s-WH modules Under the concept of localization. In Section 3, we have established significant relationships between s-WH modules and various other concepts. By exploring this connections, we have deepened our understanding of s-WH modules and their place within the broader context of module theory. Our research demonstrates that every s-WH module is δ-weakly Hopfian. We give a cases that make the concepts of WH, δ-weakly Hopfian and s-WH modules are identical.

2. s-WH modules and some basic properties

Definition 2.1. A non-zero R-module M is called strongly weakly Hopfian, for short s-WH if any e-small epimorphism $g \in End(M)$ is an automorphism. Moreover, a ring R is called s-WH if, R as an R-module is an s-WH module**.**

Remarks and Examples 2.2.
(1) Evidently, every Hopfian R-module is an s-WH.
(2) Every Noetherian module is s-WH.
Proof. It follows directly by ([14], Lemma 4, p. 42), and (1). □
(3) Each of $\mathbb{Q}$ and $\mathbb{Z}$ are s-WH, because the only rings homomorphism of them is the identity map.

(4) From [11], the modules $\mathbb{Q}$ as $\mathbb{Z}$-module and $\mathbb{Q}$ as $\mathbb{Q}$-module are Hopfian, so they are s-WH, by (1).

Example 2.3. The $\mathbb{Z}$-module $\mathbb{Z}_n$ is s-WH.
Proof. Let $f \in End(\mathbb{Z}_n)$ be an e-small $\mathbb{Z}$-epimorphism. It follows that $f(\bar{x}) = \bar{x}a$ where $\gcd(a, n) = 1$ for all $a \in \mathbb{Z}^+$ and $a < n$. Evidently, $kerf = 0$, so $\mathbb{Z}_n$ is s-WH $\mathbb{Z}$-module.

The module which introduced in the next example is not an s-WH.

Example 2.4. Consider $M = \mathbb{Z}_{p^\infty}$ as $\mathbb{Z}$-module. M is hollow. So, we have an e-small kernel for every surjective endomorphism of M. But, we can have a surjective endomorphism of M which is not an automorphism, from the multiplication by p. i.e., $f(\bar{x}) = p(\bar{x})$, where $\bar{x} \in M$.

Proposition 2.5. Every R-module M with $Rad_e(M) = 0$ is an s-WH.
Proof. Suppose that M is an R-module with $Rad_e(M) = 0$ and $f \in End(M)$ be an e-small R-epimorphism. Therefore, $kerf \ll_e M$, and $kerf \subseteq Rad_e(M)$. So, $kerf = 0$. Thus, f is an automorphism and M is an s-WH. □

Remarks 2.6.
(1) As an application of Proposition 2.5, we have the $\mathbb{Z}$ as $\mathbb{Z}$-module is s-WH. In fact, $Rad_e(\mathbb{Z}) = 0$.
(2) In general, the converse of Proposition 2.5, is not true, such as the $\mathbb{Z}_{24}$ as $\mathbb{Z}$-module is s-WH. (See Examples 2.3). While $Rad_e(\mathbb{Z}_{24}) = 2\mathbb{Z}_{24}$, that means $Rad_e(\mathbb{Z}_{24}) \neq \bar{0}$.

Proposition 2.7. Let M be a projective R-module and $\delta(M) = 0$. Then M is an s-WH.
Proof. Assume that M is a projective R-module with $\delta(M) = 0$. Let $f \in End(M)$ be an e-small R-epimorphism. Thus $kerf \ll_e M$, and hence $kerf \ll_\delta M$ see ([2], p.1053). Therefore, $kerf \subseteq \delta(M)$. Hence, $kerf = 0$. Thus f is an automorphism and M is an s-WH. □

Corollary 2.8. If R is a ring such that $\delta(R) = 0$. Then R is an s-WH.
Proof. Since $R = \langle 1 \rangle$ is a free R-module, so it is projective. We have the result by Proposition 2.7. □

Proposition 2.9. If M is an indecomposable R-module with $Rad(M) = 0$. Then M is an s-WH.
Proof. Suppose that M is an indecomposable R-module with $Rad(M) = 0$. Let $f \in End(M)$ be an e-small R-epimorphism. Therefore, $kerf$ is a proper e-small submodule of M (since M is indecomposable), [15], implies $kerf \ll M$ and $kerf \subseteq Rad(M)$, so $kerf = 0$. Thus, f is an automorphism and M is an s-WH. □

Corollary 2.10. If M is a uniform R-module such that $Rad(M) = 0$. Then M is an s-WH.
Proof. Assume that M is a uniform module, thus M is an indecomposable module by([16], Examples 3.51(1)). So, Proposition 2.9 implies the result. □

Proposition 2.11. If M is a weak duo R-module and $Rad(M) = 0$. Then M is an s-WH.
Proof. Suppose that $f \in End(M)$ is an epimorphism and $kerf \ll_e M$. Try to show that $kerf \ll M$. Assume that $M = kerf + H$, for some $H \leq M$. Since $kerf \ll_e M$, then $M = N \oplus H$, for some semisimple submodule N of M, by [2]. Therefore, $H \leq^{\oplus} M$, hence it is fully invariant, since M is a weak duo. Therefore, $M = f(M) = f(kerf + H) = f(H) \subseteq M$. Thus $f(H) = M$. Hence, $M = f(H) \subseteq H$ that implies $M = H$ and $kerf \ll M$. Thus, $kerf \subseteq Rad(M)$, it follows $kerf = 0$. Thus, f is an automorphism and M is an s-WH. □

Corollary 2.12. Let M be a duo R-module with $Rad(M) = 0$. Then M is an s-WH.
Proof. Since any duo R-module is weak duo R-module then, Proposition 2.11 implies the result. □

Proposition 2.13. A direct summand of a s-WH module is an s-WH.
Proof. Let M be a s-WH module and $N \leq^{\oplus} M$. So, $M = N \oplus L$ for some $L \leq M$. Assume that $f \in End(N)$ is an e-small epimorphism. Consider $I_L: L \to L$ is an identity map over L. Thus $f \oplus I_L: M \to M$ with $f \oplus I_L(n + l) = f(n) + l$ for all $n \in N$ and $l \in L$ is a surjective, since $f \oplus I_L(M) = f \oplus I_L(N \oplus L) = f(N) \oplus I_L(L) = N \oplus L = M$. Then by [2], we have that $ker(f \oplus I_L) = kerf \oplus ker(I_L) = kerf \oplus 0 \ll_e N \oplus L = M$, $i.e.$, $f \oplus I_L$ is an e-small epimorphism. Since M is an s-WH, so $f \oplus I_L$ is an automorphism of M. That is $ker(f \oplus I_L) = 0$ implies $kerf = 0$. Thus, N is an s-WH, since f is an automorphism of N. □

Proposition 2.14. Let $M = M_1 \oplus M_2$ such that M_1 and M_2 are fully invariant under every surjection of M. Then M is an s-WH if and only if M_i is an s-WH, for all $i = 1,2$.
Proof. ⟹) Follows directly by Proposition 2.13.
⟸) Let $f: M \to M$ be an e-small R-epimorphism, then $f|_{M_i}: M_i \to M_i$ is an R-epimorphism for all $i = 1,2$, and by assumption M_1, M_2 are fully invariant submodules. Since $kerf = ker\,(f|_{M_1} \oplus f|_{M_2}) = (kerf|_{M_1}) \oplus (kerf|_{M_2}) \ll_e M_1 \oplus M_2 = M$, then $ker\,f|_{M_1} \ll_e M_1$ and $kerf|_{M_2} \ll_e M_2$ by [2]. That means $f|_{M_1}$ and $f|_{M_2}$ are an e-small R-epimorphisms. By assumption $f|_{M_1}$ and $f|_{M_2}$ are automorphisms. Therefore, $kerf = ker\,(f|_{M_1} \oplus f|_{M_2}) = 0 \oplus 0 = 0$. Thus, f is an automorphism. Hence, M is an s-WH. □

Corollary 2.15. Let $M = \oplus_{i=1}^{n} M_i$ such that M_i is fully invariant under every surjection of M for all $i = 1,2,\dots,n$. Then M is an s-WH if and only if M_i is a s-WH, for all $i = 1,2,\dots,n$.

Corollary 2.16. If M is a weak duo module and all its direct summands are under any surjection of M. Then M is an s-WH if and only if any direct summand of M is an s-WH.
Proof. By Proposition 2.14, since any direct summand of M is fully invariant, as M is a weak duo module. □

Proposition 2.17. Let $M = M_1 \oplus M_2$ be an R-module such that $r_R(M_1) \oplus r_R(M_2) = R$. Then M is an s-WH if and only if M_i is an s-WH, for all $i = 1,2$.
Proof. If part follows directly by Proposition 2.13.
The only if part. Assume that $f \in End(M)$ is a surjective and $kerf \ll_e M$. Since $r_R(M_1) \oplus r_R(M_2) = R$ and $Imf \leq M_1 \oplus M_2$, then by [17], there exists $X \leq M_1$ and $Y \leq M_2$ such that $Imf = X \oplus Y$. Thus, $f(M)$= $f(M_1 \oplus M_2) = f(M_1) \oplus f(M_2) = X \oplus Y = Imf|_{M_1} \oplus Imf|_{M_2}$, so $Imf|_{M_1} \leq M_1$ and $Imf|_{M_2} \leq M_2$. Thus $f|_{M_i}$ is a surjective for all $i = 1,2$. And we have that $kerf$=$(kerf|_{M_1}) \oplus (kerf|_{M_2}) \ll_e M_1 \oplus M_2 = M$, therefore $ker\,f|_{M_1} \ll_e M_1$ and $kerf|_{M_2} \ll_e M_2$ by [2]. That means $f|_{M_1}$ and $f|_{M_2}$ are an e-small R-epimorphisms. By assumption $f|_{M_1}$ and $f|_{M_2}$ are automorphisms that implies $kerf|_{M_1} = kerf|_{M_2} = 0$. If $f(m_1 + m_2) = 0$, then $f(m_1) + f(m_2) = 0$, so $m_1 = m_2 = 0$, i.e., $kerf = 0$. Thus, M is an s-WH. □

Proposition 2.18. The following are equivalent for M as R-module.
(1) M is s-WH;
(2) For all e-small submodule N of M, $M/N \cong M$ if and only if $N = 0$.

Proof. $(1) \Longrightarrow)(2)$ Assume that $N = 0$. Then trivially $M/N \cong M$. Suppose that $M/N \cong M$ with $N \ll_e M$. Let $\psi: M/N \to M$ be an R-isomorphism. Consider a canonical R-epimorphism $\pi: M \to M/N$. Then $\psi\pi$ is a surjective endomorphism of M with $ker(\psi\pi) = \pi^{-1}(ker\psi) = \pi^{-1}(N) = N$, that is $\psi\pi$ is an e-small R-epimorphism. Therefore, $\psi\pi$ is an automorphism. i.e., $ker(\psi\pi) = 0$ by (1). Then $N = 0$.
$(2) \Longrightarrow)(1)$ Let $f \in End(M)$ be an e-small R-epimorphism. Thus $kerf \ll_e M$. We have that $M/kerf \cong M$ by the First Isomorphism Theorem and by (2), $kerf = 0$, so f is an R-automorphism. Hence, M is an s-WH. □

Proposition 2.19. Let M be a module, consider the following:
(1) M is s-WH.
(2) If $M \cong M \oplus N$, then $N = 0$, for some semisimple module N.
Then $(1) \Longrightarrow)(2)$. And if M is projective, we have $(2) \Longrightarrow)(1)$.

Proof. $(1) \Longrightarrow)(2)$ Assume that M is an s-WH module, where $M \cong M \oplus N$ for some semisimple module N. It follows that $M = K \oplus L$ where $K \cong M$ and $L \cong N$. From [2], we deduce that L is an e-small submodule of M. We have $M/L \cong K \cong M$. Thus, by Proposition 2.18, $L = 0$, hence $N = 0$.
$(2) \Longrightarrow)(1)$ Conversely, suppose that M is projective and $f \in End(M)$ is an epimorphism, where $kerf \ll_e M$. Thus f split, that is $M = T \oplus kerf$, for some $T \leq M$. By the First Isomorphism Theorem, we have that $T \cong M/kerf \cong M$. By [2], $M = S \oplus T$ where S is a semisimple submodule of $kerf$. By the modular law, $kerf = kerf \cap M = kerf \cap (S \oplus T) = S \oplus (kerf \cap T) = S$. It follows that $M \cong M \oplus kerf$ and $kerf$ is semisimple. By (2), $kerf = 0$ and M will be an s-WH module. □

We will offer the following condition (E^*) for any R-module M:
(E^*) If $f: M \to M'$ and $g: M' \to M''$ are any two R-endomorphisms, then f and g are e-small if and only if gf is e-small.

Proposition 2.20. If M is a module with the property that for any $g \in End(M)$, there exists an $n \in \mathbb{Z}^+$ such that $kerg^n \cap Img^n = 0$, then M is an s-WH.
Proof. Let $g \in End(M)$ be an e-small epimorphism. By assumption, there is an integer $n \geq 1$ such that $kerg^n \cap Img^n = 0$. It follows that $g^n \in End(M)$ is an epimorphism, i.e., $Img^n = M$. Thus, $kerg^n \cap Img^n = kerg^n \cap M = kerg^n = 0$. But we know that $kerg \leq kerg^n$, which implies $kerg = 0$. Therefore, g is an automorphism. Hence, M is an s-WH. □

Corollary 2.21. Let M be an R-module satisfies (E^*) property. If M has ACC on e-small submodules, then M is an s-WH.

Proposition 2.22. Let M be an R-module. If for any R-epimorphism $\varphi: M \to M$, there exist $n \geq 1$ such that $ker\varphi^n = ker\varphi^{n+i}$ for all $i \in \mathbb{Z}^+$, then M is an s-WH.
Proof. Let $\varphi \in End(M)$ be any surjective. We claim that $ker\,\varphi^n \cap Im\varphi^n = 0$. Let $y \in ker\,\varphi^n \cap Im\varphi^n$. Thus $\varphi^n(y) = 0$ and $y = \varphi^n(x)$ for some $x \in M$. Hence, $\varphi^{2n}(x) = \varphi^n(y) = 0$. Hence, $x \in ker\varphi^{2n}$. But from our assumption we have that $ker\varphi^n = ker\varphi^{n+n} = ker\varphi^{2n}$. So, $x \in ker\varphi^n$. Therefore, $0 = \varphi^n(x) = y$. Hence, $ker\,\varphi^n \cap Im\varphi^n = 0$. Since φ is a surjective, so $Im\varphi^n = M$, thus $ker\varphi^n = 0$. But $ker\varphi \subseteq ker\varphi^n$. So, $ker\varphi \ll_e M$. Therefore, M is an s-WH. □

Proposition 2.23. Let M be an R-module has (E^*) property and N be any non-zero e-small submodule of M, if M/N is s-WH, then M is an s-WH.

Proof. Assume M is not s-WH, then there is an e-small epimorphism $f \in End(M)$ that it is not automorphism, ($kerf \neq 0$). From 1st isomorphism theorem there is an R-isomorphism $\varphi: M/kerf \to M$. Consider $\pi: M \to M/kerf$ the canonical map, then $ker\pi = kerf \ll_e M$. Therefore, π is an e-small epimorphism. It follows that $\pi\varphi: M/kerf \to M/kerf$ is an e-small epimorphism, by hypothesis, which is not isomorphism (since $ker(\pi\varphi) = \varphi^{-1}(ker\pi) = \varphi^{-1}(kerf) \neq 0_{M/kerf}$), but M/N is s-WH, which is a contradiction. Hence, $kerf = 0$ and M is an s-WH. □

Theorem 2.24. Let M be a uniform quasi-projective R-module has (E^*) property. Then M is an s-WH if and only if M/N is s-WH, with N is an e-small fully invariant submodule of M.
Proof. Assume that M is s-WH and N is an e-small fully invariant submodule of M. If $f: M/N \to M/N$ is an e-small epimorphism. Consider the e-small canonical epimorphism $\pi: M \to M/N$ (as $ker\pi = N \ll_e M$), so $f\pi: M \to M/N$ is an e-small epimorphism, as M has (E^*) property. Since M is quasi projective, then there exists an endomorphism g of M such that $\pi g = f\pi$. This equality implies that g is an epimorphism, as M is uniform. Since, $f\pi$ is e-small. Then is πg is e-small that implies g is e-small, since M has (E^*) property. Since M is s-WH, then g is an automorphism. For all $m \in M$, we deduce that $f(m+N) = f\pi(m) = \pi g(m) = g(m) + N$, and then $kerf = \{m + N \in M/N | \, f(m+N) = N\} = \{m + N \in M/N | \, g(m) + N = N\} = \{m + N \in M/N | g(M) \in N\} = K/N$, where $K = \{m \in M | \, g(m) \in N\}$ and $N \subseteq K = g^{-1}(N)$. Since N is fully invariant in M and $g^{-1} \in End(M)$, then $g^{-1}(N) \subseteq N$, thus $K = g^{-1}(N) = N$. Hence, $kerf = K/N = 0_{M/N}$ and M/N, is an s-WH. The converse is clear when $N = 0$. □

Definition 2.25. [18] Let R be a ring with identity 1 a subset S of a ring R is called multiplicatively closed set if the following two conditions hold:
(1) $1 \in S$.
(2) For all u and v in S, the product $uv \in S$.

Definition 2.26. [18] Let M be an R-module. Let S be a multiplicatively closed set in R. Let T be the set of all ordered pairs (x, s) where $x \in R$ and $s \in S$. Define a relation on T by $(x,s) \sim (\acute{x}, \acute{s})$ if there exists $t \in S$ such that $t(s\acute{x} - \acute{s}x) = 0$. This is an equivalence relation on T, and we denote the equivalence class of (x, s) by x/s . Let $S^{-1}M$ denote the set of equivalence classes of T with respect to this relation. We can make $S^{-1}M$ into an R-module by setting $x/s + y/t = (tx + sy)/st$, $a(x/s) = a\,x/s$, $a \in R$. W The R-module $S^{-1}M$ is called a quotient module (localization of module), or a module of quotient. Note that if $0 \in S$, then $S^{-1}M = 0$.

Definition 2.27. [19] Let M be an R-module and R is a commutative ring. An element $r \in R$ is called prime to L, where $L \leq M$, if $rm \in L$ ($m \in M$) implies that $m \in L$.
The set of all elements of R that are not prime to L, denote by $\mathcal{L}(L)$, i.e., $\mathcal{L}(L) = \{r \in R | \, rm \in L \text{ for some } m \in M \setminus L\}$.

In the next results, we examine the behavior of the s-WH under the concept of localization.

Proposition 2.28. Suppose that M be an R-module and S a multiplicative closed subset of R such that $\mathcal{L}(L) \cap S = \emptyset$ for any $L \leq M$. If $S^{-1}M$ is a s-WH as $S^{-1}R$-module, then M is an s-WH as R-module.
Proof. Let $f: M \to M$ be an e-small R-epimorphism. Define $S^{-1}R$-endomorphism $S^{-1}f: S^{-1}M \to S^{-1}M$ by $S^{-1}f\left(\frac{m}{s}\right) = \frac{f(m)}{s}$ for all $m \in M$, $s \in S$. Then we have $Im(S^{-1}f) =$

$S^{-1}(Imf) = S^{-1}M$, then $S^{-1}f$ is an $S^{-1}R$-epimorphism. Since $kerf \ll_e M$, thus $ker(S^{-1}f) = S^{-1}(kerf) \ll_e S^{-1}M$ from ([20], Lemma 2.3.3). As $S^{-1}M$ is s-WH. Therefore, $ker(S^{-1}f) = S^{-1}(kerf) = S^{-1}(0)$, then $kerf = 0$ by ([20], Lemma 2.3.1.) Hence, M is s-WH. □

3. s-WH modules and related concepts

Many relations between s-WH modules and other types of modules are introduced in this section, such as generalized hollow, semisimple and nonsingular uniform modules. We give a case that make the concepts WH, δ-weakly Hopfian and s-WH modules are identical, we give two cases that make the concepts Hopfian and s-WH are equivalent. Also, we put a condition on co-Hopfian ring to become an s-WH ring.

Recall that a module M is called generalized hollow if any proper submodule of M is an e-small [21].

Proposition 3.1. Let M be a non-zero R-module, if M is a generalized Hollow module. Then M is an s-WH if and only if M is a Hopfian.
Proof. Let M be an s-WH R-module. Let $f \in End(M)$ be an R-epimorphism, so $kerf \subset M$ (since, if $kerf = M$ then $f = 0$, a contradiction), Then $kerf \ll_e M$, as M is generalized Hollow. Since M is s-WH, so $kerf = 0$. Hence, M is a Hopfian R-module, since f is an automorphism. Conversely, follows by Remarks and Examples 2.2(1). □

Proposition 3.2. Every s-WH module is a δ-weakly Hopfian.
Proof. Let M be a s-WH R-module. If $f \in End(M)$ is a δ-small R-epimorphism, then $kerf \ll_\delta M$, and then $kerf \ll_e M$, by ([2], p.1052). Since M is s-WH, then $kerf = 0$. Therefore, M is a δ-weakly Hopfian R-module, since f is an isomorphism. □

Corollary 3.3. Every s-WH module is WH.
Proof. Since every δ-weakly Hopfian is WH from [13]. Then the result is followed by Proposition 3.2. □

Now, we will give the case that makes the concepts WH, δ-weakly Hopfian and s-WH modules identical.

Proposition 3.4. If M is a non-zero indecomposable R-module. Then the following are equivalent.
(1) M is s-WH;
(2) M is δ-weakly Hopfian;
(3) M is WH.
Proof. $(1) \Longrightarrow)(2)$ By Proposition 3.2.
$(2) \Longrightarrow)(3)$ By [13].
$(3) \Longrightarrow)(1)$ Assume that M is a WH R-module, let $f \in End(M)$ is an e-small epimorphism. If $kerf = M$, then $f = 0$, which it is a contradiction. Thus, $kerf$ is a proper e-small submodule of M, and since M is indecomposable, [15], implies $kerf \ll M$, that means $f \in End(M)$ is a small R-epimorphism. Since M is a WH R-module, then f is an automorphism. Hence, M is an s-WH. □

Corollary 3.5. The following are equivalent for a non-zero uniform R-module M.
(1) M is s-WH;
(2) M is δ-weakly Hopfian;

(3) M is WH.

Proof. Assume that M is a uniform module, thus M is an indecomposable module by ([16] ,Examples 3.51(1)). Thus, Proposition 3.4 implying the result. □

Proposition 3.6. Let M be a uniform and torsion-free module. Then M is an s-WH.

Proof. Let $f: M \to M$ be an e-small R-epimorphism. Let $0 \neq x \in M \backslash kerf$, $f(x) \neq 0$, so $-x \in M$ and $f(-x) = f(x).-1 \neq 0$, i.e., $-x \in M \backslash kerf$. For any $r \in R$, $f(xr) = f(x)r$. Since M is an torsion-free R-module, it follows that $f(x)r \neq 0$ and then $xr \in M \backslash kerf$. Thus, $(M \backslash kerf) \cup \{0\}$ is a submodule of M and so $(M \backslash kerf) \cup \{0\} \trianglelefteq M$, as M is uniform. As $(M \backslash kerf) \cup \{0\} + kerf = M$ and f an e-small R-epimorphism, i.e., $kerf \ll_e M$, thus $(M \backslash kerf) \cup \{0\} = M$, so $kerf = 0$. Hence, M is an s-WH. □

Example 3.7. The reverse of Proposition 3.6, is not true generally. Consider the $\mathbb{Z}$-module $\mathbb{Z}_{pq}$ where p,q are prime numbers. By Examples 2.3, $\mathbb{Z}_{pq}$ is an s-WH, but nor uniform neither torsion-free $\mathbb{Z}$-module.

Theorem 3.8. For a projective R-module M, the following are equivalent.

(1) M is s-WH;

(2) if $f \in End(M)$ has a right inverse in $End(M)$ and $kerf$ is a semisimple, then f has a left inverse in $End(M)$;

(3) if $f \in End(M)$ has a right inverse in $End(M)$ and $kerf \ll_e M$, then f has a left inverse in $End(M)$;

(4) if $f \in End(M)$ has a right inverse in $End(M)$ and $(1 - gf)M \ll_e M$, then f has a left inverse in $End(M)$;

(5) if $f \in End(M)$ is a surjective and $kerf$ is semisimple, then f has a left inverse in $End(M)$.

Proof. It is clear that $f \in End(M)$ is a surjective if and only if $fg = 1$ for some $g \in End(M)$. Thus, $kerf = (1 - gf)M$, to see this: let $x \in kerf \implies f(x) = 0 \implies (1 - gf)(x) = x - gf(x) = x - g(0) = x \implies x \in (1 - gf)M$. Now, assume that $y \in (1 - gf)M \implies y = (1 - gf)(x)$ for some $x \in M \implies y = x - gf(x) \implies f(y) = f(x) - fgf(x) = f(x) - 1(f(x)) = f(x) - f(x) = 0 \implies y \in kerf$. So $M = kerf \oplus (gf)M = kerf \oplus Img$, since $kerf + (gf)M = (1 - gf)M + (gf)M = M$, also if $m \in kerf \cap Img \implies f(m) = 0$ and $m = g(a)$, for some $a \in M \Longrightarrow 0 = f(m) = f(g(a)) = fg(a) = 1(a) = a \Longrightarrow m = g(a) = g(0) = 0$.

(1) ⟹)(2) Assume that $f \in End(M)$ contain a right inverse with $kerf$ is semisimple. Thus $fg = 1$ for some $g \in End(M)$. Then g is an injective, i.e., $kerg = 0$. From 1st isomorphism theorem, $M \cong M/0 = M/kerg \cong Img$. By above argument, we have $M = Img \oplus kerf \cong M \oplus kerf$, i.e., $M \cong M \oplus kerf$ and $kerf$ is semisimple, thus $kerf = 0$, by Proposition 2.19, that is f is an automorphism. As $fg = 1$, then $g = f^{-1}$. Hence $gf = f^{-1}f = 1$, that mean g is a left inverse of f in $End(M)$.

(2) ⟹)(3) Assume that $f \in End(M)$ contain a right inverse in $End(M)$ and $kerf \ll_e M$. Since $M = kerf \oplus Img$, [2], implies $kerf$ is semisimple. From (2), f has a left inverse in $End(M)$.

(3) ⟹)(4) Since $kerf = (1 - gf)M$, (3) implies (4).

(4) ⟹)(5) Let $f \in End(M)$ be a surjective and $kerf$ is semisimple, then f has a right inverse in $End(M)$. By above argument, we have $kerf = (1 - gf)M$ and $M = kerf \oplus Img$. By [2], $kerf = (1 - gf)M \ll_e M$, then f has a left inverse in $End(M)$, by (4).

(5) ⟹)(1) Assume that if $f \in End(M)$ is a surjective and $kerf \ll_e M$. Hence, f has a right inverse in $End(M)$. By above argument, $M = kerf \oplus Img$. $kerf$ is semisimple from [2], so f

contain a left inverse in $End(M)$ by (5). That is $hf = 1$ for some $h \in End(M)$. Thus $f \in End(M)$ is an injective. Hence, it is an automorphism. Therefore, (1) holds. □

Proposition 3.9. Let M be a semisimple module. Then M is s-WH if and only if it is Hopfian.
Proof. Suppose that M is an s-WH module. Let $f: M \to M$ be an R-epimorphism. As M is a semisimple module, then by [22], we get $kerf \ll_e M$, i.e., f is an e-small R-epimorphism and so f is an automorphism. Hence, M is Hopfian. Conversely, follows by Remarks 2.2(1). □

Proposition 3.10. Every co-Hopfian quasi-projective module is an s-WH.
Proof. Suppose that M is a co-Hopfian quasi-projective module and let $\varphi: M \to M$ be an e-small epimorphism. Since M is quasi-projective, so there is an $f \in End(M)$ such that $\varphi f = I_M$. As I_M is a monomorphism, then so is f. As M is a co-Hopfian module, thus f is an epimorphism. Since $0 = kerI_M = ker(\varphi f) = f^{-1}(ker\varphi)$, then $0 = f(0) = f(f^{-1}(ker\varphi)) = ker\varphi$, that means φ is an automorphism. Hence, M is an s-WH. □

Example 3.11. The reverse of Proposition 3.10, need not be true in general. Examples 2.6(1) shows that the $\mathbb{Z}$-module $\mathbb{Z}$ is s-WH. But we know that $\mathbb{Z}$-module $\mathbb{Z}$ is quasi-projective not co-Hopfian see [11].

Corollary 3.12. Every projective co-Hopfian module is an s-WH.
Proof. Clear by Proposition 3.10. □

Corollary 3.13. Every co-Hopfian ring is a s-WH ring.
Proof. Suppose that R is a co-Hopfian ring. As $R = \langle 1 \rangle$ is a free R-module, so it is projective. Then the result is followed by Corollary 3.12. □

Proposition 3.14. If M is a nonsingular uniform module, then M is an s-WH.
Proof. Let M be a nonsingular uniform module. Suppose that $\varphi \in End(M)$ is an e-small epimorphism, i.e., $ker\varphi \ll_e M$. Assume $ker\varphi \neq 0$. We have $Ker\varphi \trianglelefteq M$ because M is uniform. Thus $M/ker\varphi$ is singular by ([1], Proposition 1.21). From the First Isomorphism Theorem $M/ker\varphi \cong M$. This is a contradiction because $M/ker\varphi$ is singular and nonsingular. Hence, $ker(\varphi) = 0$, so φ is an automorphism. Therefore, M is an s-WH. □

Remarks 3.15.
(1) We note that Proposition 3.14, is another proof for Example 2.6(1), of $\mathbb{Z}$-module $\mathbb{Z}$ being s-WH, in fact $\mathbb{Z}$ as $\mathbb{Z}$-module is nonsingular and uniform.
(2) The reverse of Proposition 3.14, need not be true in general. Examples 2.3 shows that the $\mathbb{Z}$-module $\mathbb{Z}_6$ is s-WH. But we know that $\mathbb{Z}_6$ nor nonsingular neither uniform as $\mathbb{Z}$-module.

4. Conclusions

We defined a new concept of modules called s-WH which is a proper generalized of Hopfian. It is shown and investigate some different properties and examples of this class.

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