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On Efficient Method For Fractional-Order Two-Dimensional Navier-Stokes Equations

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Abstract

For the solution of fractional-order two-dimensional Navier-Stokes equations (FOTDNSEs), the current work proposes a novel variant of the Laplace Adomian decomposition approach with the Atangana-Baleanu fractional operator in the Caputo sense (ABCFO). In this approach, the solution is considered as a Taylor series expansion that converges rapidly to the exact solution. Only two components are required for the new approximation analytical technique. When compared to other strategies, the current method is very simple, requires less calculation, and is extremely accurate.

Keywords: Navier-Stokes equations; Laplace transform; Adomian decomposition method; Atangana-Baleanu fractional operator.

حول طريقة فعالة لمعادلات نيفر_ستوكس ثنائية البعد ذات الرتبة الكسرية

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الخلاصة

لحل معادلات نيفر_ستوكس ثنائية البعد ذات الرتبة الكسرية (FOTDNSEs), يقترح هذا العمل متغير جديد لطريقة تحليل ادوميان ولابلاس مع الؤثر الكسري اتنجانا–بالينو (ABFO). في هذه الطريقة يفرض الحل كتوسيع سلسلة تايلر الذي يتقارب بسرعة مع الحل الدقيق. مطلوب مكونين فقط للتقنية التحليلية التقريبية الجديدة. عند مقانة هذه الطريقة بالاستراتيجيات الاخرى فان الطريقة الحالية تعتبر بسيطة للغاية وتتطلب حسابات اقل ودقيقة جدا.

1. Introduction

In recent years, fractional differential equations have sparked a lot of interest, and they have been studied and applied to a lot of real-world situations in a variety of fields. One reason for this unpopularity might be that fractional derivatives have numerous nonequivalent definitions [1], [2]. Another issue is that, due to their nonlocal nature, fractional derivatives have no obvious geometrical meaning. However, in the last 12 years, scientists have begun to pay considerably more attention to the fractional calculus. With the use of fractional derivatives, it was discovered that a variety of applications, particularly multidisciplinary applications [3], [4] may be neatly described.

For the solution of linear and nonlinear FPDEs, a variety of numerical and analytical strategies have been proposed. For example, the Adomian decomposition method (ADM) [5], [6], homotopy analysis method (HAM) , variational iteration method (VIM) [7], [8], homotopy analysis transform method (HATM) [9], reduced differential transform method (RDTM) [9], Sumudu variational iteration method (SVIM) [10], [11], Laplace homotopy perturbation method (LHPM) [12], Laplace variational iteration method (LVIM) [13], Sumudu homotopy perturbation method (SHPM) [14], and other methods [15]–[22]. Our aim is to present the coupling method of the Laplace transform (LT) and ADM, which is called the Laplace Adomian decomposition method (LADM), and used to solve the fractional-order two-dimensional Navier-Stokes equations. The fractional-order two-dimensional Navier-Stokes equations (FOTDNSEs) are a modified version of the classical Navier-Stokes equations that incorporate fractional derivatives. These equations are used to describe the motion of fluid in a two-dimensional domain taking into account non-local and memory effects.

The fractional-order two-dimensional Navier-Stokes equations with Atangana-Baleanu fractional operator in the Caputo sense can be written as follows:

$$
\begin{aligned}
{}^{\mathrm{ABC}}D_t^{\alpha} u(x,t) &= \rho_0\, \Delta^2 u - u u_x - v u_x + g,\\
{}^{\mathrm{ABC}}D_t^{\alpha} v(x,t) &= \rho_0\, \Delta^2 v - u v_x - v v_x - g,
\end{aligned} \tag{1}
$$

with initial conditions

$$
\begin{aligned}
u(x,0) &= f_1(x,y),\\
v(x,0) &= f_2(x,y),
\end{aligned} \tag{2}
$$

where $\rho_0, t,$ and g denote the constant density, time, and pressure, respectively. x, y are the spatial components. The functions $f_1(x,y)$ and $f_2(x,y)$ are depending only on x, y.

The fractional derivatives in the FOTDNSEs introduce memory effects into the fluid flow, accounting for the history of the system. The fractional orders α and β determine the degree of memory and non-locality incorporated into the equations. Different choices of fractional orders lead to different physical behaviours of the fluid flow. Solving the FOTDNSEs requires specialized techniques for handling fractional derivatives, such as fractional calculus or numerical methods that are specifically designed for the fractional-order equations. The FOTDNSEs have been studied in various research fields, including fluid dynamics, viscoelastic fluids, and complex systems. To better understand the behaviour of fluids with memory effects, see [23]–[27]. The current work proposes a novel variant of the Laplace Adomian decomposition approach with the Atangana-Baleanu fractional operator in the Caputo sense (ABCFO). The paper has been organized as follows: The basic definitions of the fractional calculus are given in Section 2. An analysis of the used method is given in Section 3. An illustrative two examples that show the effectiveness of the proposed method are given in Section 4. Finally the conclusion is given in Section 5.

2. Preliminaries

Definition 2.1. The Atangana-Baleanu fractional derivative in the Caputo sense (ABCFD) of order α is defined as follows [28], [29]:

$$ {}^{ABC}_{\ a}D_t^{\alpha} u(t) = \frac{M(\alpha)}{1-\alpha}\int_{a}^{t} E_{\alpha}\left(\frac{-\alpha(t-x)^{\alpha}}{\alpha-1}\right) u'(x)dx, \tag{3}$$

where $0 < \alpha < 1$ and $M(0) = M(1) = 1$. $M(\alpha)$ is a normalization function.

The characteristics of eq.(3) are defined as follows:

1. ${}^{ABC}_{\ a}D_t^{\alpha}\ c = 0,$ where c is a constant.
2. $L\{{}^{ABC}_{\ a}D_t^{\alpha}\ u\ (x\ ,t\ )\} = \dfrac{s^{\alpha} L\, u\ (\ x\ ,t\ )}{s^{\alpha}(1-\ \alpha\ )\ +\ \alpha} - \dfrac{s^{\alpha-1}\, L\, u\ (\ x\ ,0)}{s^{\alpha}(1-\ \alpha\ )\ +\ \alpha}.$

Definition 2.1. The Atangana-Baleanu fractional integral (ABFI) of order α is defined as follows [18], [30]:

$$ {}^{AB}_{\ a}I_{\ t}^{\ \alpha}\ u\ (\ t\ ) = \frac{1-\ \alpha}{M\ (\ \alpha\ )} u\ (\ t\ ) + \frac{\alpha}{M\ (\ \alpha\ )}\ \frac{1}{\Gamma(\alpha\ )}\ \int_{a}^{t} (\ t-x\ )^{\alpha-1}\, u\ (\ x\ )\ dx. \tag{4}$$

The properties of eq.(5) is defined as follows [31]–[33]:

1. ${}^{AB}_{\ a}I_{\ t}^{\ \alpha}\ {}^{AB}_{\ a}D_t^{\alpha}\ \ u\ (\ t\ )\ \ = \ u\ (\ t\ ) - u(0).$
2. ${}^{AB}_{\ a}I_{\ t}^{\ \alpha}\ \ c\ = \ \frac{c}{M\ (\ \alpha\ )}\ (\ 1-\ \alpha + \ \frac{t^{\alpha}}{\Gamma(\alpha\ )}\ ).$

3. The Laplace-Adomian Decomposition Technique.

The LADM is discussed in this section for the solution of nonhomogeneous fractional nonlinear PDEs

$$\begin{aligned} &{}^{AB}_{\ a}D_t^{\alpha}\ u\ (\ x,y,t\ ) + R_1\ (u,v) + N_1\ (u,v) = g_1\ (x,y,t\ ),\\ &{}^{AB}_{\ a}D_t^{\alpha}\ v\ (\ x,y,t\ ) + R_2\ (u,v) + N_2\ (u,v) = g_2\ (x,y,t\ ), \end{aligned} \tag{5}$$

where $0 < \alpha \leq 1,\ t > 0,\ {}^{AB}_{\ a}D_t^{\alpha}\ u$ and ${}^{AB}_{\ a}D_t^{\alpha}\ v$ are Atangana-Baleanu operators, R_1, R_2 and N_1, N_2 are linear and nonlinear operators, respectively.

With the initial conditions

$$\begin{aligned} u(x,y,0) &= f_1(x,y),\\ v(x,y,0) &= f_2(x,y) \end{aligned} \tag{6}$$

Using the Laplace transform (LT) differentiation property to eq.(5), we get

$$\begin{aligned} &L\{{}^{AB}_{\ a}D_t^{\alpha}\ u\ (\ x,y,t\ )\} + L\{R_1(u,v) + N_1\ (u,v)\} = L\{g_1\ (x,y,t\ )\},\\ &L\{{}^{AB}_{\ a}D_t^{\alpha}\ v\ (\ x,y,t\ )\} + L\{R_2(u,v) + N_2\ (u,v)\} = L\{g_2\ (x,y,t\ )\}, \end{aligned} \tag{7}$$

These are equivalent to

$$\begin{aligned} &\frac{s^{\alpha} L\ \{u(x,y,t)\}}{s^{\alpha}(1-\alpha\ )+\alpha} - \frac{s^{\alpha-1}\ u\ (x,y,0)}{s^{\alpha}(1-\alpha\ )+\alpha} = \ L\{g_1\} - L\ \{R_1\ [(u,v)\ ] + N_1[\ (u,v)]\},\\ &\frac{s^{\alpha} L\ \{v(x,y,t)\}}{s^{\alpha}(1-\alpha\ )+\alpha} - \frac{s^{\alpha-1}\, v\ (x,y,0)}{s^{\alpha}(1-\alpha\ )+\alpha} = \ L\{g_2\} - L\ \{R_2\ [(u,v)\ ] + N_2[(u,v)]\}, \end{aligned} \tag{8}$$

Now applying the inverse of LT of (8), we have

$$u(x,y,t) = f_1 + L^{-1}\left(\left(1-\ \alpha+\frac{\alpha}{s^{\alpha}}\right) L\{g_1\}\right)$$
$$- L^{-1}\left(\left(1-\ \alpha+\frac{\alpha}{s^{\alpha}}\right) L\,\{R_1\,[(u,v)\,] + N_1[\,(u,v)]\}\right),$$
$$v(x,y,t) = f_2 + L^{-1}\left(\left(1-\ \alpha+\frac{\alpha}{s^{\alpha}}\right) L\{g_2\}\right)$$
$$- L^{-1}\left(\left(1-\ \alpha+\frac{\alpha}{s^{\alpha}}\right) L\,\{R_2\,[(u,v)\,] + N_2[\,(u,v)]\}\right). \quad (9)$$

The infinite series that are shown here reflects the LADM solution of $u\,(\mathrm{x,y,t})$ and $v(\mathrm{x,y,t})$ as follows:

$$u(x,y,t) = \sum_{n=0}^{\infty} u_n(x,y,t),$$
$$v(x,y,t) = \sum_{n=0}^{\infty} v_n(x,y,t), \quad (10)$$

the problem's nonlinear terms may be written as an Adomian polynomial as follows:

$$N_1[\,(u,v)] = \sum_{n=0}^{\infty} A_n,$$
$$N_2[\,(u,v)] = \sum_{n=0}^{\infty} B_n, \quad (11)$$

By adding eq.(10) and eq.(11) into eq.(9), we get

$$\sum_{ni=0}^{\infty} u_n(\mathrm{x,y,t}) = f_1 + L^{-1}\left(\left(1i-\alpha+\frac{i\alpha}{s^{\alpha}}\right) L\{g_1\}\right)$$
$$- L^{-1}\left(\left(1-i\alpha+\frac{\alpha}{s^{\alpha}}\right) L\left\{R_1\left(\sum_{n=0}^{\infty} u_n, \sum_{n=0}^{\infty} v_n\right) + \sum_{n=0}^{\infty} A_n\right\}\right),$$
$$\sum_{in=0}^{\infty} v_n(x,y,t) = f_2 + L^{-1}\left(\left(1-i\alpha+\frac{\alpha}{s^{\alpha}}\right) L\{g_2\}\right)$$
$$- L^{-1}\left(\left(1-i\alpha+\frac{\alpha}{s^{\alpha}}\right) L\left\{R_2\left(\sum_{n=0}^{\infty} u_n, \sum_{n=0}^{\infty} v_n\right) + \sum_{n=0}^{\infty} B_n\right\}\right). \quad (12)$$

When both sides of eq.(12) are compared, we get:

$$u_0(x,y,t) = f_1 + L^{-1}\left(\left(1i-\alpha+\frac{\alpha}{s^{\alpha}}\right) L\{g_1\}\right),$$
$$v_0(x,y,t) = f_2 + L^{-1}\left(\left(1i-\alpha+\frac{\alpha}{s^{\alpha}}\right) L\{g_2\}\right),$$
$$u_1(x,y,t) = -L^{-1}\left(\left(1i-\alpha+\frac{\alpha}{s^{\alpha}}\right) L\{\,R_1\,(u_0,v_0) + A_0\}\right)$$
$$v_1(x,y,t) = -L^{-1}\left(\left(1-i\alpha+\frac{\alpha}{s^{\alpha}}\right) L\{\,R_2\,(u_0,v_0) + B_0\}\right),$$
$$u_2(x,y,t) = -L^{-1}\left(\left(1i-\alpha+\frac{\alpha}{s^{\alpha}}\right) L\{\,R_1\,(u_1,v_1) + A_1\}\right),$$
$$v_2(x,y,t) = -L^{-1}\left(\left(1i-\alpha+\frac{\alpha}{s^{\alpha}}\right) L\{\,R_2\,(u_1,v_1) + B_1\}\right),$$

$$\vdots$$

$$u_{n+1}(x,y,t) = -L^{-1}\left(\left(1i - \alpha + \frac{\alpha}{s^{\alpha}}\right) L\{ R_1\,(u_n, v_n) + A_n\}\right),$$

$$v_{n+1}(x,y,t) = -L^{-1}\left(\left(1i - \alpha + \frac{\alpha}{s^{\alpha}}\right) L\{ R_2\,(u_n, v_n) + B_n\}\right). \quad (13)$$

Thus, the approximate solution of eq.(5) is:

$$u(x,t) = u_0 + u_1 + u_2 + \cdots$$
$$v(x,t) = v_0 + v_1 + v_2 + \cdots \quad (14)$$

Theorem 1 . Suppose that $u_n(x,t)$ and $u\,(x,t)$ are defined in a Hilbert space (H,$\|.\|$). Then the FLADM series solution $\sum_{n=0}^{\infty} u_n(x,t)$ converges to the solution of eq.(1), if $0 < \lambda \leq 1$.

Proof: Assume that $\{A_n\}$ is a sequence of partial sums of the series $u\,(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$ we assume the following:

$$\begin{aligned}\|A_{n+1}(x,t) - A_n(x,t)\| &= \|u_{n+1}(x,t)\| \\ &\leq \lambda\,\|u_n\,(x,t)\| \\ &\leq \lambda^2\|\,u_{n-1}\,(x,t)\| \\ &\vdots \\ &\leq \lambda^{n+1}\|\,u_0\,(x,t)\|\end{aligned}$$

Now, for every $, m \in N, n > m,$ we have

$$\begin{aligned}\|A_n - A_j\| &= \|(A_n - A_{n-1}) + (A_{n-1} - A_{n-2}) + \ldots + (A_{j+1} - A_m)\| \\ &\leq \|(A_n - A_{n-1})\| + \|(A_{n-1} - A_{n-2})\| + \ldots + \|(A_{j+1} - A_m)\| \\ &\leq (\lambda^n + \lambda^{n-1} + \cdots \ldots + \lambda^{m+1})\|(u_0)\| \\ &\leq \lambda^{m+1}(\lambda^{n-m-1} + \lambda^{n-m-2} + \cdots \ldots + \lambda + 1)\|(u_0)\| \\ &\leq \frac{1 - \lambda^{n-m}}{1 - \lambda}\,\lambda^{m+1}\,\|u_0\|\,.\end{aligned}$$

Since $0 < \lambda \leq 1$, then we have $\lim_{n,m\to\infty}\|A_n - A_j\| = 0$.

Therefore, $\{A_n\}_{n=0}^{\infty}$ is a Cauchy sequence in the Hilbert space H and it implies that the series solution $u(x\,,t) = \sum_{n=0}^{\infty} u_n(x,t)$ is convergent. This completes the proof of Theorem.

Theorem 2. The maximum absolute truncated error of the series solution (9) of the nonlinear fractional differential equation (5) is estimated as follows:

$$\|u(x,t) - \sum_{n=0}^{\infty} u_n(x,t)\| \leq \left(\frac{\mathbf{1}}{\mathbf{1} - \boldsymbol{\lambda}}\right)\lambda^{m+1}\,\|u_0\|.$$

Proof:

$$\|A_n - A_m\| \leq \frac{1 - \lambda^{n-m}}{1 - \lambda}\,\lambda^{m+1}\,\|u_0\|.$$

For n $\geq m$, Now, as $n \to \infty$ *then* $A_n \to u(\,x\,,t\,)$, so

$$\left\|u(x,t) - \sum_{n=0}^{m} u_n(x,t)\right\| \leq \frac{1 - \lambda^{n-m}}{1 - \lambda}\lambda^{m+1}\,\|u_0\|.$$

Since $0 < \lambda \leq 1$, we have $1 - \lambda^{n-m} < 1$.

Therefore, the above inequality becomes

$$\left\|u(x,t) - \sum_{n=0}^{m} u_n(x,t)\right\| \leq \frac{1}{1 - \lambda}\lambda^{m+1}\,\|u_0\|.$$

4. Illustrate Examples

Example 4.1. Consider the time fractional-order two dimensional Navier-Stokes equation:

$$^{\mathrm{ABC}}D_t^{\alpha}u + uu_x + vu_x = p\,[u_{xx} + u_{yy}] + q,$$
$$^{\mathrm{ABC}}D_t^{\alpha}v + uv_x + vv_x = p\,[v_{xx} + v_{yy}] - q, \quad (15)$$

with initial conditions

$$u\,(x,y,0) = -e^{x+y},$$
$$v\,(x,y,0) = e^{x+y}, \tag{16}$$

Equation (15) is obtained using the iterative technique according to eq.(13)

$$u_0(x,y,t) = -e^{x+y},$$
$$v_0(x,y,t) = e^{x+y},$$
$$u_1(x,y,t) = L^{-1}\left\{\left(1-\alpha+\frac{\alpha}{s^\alpha}\right)L\left\{p\left[(u_0)_{xx}+(u_0)_{yy}\right]+q-A_0-B_0\right\}\right\},$$
$$v_1(x,y,t) = L^{-1}\left\{\left(1-\alpha+\frac{\alpha}{s^\alpha}\right)L\left\{\,p\left[(v_0)_{xx}+(v_0)_{yy}\,\right]-q-C_0-H_0\right\}\right\},$$

where

$$A_0 = u_0u_{0x} = e^{2(x+y)},$$
$$B_0 = v_0u_{0x} = -e^{2(x+y)},$$
$$C_0 = u_0v_{0x} = -e^{2(x+y)},$$
$$H_0 = v_0v_{0x} = \;\; e^{2(x+y)},$$

then, we have

$$u_1(x,y,t) = L^{-1}\left\{\left(1-\alpha+\frac{\alpha}{s^\alpha}\right)L\{-2p\,e^{x+y}\}+q\right\}$$
$$= \;-2p\,e^{x+y}\left[1-\alpha+\frac{t^\alpha}{\Gamma(\alpha)}\right]+q\left[1-\alpha+\frac{t^\alpha}{\Gamma(\alpha)}\right].$$
$$v_1(x,y,t) = L^{-1}\left\{\left(1-\alpha+\frac{\alpha}{s^\alpha}\right)L\{2p\,e^{x+y}-q\}\right\}$$
$$= 2p\,e^{x+y}\left[1-\alpha+\frac{t^\alpha}{\Gamma(\alpha)}\right]-q\left[1-\alpha+\frac{t^\alpha}{\Gamma(\alpha)}\right],$$
$$u_2(x,y,t) = L^{-1}\left\{\left(1-\alpha+\frac{\alpha}{s^\alpha}\right)L\left\{p\left[(u_1)_{xx}+(u_1)_{yy}\right]+q-A_1-B_1\right\}\right\},$$
$$v_2(x,y,t) = L^{-1}\left\{\left(1-\alpha+\frac{\alpha}{s^\alpha}\right)L\left\{\,p\left[(v_1)_{xx}+(v_1)_{yy}\,\right]-q-C_1-H_1\right\}\right\},$$

where

$$A_1 = u_1u_{0x}+u_0u_{1x}$$
$$= 4e^{2(x+y)}\left[1-\alpha+\frac{t^\alpha}{\Gamma(\alpha)}\right]-e^{x+y}[q]\left(1-\alpha+\frac{t^\alpha}{\Gamma(\alpha)}\right),$$
$$B_1 = v_1u_{0x}+v_0u_{1x}$$
$$= -4e^{2(x+y)}\left[1-\alpha+\frac{t^\alpha}{\Gamma(\alpha)}\right]+e^{x+y}[q]\left(1-\alpha+\frac{t^\alpha}{\Gamma(\alpha)}\right),$$
$$C_1 = u_1v_{0x}+u_0v_{1x}$$
$$= -4e^{2(x+y)}\left[1-\alpha+\frac{t^\alpha}{\Gamma(\alpha)}\right]+e^{x+y}[q]\left(1-\alpha+\frac{t^\alpha}{\Gamma(\alpha)}\right),$$
$$H_1 = v_1v_{0x}+v_0v_{1x}$$
$$= 4e^{2(x+y)}\left[1-\alpha+\frac{t^\alpha}{\Gamma(\alpha)}\right]-e^{x+y}[q]\left(1-\alpha+\frac{t^\alpha}{\Gamma(\alpha)}\right),$$

then, we have

$$u_2(x,y,t) = L^{-1}\left\{\left(1-\alpha+\frac{\alpha}{s^\alpha}\right)L[-4e^{x+y}]\left(1-\alpha+\frac{t^\alpha}{\Gamma\alpha}\right)-q\right\}$$
$$= -4p^2\,e^{x+y}\left[(1-\alpha)^2+2(1-\alpha)\frac{t^\alpha}{\Gamma(\alpha)}+\frac{\alpha^2t^{2\alpha}}{\Gamma(2\alpha+1)}\right]+q\left[1-\alpha+\frac{t^\alpha}{\Gamma(\alpha)}\right].$$
$$v_2(x,y,t) = L^{-1}\left\{\left(1-\alpha+\frac{\alpha}{s^\alpha}\right)L\left\{4p^2\,e^{x+y}\left(1-\alpha+\frac{t^\alpha}{\Gamma(\alpha)}\right)-q\right\}\right\}.$$
$$= \;4p^2\;\,e^{x+y}\;\left((1-\alpha)^2+2(1-\alpha)\frac{t^\alpha}{\Gamma(\alpha)}+\frac{\alpha^2t^{2\alpha}}{\Gamma(2\alpha+1)}\right)$$
$$-\,q\left(1-\alpha+\frac{t^\alpha}{\Gamma(\alpha)}\right)$$

⋮

The exact solution of eq.(15) at $\alpha = 1$ and $q = 0$,

$$u(x, y, t) = -e^{x+y+2pt},$$
$$v(x, y, t) = e^{x+y+2pt}.$$

The behavior of solutions to the exact and analytical findings using the beginning circumstances that is given in eq.(1) is shown in Figures 1 and 6. (16). The exact and approximate solutions of u at α = 1 are shown in Figure 1 in proximity. For alternative values of α = 0.8 and,0.9 for u, see Figures 2 and 3. The graphs of the approximate and exact solutions of u for various values of and when x is constant are shown in Figure 4. The exact and approximate solutions of v at α = 1 are shown in Figure 6. For alternative values of α = 0.8 and,0.9 for v, see Figures 7 and 8. The graphs of the approximate and exact solutions of v for various values of and when x is fixed are shown in Figure 9. Each problem's fractional outcomes are examined to see if they converge to an integer-order result.

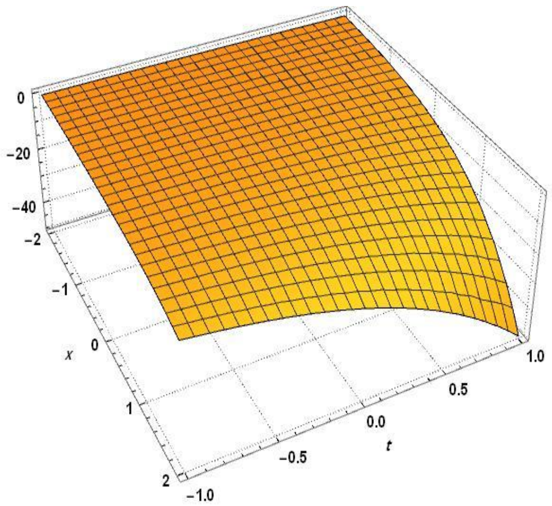

Figure 1: The precise and approximate solutions of u for α =1 for eq (15).

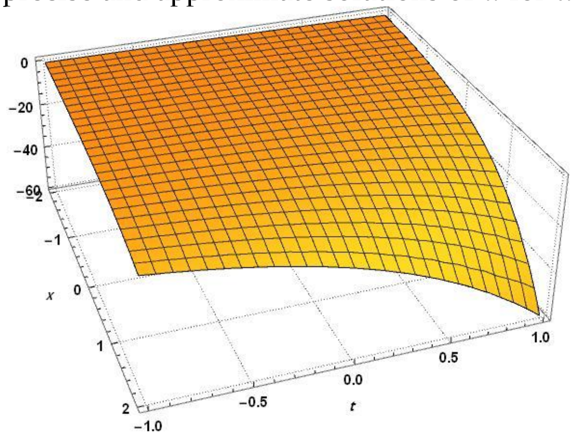

Figure 2: The surface graph of the approximate solution of u when $\alpha = 0.9$ for eq.(15).

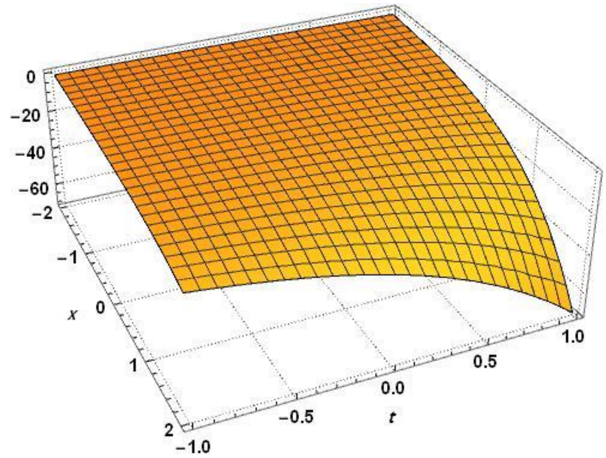

Figure 3: The surface graph of the approximate solution of u when $\alpha = 0.8$ for eq.(15).

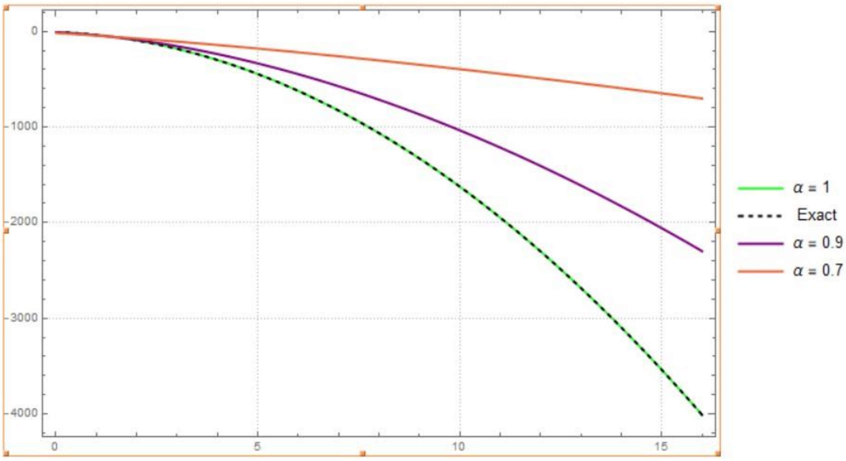

Figure 4: Plots of the exact and approximate solutions u for different values of α with fixed value $x = 1$ for eq.(15).

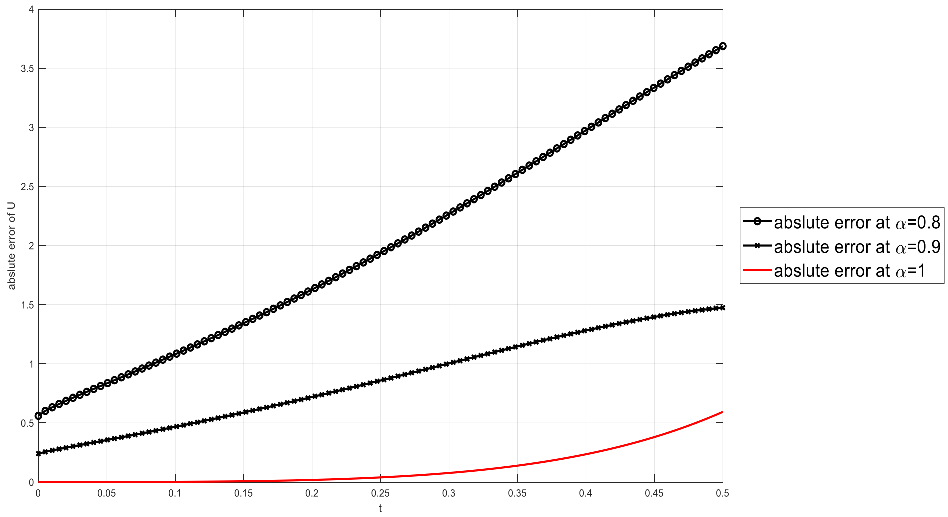

Figure 5: Plots of the absolute error of solutions u for different values of α for eq.(15).

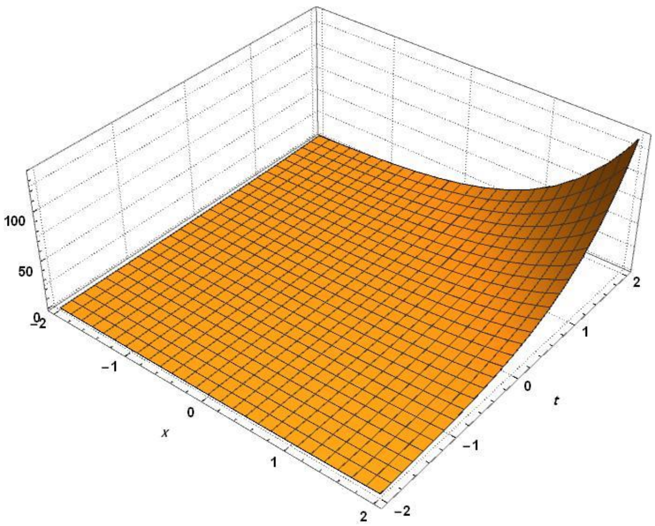

Figure 6: The surface graph of the exact and approximate solutions of when $\alpha = 1$ for eq.(15).

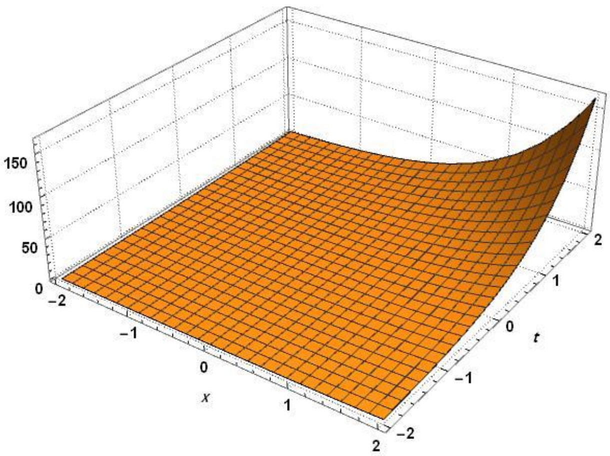

Figure 7: The surface graph of the approximate solution of v when $\alpha = 0.9$ for eq.(15).

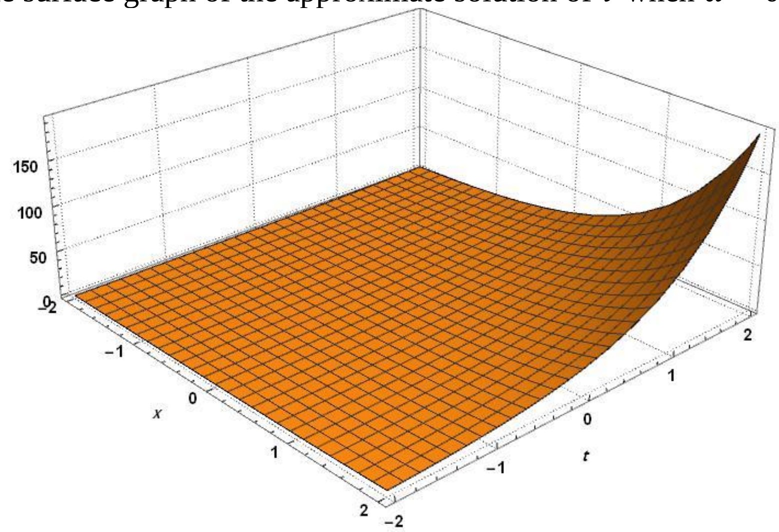

Figure 8: The surface graph of the approximate solution of v when $\alpha = 0.8$ for eq.(15).

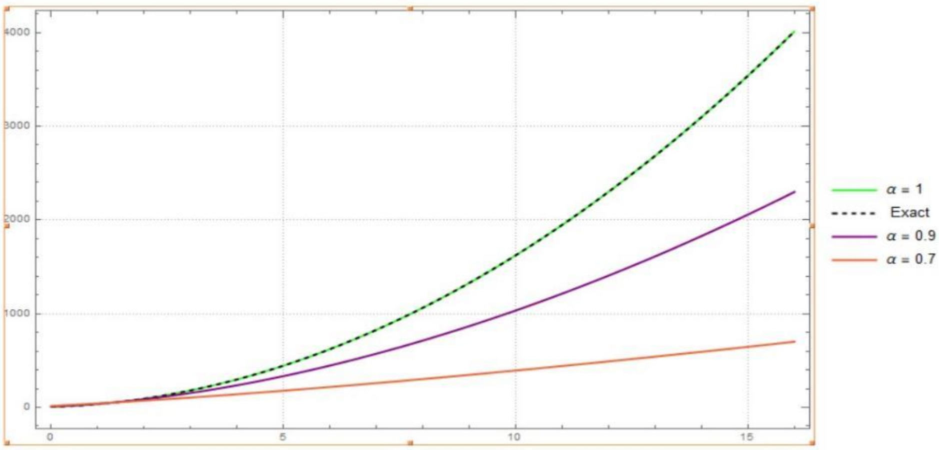

Figure 9: Plots of the exact and approximate solutions v for different values of α with fixed value x = 1 for eq.(15).

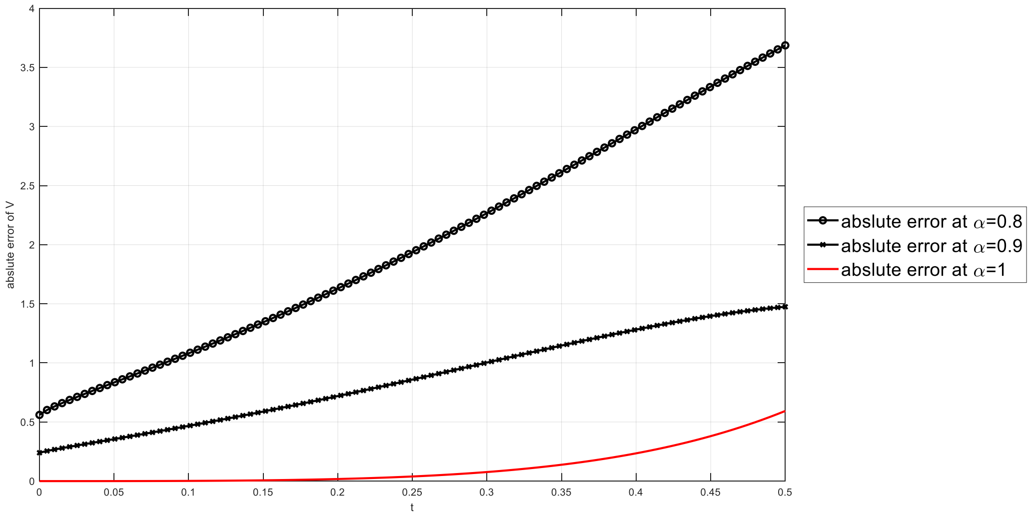

Figure 10: Plots of the absolute error of solutions v for different values of α for eq.(15).

Example 4.2. Consider the time fractional-order two dimensional Navier-Stokes equation:

$$^{ABC}D_t^{\alpha}u + uu_x + vu_x = p[u_{xx} + u_{yy}] + q,$$
$$^{ABC}D_t^{\alpha}u + uv_x + vv_x = p[v_{xx} + v_{yy}] - q, \quad (17)$$

with the initial conditions

$$u(x,y,0) = -sin(x+y),$$
$$v(x,y,0) = sin(x+y), \quad (18)$$

Using the iterative method according to eq.(13) in eq.(17), we get

$$u_0(x,y,t) = -sin\,(x+y)$$
$$v_0(x,y,t) = sin\,(x+y)$$
$$u_1(x,y,t) = L^{-1}\left\{\left(1-\alpha+\frac{\alpha}{s^{\alpha}}\right)L\{[(u_0)_{xx} + (u_0)_{yy}] + q - A_0 - B_0\}\right\},$$
$$v_1(x,y,t) = L^{-1}\left\{\left(1-\alpha+\frac{\alpha}{s^{\alpha}}\right)L\{[(v_0)_{xx} + (v_0)_{yy}] - q - C_0 - H_0\}\right\},$$

where

$$A_0 = u_0u_{0x} = sin(x+y)\,cos(x+y),$$
$$B_0 = v_0u_{0x} = -sin(x+y)\,cos(x+y),$$
$$C_0 = u_0v_{0x} = -sin(x+y)\,cos\,(x+y),$$
$$H_0 = v_0v_{0x} = sin(x+y)\,cos\,(x+y),$$

then, we have

$$u_1(x\,,y\,,t\,) = L^{-1}\left\{\left(1-\alpha+\frac{\alpha}{s^{\alpha}}\right)L\{2p\,sin(x+y)+q\}\right\}$$
$$= L^{-1}\left\{[2p\;\;sin(x+y)]\left((1-\alpha)\frac{1}{s}+\frac{\alpha}{S^{\alpha+1}}\right)+(q)((1-\alpha)\frac{1}{s}+\frac{\alpha}{S^{\alpha+1}})\right\}$$

$$=\;2p\,sin(x+y)\left[1-\alpha+\frac{\alpha\,t^{\alpha}}{\Gamma(\alpha+1)}\right]+q\left[1-\alpha+\frac{\alpha\,t^{\alpha}}{\Gamma(\alpha+1)}\right]$$
$$=\;2p\,sin(x+y)\left[1-\alpha+\frac{t^{\alpha}}{\Gamma(\alpha)}\right]+q\left[1-\alpha+\frac{t^{\alpha}}{\Gamma(\alpha)}\right].$$

$$v_1\,(\,x\,,y\,,t\,) = L^{-1}\left\{\left(1-\alpha+\frac{\alpha}{s^{\alpha}}\right)L\{-2p\,sin(x+y)-q\}\right\}$$
$$= L^{-1}\left\{\left(1-\alpha+\frac{\alpha}{S^{\alpha}}\right)([-2p\,sin(x+y)]\frac{1}{s}-(q)\frac{1}{s})\right\}$$
$$= L^{-1}\left\{[-2p\;\;sin(x+y)]\left((1-\alpha)\frac{1}{s}+\frac{\alpha}{S^{\alpha+1}}\right)-(q)((1-\alpha)\frac{1}{s}+\frac{\alpha}{S^{\alpha+1}})\right\}$$
$$=-\,2p\,sin(x+y)\left[1-\alpha+\frac{t^{\alpha}}{\Gamma(\alpha)}\right]-q\left[1-\alpha+\frac{t^{\alpha}}{\Gamma(\alpha)}\right].$$

$$u_2(x,y,t) = L^{-1}\left\{\left(1-\alpha+\frac{\alpha}{s^{\alpha}}\right)L\left\{\left[(u_1)_{xx}+(u_1)_{yy}\right]+q-A_1-B_1\right\}\right\},$$
$$v_2(x,y,t) = L^{-1}\left\{\left(1-\alpha+\frac{\alpha}{s^{\alpha}}\right)L\left\{\left[(v_1)_{xx}+(v_1)_{yy}\right]-q-C_1-H_1\right\}\right\},$$

where

$$A_1 = u_1u_{0x}+u_0u_{1x}$$
$$=-4p\,sin(x+y)\,cos(x+y)\left[1-\alpha+\frac{t^{\alpha}}{\Gamma(\alpha)}\right]-qco\,s(x+y)\left[1-\alpha+\frac{t^{\alpha}}{\Gamma(\alpha)}\right],$$
$$B_1 = v_1u_{0x}+v_0u_{1x}$$
$$=4p\,sin(x+y)\,cos(x+y)\left[1-\alpha+\frac{t^{\alpha}}{\Gamma(\alpha)}\right]+q\;cos\;(x+y)\left[1-\alpha+\frac{t^{\alpha}}{\Gamma(\alpha)}\right],$$
$$C_1 = u_1v_{0x}+u_0v_{1x}$$
$$=4p\,sin(x+y)\,cos(x+y)\left[1-\alpha+\frac{t^{\alpha}}{\Gamma(\alpha)}\right]+qco\,s(x+y)\left[1-\alpha+\frac{t^{\alpha}}{\Gamma(\alpha)}\right],$$
$$H_1 = v_1v_{0x}+v_0v_{1x}$$
$$=-4p\,sin(x+y)\,cos(x+y)\left[1-\alpha+\frac{t^{\alpha}}{\Gamma(\alpha)}\right]-qco\,s(x+y)\left[1-\alpha+\frac{t^{\alpha}}{\Gamma(\alpha)}\right],$$

then, we have

$$u_2 = L^{-1}\left\{\left(1-\alpha+\frac{\alpha}{s^{\alpha}}\right)L[-4p^2\,sin(x+y)]\left[1-\alpha+\frac{t^{\alpha}}{\Gamma(\alpha)}\right]+q\right\}$$
$$= L^{-1}\left\{\left(1-\alpha+\frac{\alpha}{s^{\alpha}}\right)\left[-4p^2\;sin\;(x+y)\left[(1-\alpha)\frac{1}{S}+\frac{\Gamma(\alpha+1)}{\Gamma(\alpha)}.\frac{1}{S^{\alpha+1}}+q\frac{1}{s}\right]\right]\right\}$$
$$=\;L^{-1}\left\{-4p^2\,sin(x+y)\left[\left[(1-\alpha)^2\frac{1}{S}+\frac{2\alpha}{S^{\alpha+1}}(1-\alpha)+\frac{\alpha^2}{S^{2\alpha+1}}\right]+q((1-\alpha)\frac{1}{S}+\frac{\alpha}{S^{\alpha+1}}\right]\right\}$$
$$=\;4p^2\,sin(x+y)\left[(1-\alpha)^2+2(1-\alpha)\frac{t^{\alpha}}{\Gamma(\alpha)}+\frac{\alpha^2t^{2\alpha}}{\Gamma(2\alpha+1)}\right].$$

$$v_2 = L^{-1}\left\{\left(1-\alpha+\frac{\alpha}{s^{\alpha}}\right)L\left[4p^2\,sin(x+y)\left[1-\alpha+\frac{t^{\alpha}}{\Gamma(\alpha)}\right]-q\right]\right\}$$
$$= L^{-1}\left\{\left(1-\alpha+\frac{\alpha}{s^{\alpha}}\right)\left[4p^2\;sin\;(x+y)\left[(1-\alpha)\frac{1}{S}+\frac{\Gamma(\alpha+1)}{\Gamma(\alpha)}.\frac{1}{S^{\alpha+1}}-q\frac{1}{s}\right]\right]\right\}$$

$$= L^{-1}\left\{4p^2 \sin(x+y)\left[\left[(1-\alpha)^2\frac{1}{S}+\frac{2\alpha}{S^{\alpha+1}}(1-\alpha)+\frac{\alpha^2}{S^{2\alpha+1}}\right]-q[(1-\alpha)\frac{1}{S}+\frac{\alpha}{S^{\alpha+1}}]\right]\right\}$$

$$= 4p^2 \sin(x+y)\left[\,(1-\alpha)^2+2(1-\alpha)\frac{t^\alpha}{\Gamma(\alpha)}+\frac{\alpha^2 t^{2\alpha}}{\Gamma(2\alpha+1)}\right] - q\,\left(1-\alpha+\frac{t^\alpha}{\Gamma(\alpha)}\right).$$

The exact solution of eq.(14) at $\alpha = 1$ and $q = 0$,

$$u(x,y,t) = -e^{2pt}\sin(x+y),$$
$$v(x,y,t) = e^{2pt}\sin(x+y).$$

The behavior of solutions of the exact and approximate findings utilizing the beginning circumstances that are given in equation 10 and 15 is shown in Figures 12 and 18. The exact and approximate solutions of u at α = 1 are shown in Figure 10 in proximity. In Figures 12 and 13 for different values of $\alpha = 0.8$ and $, 0.9$ for u. Figure 14 show the graphs of the approximate and the exact solutions of u among different values of and when x is fixed. In Figure 16, at the value of α =1, we show the exact and approximate solutions of v. In Figures 17 and 18 for different values of $\alpha = 0.8$ and $, 0.9$ for v. Figure 19 shows the graphs of the approximate and the exact solutions of v among different values of and when x is fixed. Each problem's fractional outcomes are examined to see if they converge to an integer-order result.

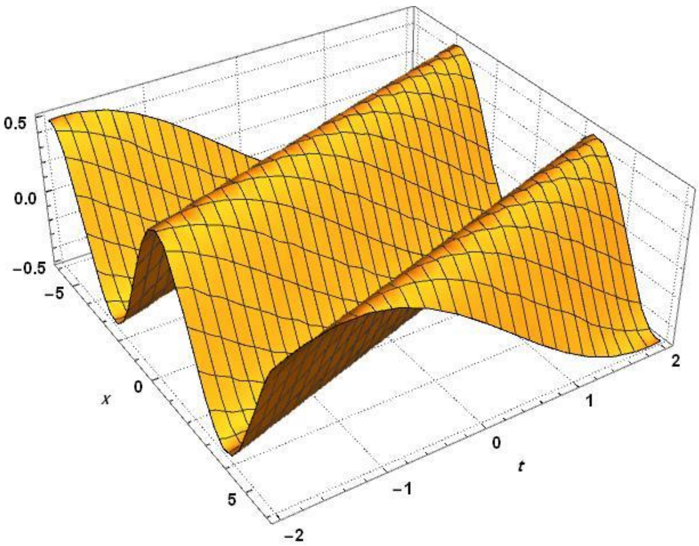

Figure 11: The surface graph of the exact and approximate solutions of u when $\alpha = 1$ for eq.(17).

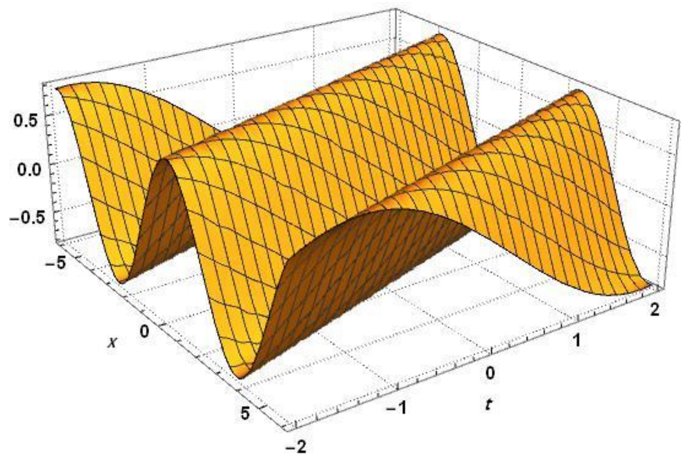

Figure 12: The surface graph of the approximate solution of u when $\alpha = 0.9$ for eq.(17).

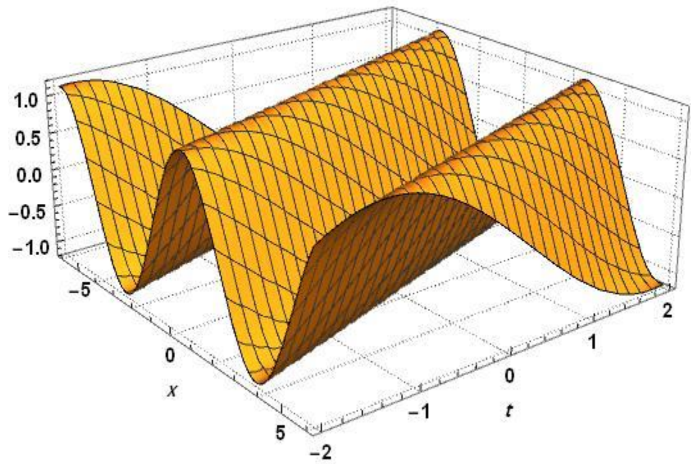

Figure 13: The surface graph of the approximate solution of u when $\alpha = 0.8$ for eq.(17).

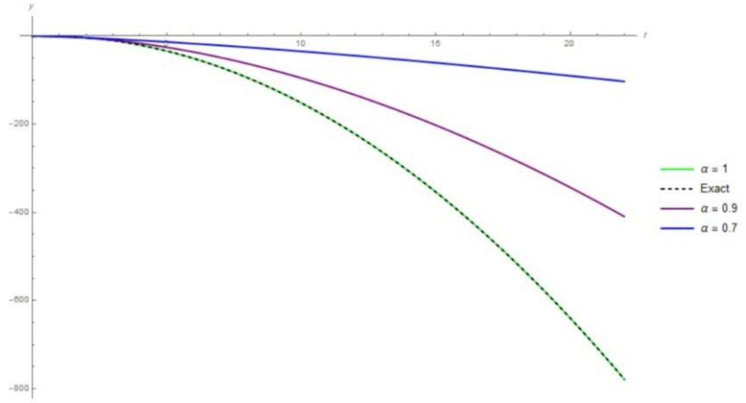

Figure 14: Plots of the exact and approximate solutions u for different values of α with fixed value $x = 1$ for eq.(17).

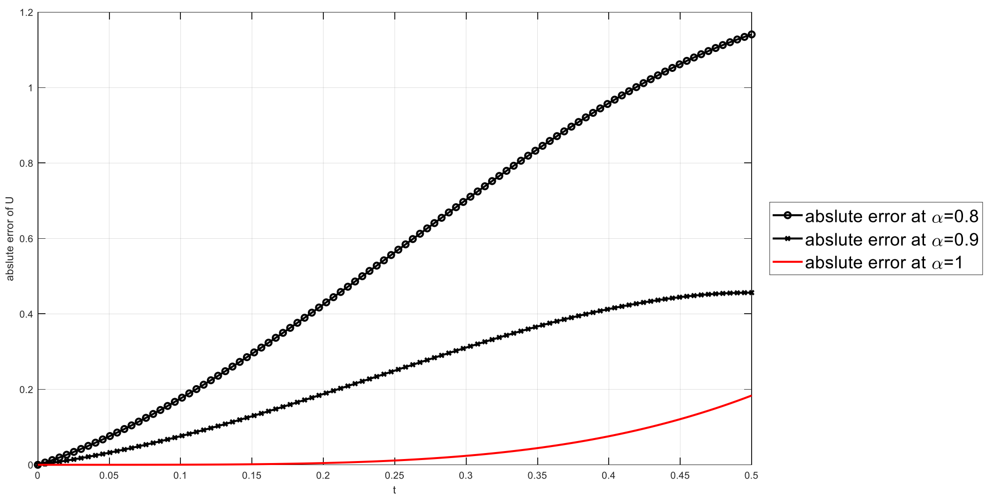

Figure 15: Plots of the absolute error of solutions u for different values of α for eq.(17).

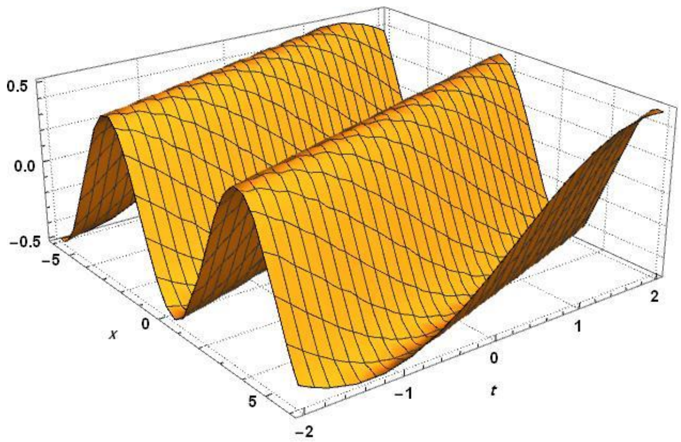

Figure 16: The surface graph of the exact and approximate solutions of when $\alpha = 1$ for eq.(17).

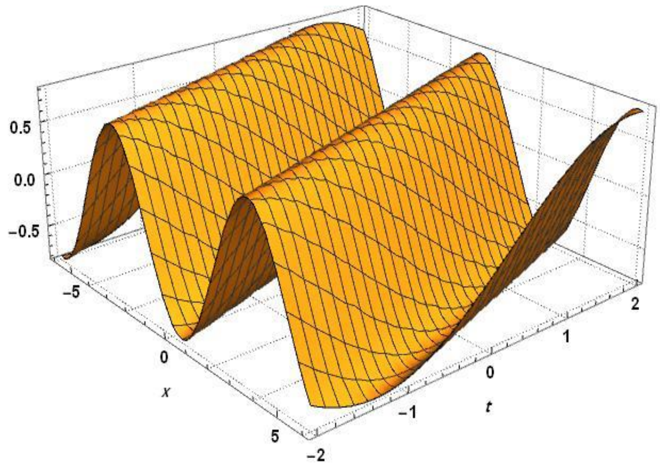

Figure 17: The surface graph of the approximate solution of v when $\alpha = 0.9$ for eq.(17).

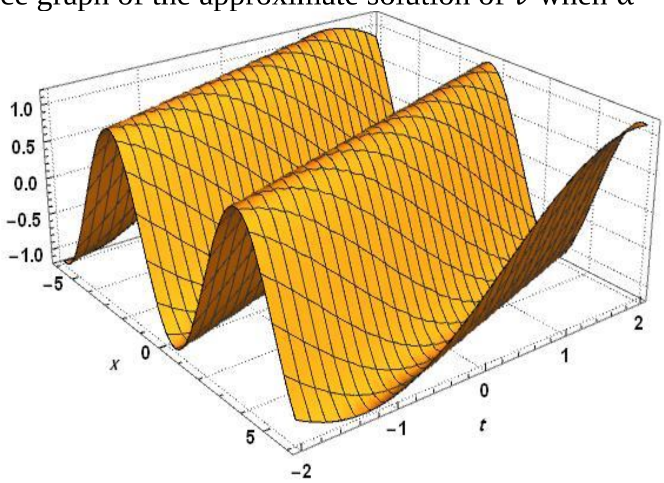

Figure 18: The surface graph of the approximate solution of v when $\alpha = 0.8$ for eq.(17).

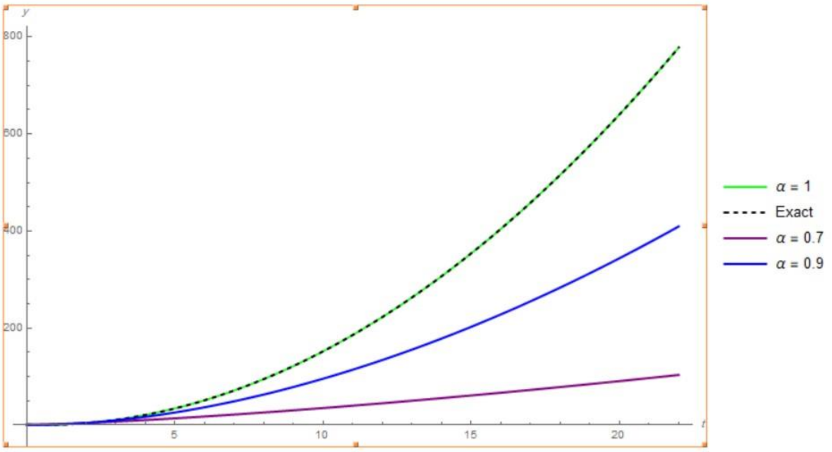

Figure 19: Plots of the exact and approximate solutions v for different values of α with fixed value $x = 1$ for eq.(17).

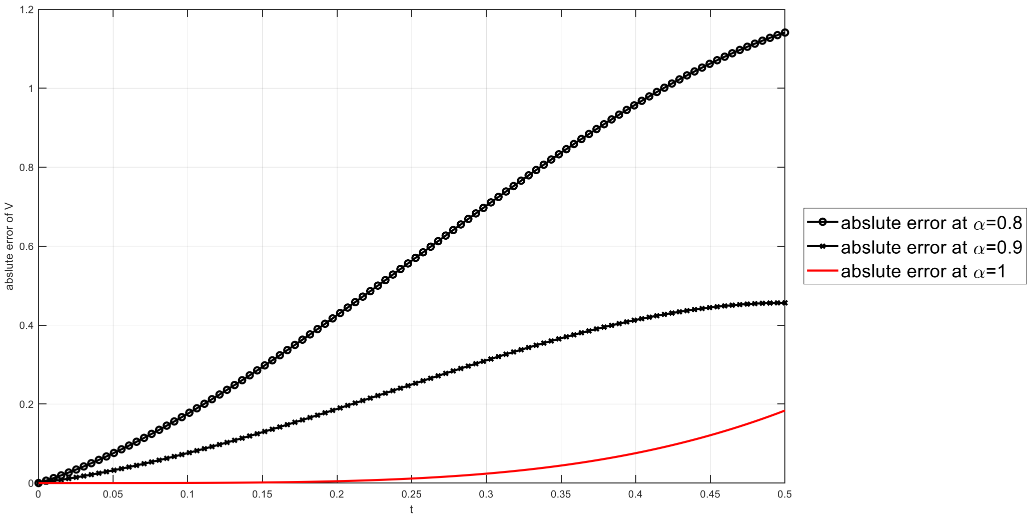

Figure 20: Plots of the absolute error of solutions v for different values of α for eq.(17).

5. Conclusions

The LADM with ABCFO is used to evaluate the fractional-order two-dimensional Navier–Stokes equations in this paper. The present technique is used to demonstrate the solutions to cases. The LADM result closely resembles the precise solution to the provided issues. The convergence of the fractional-order answers to integer-order solutions was confirmed by a graphical examination of the results. Furthermore, the proposed method is clear, simple, and low-cost to implement. It may be extended to solve additional fractional-order partial differential equations.

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