DOI: 10.24996/ijs.2019.60.10.13

ISSN: 0067-2904

The convergence of Iteration Scheme to Fixed Points in Modular Spaces

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Abstract
The aim of this paper is to study the convergence of an iteration scheme for multi-valued mappings which defined on a subset of a complete convex real modular. There are two main results, in the first result, we show that the convergence with respect to a multi-valued contraction mapping to a fixed point. And, in the second result, we deal with two different schemes for two multivalued mappings (one of them is a contraction and other has a fixed point) and then we show that the limit point of these two schemes is the same. Moreover, this limit will be the common fixed point the two mappings.

Key Words: Modular spaces, fixed points, multivalued mapping.

تقارب مخطط تكراري الى النقاط الثابتة في فضاءات الوحدات

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الخلاصة
ان الهدف من هذا البحث هو دراسة التقارب لمخطط تكراري للتطبيقات متعددة القيم المعرفة على مجموعة جزئية من فضاء وحدات حقيقي كامل ومحدب. تم تقديم نتيجتين, في الاولى تم البرهنة على تقارب بالاعتماد على تطبيق انكماشي متعدد القيم. اما في الثانية فقد تم التعامل مع مخططين تكرارية مختلفة لتطبيقين متعددة القيم (واحد منها انكماشي والاخر يمتلك نقطة صامدة) ثم البرهنة على ان المخططين تمتلكان نفس نقطة التقارب بالاضافة الى كونها نقطة صامدة مشتركة لكلا التطبيقين.

1-Introduction and Preliminaries

The notion of modular spaces was introduced by Nakano [1] in 1950 as a generalization of metric spaces and then redefined and modified by Musielak and Ortiz [2] in 1959. Many results about fixed points in these spaces were considered such as [3- 6]. Further and the most complete development of these theories are due to Orlicz, Mazur, Musielak, Luxemburg,Turpin [7] and their collaborators. In the present time the theory of modular and modular spaces is extensively applied, in particular, in the study of various Orlicz spaces [8], which in their turn have broad applications[9]. Recently, Abed [10] defined the best approximation and proved results about proximinal set, Chebysev set and existence invariant best approximation. see also [11] Now, recall the following

Definition 1.1[3] Let M be a linear space over $F(= R\ or\ ₵)$. A function$\gamma{:}M \rightarrow [0,\infty]$ is called modular if

(i) $\gamma\,(v) = 0$if and only if$v = 0$;
(ii) $\gamma(\alpha v) = \alpha(v)$ for $\alpha \in F$ with $|\alpha| =1$, for all $\alpha \in M$;
(iii) $\gamma(\alpha v + \beta u) \leq \gamma(v)+ \gamma(u)$Iff $\alpha, \beta \geq 0$, for all $u, v \in M$.

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If (iii) replaced by
(iii)` $\gamma(\alpha v+\beta u)\leq \alpha\gamma(v)+\beta\gamma(u)$, for $\alpha,\beta\geq 0$, $\alpha+\beta=1$, for all v,u $\in M$,
Then M_γ is called convex modular.
Definition 1.2 [3] A modular γ defines a corresponding modular space,$i.e.$,the spac M_γ given by

$$M_\gamma=\{v\in M:\gamma(\alpha v)\to 0\ whenever\ \alpha\to 0\}.$$

Remark 1.1[4] By condition (iii) above, if u= 0 then $\gamma(\alpha v)=\gamma\left(\frac{\alpha}{\beta}\beta v\right)\leq\gamma(\beta v)$, for all $\alpha,\beta\ in\ F$, $0<\alpha<\beta$. This shows that γ is an increasing function.
Example [12]
The Orlicz modular is defined for every measurable real function f by the formula**(1)** $\gamma(f)=\int\varphi(|\ f(t)|\ )\ dm\,(t)$
Where m denotes the Lebesgue measure in R and $\varphi:R\to[0,\infty)$ is continuous in which $\varphi(u)=0$ if and only if $u=o$ and $\varphi(t)\to\infty$ as $n\to\infty$. The modular space induced by the Orlicz modular γ_φ is called the Orlicz space L^φ
(2) The Musielak- Orlicz modular space . Let
$\gamma(f)=\int\varphi(\omega,\ f(\omega))\ d\mu(\omega)$,
Where μ is a $\sigma-$ finite measure on Ω , and $\varphi:\ \Omega\times R\to[0,\infty)$ satisfy the following :
(i) $\varphi(\omega,u)$ is a continuous even function of u which is nondecreasing for $u>0$, Such that $\varphi(\omega,0)=0$, $\varphi(\omega,u)>0$ for $u\neq 0$ and $\varphi(\omega,u)\to\infty$ as $n\to\infty$
(ii) $\varphi(\omega,u)$ is a measurable function of ω for each $u\in R$
Definition 1.3 [6] The γ-ball, $B_r(u)$centered at $u\in M_\gamma$with radius $r>0$as
$B_r(u)=\{\boldsymbol{v}\in M_\gamma;\gamma(u-v)<r\}$**.**
The class of all γ-balls in a modular space M_γ generates a topology which makes M_γ Hausdorff topological linear space. Every γ-ball is convex set, therefore every modular space is locally convex Hausdorff topological vector space [7].
Definition 1.5 [6] Let M_γ be a modular spase.
(a) A sequence $\{v_n\}\subset M_\gamma$ is said to be γ-convergent to $v\in M_\gamma$ and write $v_n\overset{\gamma}{\to}v$ if $\gamma(v_n-v)\to 0$ as n$\to\infty$.
(b) A sequence $\{v_n\}$ is called $\gamma-$ Cauchy whenever $\gamma(v_n-v_m)\to 0$ as $n,m\to\infty$.
(c) M_γ is called $\gamma-$ complete if any $\gamma-$ Cauchy sequence in M_γ is $\gamma-$ convergent.
(d) A subset $B\subset M_\gamma$ is called $\gamma-$ closed if for any sequence $\{v_n\}\subset B$ is $\gamma-$ convergent to a point in B
(e) A $\gamma-$ closed subset $B\subset M_\gamma$ is called $\gamma-$ compact if any sequence $\{v_n\}\subset B$ has a $\gamma-$ convergent subsequence.
(f) A subset $B\subset M_\gamma$ is said to be $\gamma-$ bounded if $daim_\gamma(B)<\infty$, where

$$daim_\gamma(B)=sup\{\gamma(v-u);\ v,u\in B\}$$

is called the $\gamma-$ diameter of B.
(g) The distance between $v\in M_\gamma$ and $B\subset M_\gamma$ is
$\gamma(v-B)=$inf $\{\gamma(v-u);u\in B\}$.
Definition 1.6 Let M_γ be a modular space, and A,B are two non – empty subsets in 2^{M_γ} .Let $H_\gamma(A,B)$ denotes Hausdorff distance of A and B that is defined as the following

$$H_\gamma(A,B)=max\{\ sup_{a\in A}\gamma(a-B),sup_{b\in B}\gamma(b-A)\}.$$

Lemma (1.1) Let M_γ be a modular space and let $A_n$and B_n real sequences in $CB(M_\gamma)$ Then we can choose a_n in A_n , b_n in B_n such that
$\gamma(a_n-b_n)=H_\gamma(A_n,B_n)+\varepsilon_n$, $lim_{n\to\infty}\varepsilon_n=0$ ... (1.1)
Definition (1.7) [13] or **[14]** Let A be a non-empty set and $T:A\longrightarrow 2^A$ be a multi-valued mapping, the point $x\in A$ is said to be a fixed point of T $\Leftrightarrow$ $x\in T(x)$. And x is a fixed point of a single-valued mapping T if and only if $x=T(x)$
Definition (1.8) [15] A mapping $T:A\to 2^{M_\gamma}$ such that

$$H_\gamma(Tu,Tv)\leq K\ \gamma(u-v)\quad for\ all\ u,v\in A\ \ldots(1.2)$$

is said to be multi – valued Lipschitz if there exists $k>0$ and multi – valued contraction (shortly, *m.v.c*) if $k<1$.

The following lemmas are required in the next section.

Lemma (1.2): [15] Let $\{a_n\} \subset R^+$ such that $a_{n+1} \leq (1 - \lambda_n)\ a_n + \sigma_n$ where $\lambda_n \in (0,1)$, for all $n \in N$, $\sum_{n=1}^{\infty} \lambda_n$ and $\sigma_n = o(\lambda_n)$. Then $lim_{n\to\infty} a_n = 0$.

Lemma (1.3): [16, 17] Let $\{a_n\} \subset R^+$ such that $a_{n+1} \leq (1 - \lambda_n)\ a_n + \lambda_n\ \varepsilon$
where $\lambda_n \in (0,1)$, for all $n \in N$, $\sum_{n=1}^{\infty} \lambda_n = \infty$ *and* $\varepsilon > 0$ is fixed number. Then $0 \leq lim_{n\to\infty} sup\, a_n \leq \varepsilon$.

exists a δ(ε) > 0, such that

if $\gamma(v) = \gamma(u) = 1$ *and* $(v - u) \geq \varepsilon$, *then* $\gamma\left(\frac{1}{2}(v + u)\right) \leq 1 - \delta$.

2. Main Results

Let A be a non-empty subset of M_γ , and $T: A \longrightarrow 2^A\ u_0 \in A$. If the sequence $\{u_n\} \subset A$ is defined by

$$u_{n+1} \in (1 - a_n)u_n + a_n T v_n$$

$$v_n \in (1 - \beta_n)\, u_n + \beta_n\, Tu_n,\ \forall n \geq 0 \qquad \text{…..(1.3)}$$

or

$$u_{n+1} = (1 - a_n)u_n + a_n \mu_n\ , \qquad \mu_n \in Tv_n\ ,\ \forall n \geq 0$$

$$v_n = (1 - \beta_n)\, u_n + \beta_n \xi_n\ , \qquad \xi_n \in Tu_n,\ \forall n \geq 0 \qquad \text{…(1.4)}$$

Two convergence results for iteration (1.4) are established dealing with contraction. We start with the following needed lemma:-

Theorem (2.2) : Let M_γ be a complete modular space and $\emptyset \neq A \subseteq M_\gamma$ A is a nonempty closed. If $T : A \to CB(A)$ is *m.v.c* mapping, then T has a fixed in A

Proof

Select $u_0 \in A$ and $u_1 \in T(u_0)$ By Lemma (1.1) there must exist $u_2 \in T(u_1)$ such that

$$\gamma\,(u_1 - u_2) \leq H_\gamma\,(T\,(u_0)\,,\ T\,(u_1)) + K$$

similary , there exists $u_3 \in T(u_2)$ such that

$$\gamma\,(u_2 - u_3) \leq H_\gamma\,(T\,(u_1)\,,\ T\,(u_2)) + K^2$$

By induction, there is the sequence $\{u_n\}$ *in* A such that $\forall i, \in N$, $u_{i+1} \in T\,(u_i)$ and

$$\gamma\,(u_i - u_{i+1}) \leq k^i\, d\,(u_0\,, u_1) + i\, k^i$$

Therefore ,

$$\sum_{i=0}^{\infty} \gamma\,(u_i - u_{i+1}) \leq \gamma\,(u_0 - u_1)\,(\textstyle\sum_{i=0}^{\infty} k^i) + \sum_{i=0}^{\infty} i\, k^i$$

This prove that $\{u_n\}$ is a cauchy sequence. so, since A is complete there exist $u \in A$ such that $\lim_{n\to\infty} u_n = u$. Also, the continuity of T lied to $lim_{n\to\infty} H_\gamma\,(T(u_n), T(u)) = 0$

Since $u_n \in T\,(u_{n-1})$, then

$$lim_{n\to\infty} dis\,t(u_n\ , T(u)) = lim_{n\to\infty} inf\ \{\gamma\,(u_n - v) : v \in T\,(u)\} = 0$$

This implies that

$dist\,(u\,, T(u)) = inf\ \{\gamma\,(u - v) : v \in T\,(u)\} = 0$ And, the closeness of $T\,(u)$, it must be the case that $u \in T(u)$

Theorem (2.3) Let M_γ be a complete convex real modular space, let $\emptyset \neq A \subseteq M_\gamma$ and A be convex and closed subset of M_γ and T: A⟶ CB(A) be a *m.v.c* mapping. Let $\{\alpha_n\}$, $\{\beta_n\} \subseteq (0,1)$satisfying:

(i) $0 < \alpha_n, \beta_n < 1$

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$

then for $u_0 \in A$, the sequence $\{u_n\}$ in (1.4) converges to a fixed point of T.

Proof:

The existence of the fixed point follows from the Theorem (2.2). Let $p \in A$ be a fixed point of T. By conditions (1.4), (1.1) and (1.2), we get

$$u_{n+1} - p = (1 - \alpha_n)u_n + \alpha_n\, \mu_n - p \qquad where\ \ \mu_n \in Tv_n$$

$$\gamma\,(u_{n+1} - p) = \gamma((1 - \alpha_n)u_n + \alpha_n\, \mu_n - ((1 - \alpha_n)p + \alpha_n\, p))$$

$$= \gamma\,((1 - \alpha_n)(u_n - p) + \alpha_n\,(\mu_n - p))$$

$$\leq (1 - \alpha_n)\gamma\,(u_n - p) + \alpha_n\, \gamma\,(\mu_n - p)$$

$\leq (1- \alpha_n) \gamma (u_n - p) + \alpha_n H_\gamma (T v_n , Tp) + \alpha_n \varepsilon_n$

$\leq (1- \alpha_n) \gamma (u_n - p) + \alpha_n k \gamma (v_n - p) + \alpha_n \varepsilon_n$

Again , from conditions (1.4), (1.1) and (1.2), we get

$\gamma(u_{n+1} - p) = (1- \alpha_n) \gamma (u_n - p) + \alpha_n k \gamma ((1- \beta_n)u_n + \beta_n \varepsilon_n - (1- \beta_n)p - \beta_n p) + \alpha_n \varepsilon_n$, *where* $\varepsilon_n \in Tu_n$

$\leq (1- \alpha_n) \gamma (u_n - p) + \alpha_n k (1- \beta_n)\gamma (u_n - p) + \alpha_n k\beta_n \gamma (\varepsilon_n - p) + \alpha_n \varepsilon_n$

$\leq (1-\alpha_n) \gamma (u_n - p) + \alpha_n k(1- \beta_n)\gamma (u_n - p) + \alpha_n k\beta_n H_\gamma (T u_n ,Tp) + \alpha_n \varepsilon_n + \alpha_n k \beta_n \varepsilon_n$

$\leq (1-\alpha_n) \gamma (u_n - p) + \alpha_n k(1- \beta_n)\gamma (u_n - p) + \alpha_n k^2\beta_n \gamma (u_n - p) + \alpha_n \varepsilon_n + \alpha_n k \beta_n \varepsilon_n$

$\leq \big((1-\alpha_n) + \alpha_n k \big)\gamma (u_n - p) + \alpha_n \varepsilon_n + \alpha_n k \beta_n \varepsilon_n$

Thus,

$\gamma (u_{n+1} - p) = (\big(1- \alpha_n (1-k)\big)p (u_n - p) + 0 (\alpha_n)$

Let us denote

$a_n = \gamma (u_n - p)$

$\lambda_n = \alpha_n (1-k) \in (0 ,1) \qquad \forall n \geq 0$

And using lemma (1.2), we obtain $\lim_{n\to\infty} a_n = 0$. which implies that $\lim_{n\to\infty} \gamma(u_n - p) = 0$, thus $\lim_{n\to\infty} u_n = p$

Theorem (2.4): Let M_γ be a complete convex real modular space, let $\emptyset \neq A \subseteq M_\gamma$ and A be a convex and closed subset of M_γ. Let $\varepsilon > 0$ be a fixed number and $T,S : A \longrightarrow C(A)$ be two multi-valued mappings. If S is a *m.v.c* mapping and $\lim_{n\to\infty} w_n = p$, where p is a fixed point of T, for any given $w_0 \in A$, the sequence $\{w_n\}$ in (1.4) with $\{\alpha_n\}$, $\{\beta_n\}$ satisfying

(i) $0 < \alpha_n ,\beta_n < 1$

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$

and if $H (T z ,S z) \leq \epsilon$, for all $z \in A$, then p is a common fixed point of T *and* S.

Proof:

The existence of the fixed point q of S follows from the Theorem (2.2). For the mapping S from (1.4) be

$u_{n+1} = (1-\alpha_n) u_n + \alpha_n \theta_n$, *where* $\theta_n \in Sv_n$, $\forall n \geq 0$

$v_n = (1-\beta_n) u_n + \beta_n \xi_n$, *where* $\xi_n \in Su_n$, $\forall n \geq 0$

So, from conditions (1.1) and (1.2), we get

$w_{n+1} - u_{n+1} = (1-\alpha_n) (w_n - u_n) + \alpha_n(\mu_n - \theta_n)$, *where* $\theta_n \in Sv_n$

$\gamma(w_{n+1} - u_{n+1}) = \gamma ((1-\alpha_n) (w_n - u_n) + \alpha_n (\mu_n - \theta_n))$

$\leq (1-\alpha_n) \gamma(w_n - u_n) + \alpha_n \gamma(\mu_n - \theta_n)$

$= (1-\alpha_n) \gamma(w_n - u_n) + \alpha_n \gamma (\mu_n - \varpi_n + \varpi_n - \theta_n)$, *where* $\varpi_n \in Sg_n$

$\leq (1-\alpha_n) \gamma(w_n - u_n) + \alpha_n \gamma (\mu_n - \varpi_n) + \alpha_n \gamma (\varpi_n - \theta_n)$

$\leq (1-\alpha_n) \gamma(w_n - u_n) + \alpha_n H_\gamma (Tg_n , Sg_n) + \alpha_n H_\gamma (S g_n ,Sv_n) + \alpha_n b_n + \alpha_n d_n$

$= (1-\alpha_n) \gamma(w_n - u_n) + \alpha_n \varepsilon + \alpha_n k \gamma(g_n - v_n) + \alpha_n b_n + \alpha_n d_n$

$= (1-\alpha_n) \gamma(w_n - u_n) + \alpha_n \varepsilon + \alpha_n k \gamma ((1-\beta_n) w_n + \beta_n \xi_n - (1-\beta_n) u_n + \beta_n \zeta_n)). + \alpha_n b_n + \alpha_n d_n$

where $\xi_n \in Tw_n , \zeta_n \in Su_n$

$= (1-\alpha_n) \gamma(w_n - u_n) + \alpha_n \varepsilon + \alpha_n k \gamma ((1-\beta_n)(w_n - u_n) + \beta_n(\xi_n - \zeta_n) + \alpha_n b_n + \alpha_n d_n$

$\gamma(w_{n+1} - u_{n+1}) \leq (1-\alpha_n) \gamma(w_n - u_n) + \alpha_n \varepsilon + \alpha_n k(1-\beta_n) \gamma(w_n - u_n) + \alpha_n k\beta_n \gamma(\xi_n - \zeta_n) + \alpha_n b_n + \alpha_n d_n$

$= (1-\alpha_n) \gamma(w_n - u_n) + \alpha_n \varepsilon + \alpha_n k(1-\beta_n) \gamma(w_n - u_n) + \alpha_n k\beta_n \gamma (\xi_n - e_n + e_n - \zeta_n) + \alpha_n b_n + \alpha_n d_n$, *where* $e_n \in S w_n$

$\leq (1-\alpha_n) \gamma(w_n - u_n) + \alpha_n \varepsilon + \alpha_n k(1-\beta_n) \gamma(w_n - u_n) + \alpha_n k\beta_n \gamma$

$(\xi_n - e_n) + \alpha_n\, k\beta_n\, \gamma(\, e_n - \zeta_n) \;+\; \alpha_n\, b_n +\; \alpha_n\, d_n$

$\gamma(w_{n+1} \;-\; \mathrm{u}_{n+1}) \leq (1 - \alpha_n) \quad \gamma(w_n \;-\; \mathrm{u}_n\;) + \alpha_n\, \varepsilon + \quad \alpha_n\, k(1 - \beta_n\,)\, \gamma(\, w_n \;-\; u_n\;) + \alpha_n\, k\beta_n H_\gamma(T\, _{w_n,}\; S\, _{w_n}) + \alpha_n\, k\beta_n H_\gamma(\, S\, w_{n\,,}\, Sw_n\,). + \alpha_n\, b_n +\; \alpha_n\, d_n + \alpha_n\, k\beta_n c_n + \alpha_n\, k\beta_n\; f_n$

$\leq \big(1 - \alpha_n\big)\; \gamma(w_n \;-\; \mathrm{u}_n\;) + \alpha_n\, \varepsilon + \alpha_n\, k(1 - \beta_n\,)\gamma(\, w_n \;-\; u_n\;) + \alpha_n\, k\beta_n\; \varepsilon + \alpha_n\; k^2\beta_n\gamma(\, w_n \;- u_n\;) +\; \alpha_n\, b_n +\; \alpha_n\, d_n\; + \alpha_n\, k\beta_n c_n + \alpha_n\, k\beta_n\; f_n$

$\leq ((1 - \alpha_n) + \alpha_n\, k\;\; (1 - \beta_n\,) + \alpha_n k^2\beta_n\;\;)\;\; \gamma(\, w_n \;-\; u_n\;\;) + (\,1 + k\beta_n)\, \alpha_n\, \varepsilon\; + \alpha_n\, b_n +\; \alpha_n\, d_n\; + \alpha_n\, k\beta_n c_n + \alpha_n\, k\beta_n\; f_n$

$\leq ((1 - \alpha_n) + \alpha_n\, k\; (1 - \beta_n\,) + \alpha_n\, k\beta_n\,)\, \gamma(\, w_n \;-\; u_n\;\;) + 2\; \alpha_n\, \varepsilon + \alpha_n\, b_n +\; \alpha_n\, d_n\; + \alpha_n\, k\beta_n c_n + \alpha_n\, k\beta_n\; f_n$

Thus,

$\gamma(w_{n+1} \;-\; \mathrm{u}_{n+1}) \leq\; ((1 - \alpha_n(\,1\; - k\,))\gamma(\, w_n \;-\; u_n\;\;) + 2\; \alpha_n\, \varepsilon + \alpha_n\, b_n +\; \alpha_n\, d_n\; + \alpha_n\, k\beta_n c_n + \alpha_n\, k\beta_n\; f_n$

Finally

$\alpha_n\, b_n +\; \alpha_n\, d_n\; + \alpha_n\, k\beta_n c_n + \alpha_n\, k\beta_n\; f_n \;\to\; 0 \qquad as \quad n \to \infty$

Let us denote

$a_n \;=\; \gamma\,(w_n \;\;-\;\; \mathrm{u}_n\;), \lambda_n = \alpha_n(\,1\; - k\,)\;\in\;(\,0\,,1\,)\quad ;\, \forall\, n \geq 0$

and from Lemma (1.3) it follows that

$$0 \leq \limsup_{n\to\infty} a_n \leq \varepsilon$$

Since ε is arbitrary $\lim_{n\to\infty} sup a_n = 0$ and so $\lim_{n\to\infty} a_n = 0$, which implies that, and

$\lim_{n\to\infty} \gamma\,(\,x_n - u_n\,) = 0\;\; , and\;\; \lim_{n\to\infty} \gamma(x_n - u_n) = \gamma(\,p - q\,)\;\, so\, \gamma\,(\,p - q\,) = 0, hence\;\; p = q\,.$

Corollary (1):

Consider we have M_γ, A and Tas in Theorem (2.2). For any $u_0 \in A\,,\;\; u_{n+1} = (1 - \alpha_n)u_n + \alpha_n v_n,\;\; v_n \in Tu_n\,, n \geq 0$

and $\{\alpha_n\,\} \subset R^+$ satisfying:

(i) $0\; <\; \alpha_n <\; 1$

(ii) $\sum_{n=1}^{\infty} \alpha_n \;=\; \infty$

the iteration sequence $\{u_n\}$ converges to a fixed point of T.

Proof: Follows from Theorem (2.3) with $\beta_n = 0$. $n \geq 0$

By the proof of Theorem (2.2), we have that the iteration sequence $u_{n+1} = Tu_n$ converges to a fixed point of a $m.v.c.$ mapping T.

Corollary (2):

Let M_γ , A and Tas in Theorem (2.3). Let $\varepsilon\, >\, 0$ be a fixed number. If $\mathrm{T,S} : \mathrm{M} \longrightarrow \mathrm{C(M)}$ are two $m.v.c.$ mappings, $\{u_n\}$ defined by condition (1.4) with *{*α_n *}, {*β_n *}* satisfying

(i) *0 <*α_n *,*β_n *< 1*

(ii) $\sum_{n=1}^{\infty} \alpha_n \;=\; \infty$

and if H_γ $(\;T\;z\,, S\;z\,)$ for all *z*∈ *M*, then p is a common fixed point of T and S.

Proof:

Since T, S are $m.v.c.$ mappings then T, S have a fixed point p, q, and by Theorem (2.3) the iteration $\{u_n\,\}$ in condition (1.4) converges to p and then by Theorem (2.4)

$\gamma(\,p - q\,)\; = 0$

Hence

$p \;=\; q$

So, p *is* a common fixed point $of\, T$ and S. ■

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