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Almost Pure Ideals (Submodules) and Almost Regular Rings (Modules)

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Abstract

Let R be a commutative ring with identity and M be a unitary R -module. In this work we present almost pure ideal (submodule) concept as a generalization of pure ideal (submodule). Also, we generalize some properties of almost pure ideal (submodule). The study is almost regular ring (R -module).

Keyword: pure ideal, regular ring, pure submodule, regular module, almost pure submodule, almost regular module.

المثاليات (المقاسات الجزئية) النقية تقريبا و الحلقات (المقاسات) المنتظمة قريبا

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الخلاصة

لتكن R حلقة ابدالية بمحايد، ولتكن M مقياس R بمحايد. في هذا العمل قدمنا مفهوم المثاليات النقية (المقاسات الجزئية النقيه) تقريبا كتعميم لمفهوم المثاليات النقية (المقاسات الجزئية النقيه). كذلك عممنا بعض الخصائص للحلقات المنتظمة (المقاسات المنتظمة) تقريبا.

1. Introduction

Let M be a unitary R -module and let R be commutative ring with identity. A submodule N of an R -module M is called pure if $N \cap IM = IN$, for each ideal I of a ring R , E, Anderson and Fuller [1]. Fieldhous (1969) in [2] defined N to be pure in M if $rM \cap N = rN$ for each $r \in R$. And defined a ring R is regular if every ideal I is pure. Although Fieldhous generalize regular rings to regular modules in [2], Ware (1971) in [3] and Zelmanowliz (1972) in [4], also study regular modules. Nuhad S. Al-Mothafar and Ghaleb A. Humod in [5] study 2- pure submodules and 2- regular modules they define N to be 2-pure in M if $N \cap I^2M = I^2N$, for each ideal I in R .

In this paper we introduce an almost pure ideals (submodules) concept as a generalization of pure ideals (submodules). An ideal I of a ring R is said to be almost pure if $I \cap J(R) = IJ(R)$, where $J(R)$ is the Jacobson radical of R . A submodule N of an R -module M is called almost pure if $N \cap J(R)M = J(R)N$, we study some properties of almost pure submodules. Also, we get some results of almost pure submodules like a (Prop. 3.4) let M be an R -module and N be an almost pure submodule of M . If S is a submodule of M containing N , then N is an almost pure submodule of S .

And we present almost regular rings (modules) concepts as generalization of regular rings (modules) we generalize some properties of regular ideals (modules) to almost regular rings (modules).

This work includes three sections. In section two, we introduce almost pure ideals and almost regular rings concepts as a generalization of pure ideals and regular rings, we provided in (Prop. 2.9). If R is A -regular integral domain such that $J(R) \neq R$, then R is semisimple ring.

In section three, we introduce almost pure submodules and almost regular modules concepts as a generalization of pure submodules and regular modules, we present (Prop. 3.16) let M be an almost regular R -module contain a non torsion element, then R is almost regular ring.

Finally in section three, we see in proposition (3.15) that if M is almost regular R -module, then $\forall 0 \neq x \in M$ such that $\text{ann}(x) \subseteq J(R)$, then $R/\text{ann}(x)$ is almost regular ring.

2. Almost Pure Ideals and Almost Regular Rings

In this section, we introduce a generalization for pure ideals concept namely almost pure ideal. First we recall that an ideal I of a ring R is called pure if $I \cap J = IJ$ for each ideal J of R . It is well know that a ring R is (Von Neumann) regular if and only if every ideal of R is pure, equivalently for every $a \in R$ there exist $b \in R$ such that $a = aba$, [2]. We start by the following definition.

Definition (2.1):

An ideal I of a ring R is called almost pure (for short A -pure) ideal of R if $I \cap J(R) = IJ(R)$, where $J(R)$ is the Jacobson radical of R .

Remarks and examples (2.2):

1. It is clear that every pure ideal of a ring R is A -pure, but the converse is not true in general, for example: In a ring Z , the ideal $I = \langle 3 \rangle$ is A -pure, since $\langle 3 \rangle \cap \{0\} = \langle 3 \rangle \cdot \{0\}$, but $\langle 3 \rangle$ is not pure in the ring Z .
2. Every ideal generated by idempotent is A -pure. Since every ideal that generated by idempotent is pure [6], hence is A -pure by remark (1).
3. If I and J are A -pure ideals of a ring R , but $I \cap J$ need not be A -pure ideal of R . To show that if $R = Z_4 \oplus Z_2$ and $I = Z_4 \oplus 0$, $J = Z(\bar{1}, \bar{1})$ are A -pure ideals of R but $I \cap J = \{(0, \bar{0}), (2, \bar{0})\}$ is not A -pure in R since $(I \cap J) \cap J(R) = \{(0, \bar{0}), (2, \bar{0})\}$ but $(I \cap J) \cdot J(R) = \{(0, \bar{0})\}$
4. It is clear that if $J(R) = \{0\}$, then every ideal in R is A -pure. In the ring Z every ideal is A -pure.
5. It is clear that $\{0\}$ and R are always A -pure ideals of any ring R .

Recall that a ring R is called regular if every ideal in R is pure, [2]. Now, we have the following:

Definition (2.3):

A ring R is called almost regular (for short A -regular) if every ideal in R is A -pure. (i.e.) $I \cap J(R) = IJ(R) \forall I \subseteq R$.

Remarks and examples (2.4):

- 1- Every regular ring is A -regular. But the converse is not true in general the ring Z is A -regular. Since $J(Z) = 0$, but not regular.
- 2- The rings Q, Z_p^∞ , are not A -regular, since $J(Q) = Q$ and $J(Z_p^\infty) = Z_p^\infty$.

Theorem (2.5):

A ring R is A -regular if and only if $J(R)$ is pure ideal in R .

Proof

Let I be an ideal of R . Since R is A -regular, hence I is A -pure in R . (i.e) $I \cap J(R) = IJ(R)$. Thus $J(R)$ is pure in R .

The converse is clear.

Proposition (2.6)

Let $\varphi: R \rightarrow R'$ be an R -epimorphism such that $\ker \varphi \subseteq J(R)$. If R is A -regular ring, then R' is A -regular ring.

Proof: Let I' be an ideal in a ring R' , we have to show $I' \cap J(R') = I'J(R')$. Now, $I' = \varphi(I)$, where I is an ideal in a ring R , since R is A -regular ring then $I \cap J(R) = IJ(R)$. Since φ is an epimorphism with $\ker \varphi \subseteq J(R)$, then $\varphi(I \cap J(R)) = \varphi(I) \cap \varphi(J(R))$. Now, $\varphi(I) \cap \varphi(J(R)) = \varphi(I) \cap J(R')$

$$= \varphi(IJ(R))$$

$$=$$

$\varphi(I) \cdot \varphi(J(R))$ since φ is epimorphism and $\ker \varphi \subseteq J(R)$.

But $\varphi(J(R)) = J(R')$ since φ is epimorphism and $\ker \varphi \subseteq J(R)$, [7]. Thus $I' \cap J(R') = I'J(R')$.

Proposition (2.7)

Let R be a ring and I be an ideal of R contained in $J(R)$. If R/I is A -regular ring, then R is A -regular.

Proof

To show R is A -regular ring, we have to show $K \cap J(R) = KJ(R)$ for all ideal K of R , since R/I is A -regular ring and $\frac{K}{I}$ is an ideal of R/I , then $\frac{K}{I} \cap J(R/I) = (\frac{K}{I})J(R/I)$ since $\varphi: R \rightarrow R/I$ is an epimorphism and $\ker \varphi \subseteq J(R)$ then $\varphi(J(R)) = J(\varphi(R))$, [7]. hence $(K/I) \cap ((J(R))/I) = (K/I)((J(R))/I)$, then $((K \cap J(R))/I) = ((KJ(R))/I)$, since $I \subseteq J(R)$. Thus $K \cap J(R) = KJ(R)$.

Proposition (2.8):

A finite direct sum of A -regular rings R_i such that $R_i \subseteq J(R) \forall i = 1, 2, \dots, n$ is A -regular.

Proof

Let $\{R_i\}$, $i=1, 2, \dots, n$ be a finite number of A -regular ring. Let $R=R_1 \oplus R_2 \oplus \dots \oplus R_n$ be their direct sum, to show R is A -regular. Let $n=2$, then $R=R_1 \oplus R_2$. It is clear then $R/R_1 \cong R_2$, since R_2 is A -regular by proposition(2.6) R/R_1 is A -regular, hence by proposition(2.7) R is A -regular. Using mathematical induction. Assume that the statement is true for $n=k$ to show that it is true for $k+1$, (i.e) $R=R_1 \oplus \dots \oplus R_k$ is A -regular, and to show that $R=R_1 \oplus \dots \oplus R_k \oplus R_{k+1}$ is A -regular since $R/R_1 \oplus \dots \oplus R_k \cong R_{k+1}$ and R_{k+1} is A -regular, then $R/R_1 \oplus \dots \oplus R_k$ is A -regular. proposition (2.7) implies $R=R_1 \oplus R_2 \oplus \dots \oplus R_k \oplus R_{k+1}$ is A -regular.

Recall that if R is integral domain then R is pure simple, where R is pure simple if it has no pure ideal except $\{0\}$ and R , [8]. We know that if R is regular integral domain, then R is a field. But this is not true for A -regular, since Z is A -regular integral domain but not field.

Proposition (2.9)

If R is A -regular integral domain such that $J(R) \neq R$, then R is semisimple ring.

proof: Since R is A -regular, then by(2.8) $J(R)$ is pure ideal in R . But R is integral domain, hence R has no pure ideal except $\{0\}$ and R . Since $J(R) \neq R$, hence $J(R) = 0$. Thus R is semisimple

Proposition (2.10)

If I is a prime ideal in R and R is A -regular ring such that $J(R/I) \neq R/I$, then R/I is semisimple ring.

Proof

Since I is prime ideal in R/I integral domain, and by (2.6) R/I is A -regular. Since $J(R/I) \neq R/I$, then by proposition (2.9) R/I is semisimple.

3. Almost Pure Submodules and Almost Regular Modules

Recall that a sub module N of an R -module M is called pure if $N \cap IM = IN$, for each ideal I of a ring R , [1]. We start this section by the following definition.

Definition (3.1)

A submodule N of an R -module M is called almost pure, (for short A -pure), if $N \cap J(R)M = J(R)N$, where $J(R)$ is the Jacobson radical of a ring R .

Remarks and Examples (3.2)

1. It's clear that every pure submodule is A -pure, but the converse is not true in general. For example the submodule $\{\bar{0}, \bar{2}\}$ in Z_4 as Z -module is A -pure, since $\{\bar{0}, \bar{2}\} \cap J(Z)Z_4 = J(Z)\{\bar{0}, \bar{2}\} = \{\bar{0}, \bar{2}\} \cap \{0\}Z_4 = \{0\}.\{\bar{0}, \bar{2}\} = \{\bar{0}\}$. But $\{\bar{0}, \bar{2}\}$ is not pure in Z_4 as Z -module.
2. If N and K are A -pure sub modules in an R -module M , then $N \cap K$ need not be A -pure, as the following example show. If $M=Z_4 \oplus Z_2$ as Z_4 -module and $A=Z_4 \oplus 0$, $B=Z_4(1, 1)$ are A -pure submodules of M but $A \cap B = \{(0, 0), (2, 0)\}$ is not A -pure in M . Since $(2, 0) \in (A \cap B) \cap J(Z_4)(Z_4 \oplus Z_2)$ but $(2, 0) \notin J(Z_4)(A \cap B)$.
3. It is clear that every submodule in Z as Z -module is A -pure.
4. It's clear that every direct summand of any R -module M is A -pure submodule of M since every direct summand is pure submodule, hence is A -pure by Remark(1), but the converse is not true, for example the submodule $\{\bar{0}, \bar{3}, \bar{6}\}$ of the module Z_9 as Z -module is A -pure since $\{\bar{0}, \bar{3}, \bar{6}\} \cap J(Z)Z_9 = J(Z)\{\bar{0}, \bar{3}, \bar{6}\}$ but not direct summand.

Proposition (3.3)

Let M be an R -module and N be an A -pure submodule of M . If S is A -pure in N , then S is an A -pure submodule of M .

Proof

Since N is A -pure in M , then $N \cap J(R)M = J(R)N$ and S is A -pure in N , then $S \cap J(R)N = J(R)S$. Now, $S \subseteq N$, implies $J(R)S = S \cap J(R)N = S \cap (N \cap J(R)M) = (S \cap N) \cap J(R)M = S \cap J(R)M$.

Proposition (3.4)

Let M be an R -module and N be an A -pure submodule of M . If S is a submodule of M containing N , then N is an A -pure submodule of S .

Proof

Since N is A -pure submodule in M , then $N \cap J(R)M = J(R)N$ and since $N \subseteq S \subseteq M$ implies $N \cap J(R)S = N \cap (J(R)S \cap J(R)M) = (N \cap J(R)M) \cap J(R)S = J(R)N \cap J(R)S = J(R)N$.

Proposition (3.5): Let M be an R -module and N be an A -pure submodule of M . If K is a submodule of N , then $\frac{N}{K}$ is a A -pure submodule of $\frac{M}{K}$.

Proof

$$\begin{aligned} \left(\frac{N}{K}\right) \cap J(R) \left(\frac{M}{K}\right) &= \left(\frac{N}{K}\right) \cap \left(\frac{J(R)M+K}{K}\right) \\ &= \frac{N \cap (J(R)M+K)}{K} = \frac{(N \cap J(R)M) + K}{K} \quad \text{by module law} \\ &= \frac{J(R)N+K}{K} = J(R) \left(\frac{N}{K}\right). \end{aligned}$$

Recall that an R -module M is called an F -regular module if every submodule of M is pure, [9]. Now, we have the following definition.

Definition (3.6)

An R -module M is called A -regular module if every submodule of M is A -pure.

Remarks and Examples (3.7):

1. It is clear that every F -regular is A -regular R -module but the convers is not true in general. For example Z_4 as Z -module is A -regular since every submodule of Z_4 is A -pure but Z_4 is not F -regular, see remark and examples (3.2) (1).
2. The Z -module Z is A -regular since every submodules of Z is A -pure. But Z as Z -module is not F -regular.
3. The Z_9 as Z -module is A -regular since every submodule of Z_9 is A -pure. But Z_9 is not F -regular since $\{\bar{0}, \bar{3}, \bar{9}\}$ is not pure, see remark and examples (3.2) (1).
4. The Z_{12} as Z -module is A -regular since every submodule of Z_{12} is A -pure, but Z_{12} is not F -regular since the submodule $\langle \bar{2} \rangle$ is not pure.
5. Q as Z -module is A -regular since $Q \cap J(Z)Q = J(Z)Q$, but Q as Z -module is not F -regular.

Theorem (3.8):

Let M be an R -module. The following statements are equivalent:

1. M is A -regular module.
2. Every cyclic submodule of M is A -pure in M .
3. Every finitely generated submodule of M is A -pure in M .

Proof

(1) \rightarrow (2) it follows by definition (3.6).

(2) \rightarrow (1) Assume that every cyclic submodule of M is A -pure and let N be any submodule of M . Let $x \in N \cap J(R)M$, then $x \in N$ and $x \in J(R)M$ then $x \in \langle x \rangle \cap J(R)M = J(R)\langle x \rangle \subseteq J(R)N$.

(1) \rightarrow (3). It follows by definition (3.6).

(3) \rightarrow (2). It is clear.

The following proposition shows that the factor module of A -regular module is A -regular.

Proposition (3.9):

Let M be an R -module, then M is A -regular if and only if M/N is A -regular for every submodule N of M .

Proof

Let N be a submodule of an R -module M and K be any submodule of M containing N . Since M is A -regular, then K is A -pure in M , hence K/N is A -pure in M/N proposition (3.5). Thus M/N is A -regular. The converse is easily by taking $N=0$.

Corollary (3.10)

Let M and M' be R -modules and $f: M \rightarrow M'$ is an R -epimorphism. If M is A -regular R -module, then M' is A -regular.

Proof

Since $f: M \rightarrow M'$ be an R -epimorphism and M is A -regular R -module, then $M/\ker f$ is A -regular R -module by proposition (3.9). But $M/\ker f$ is isomorphic to M' by the first isomorphism theorem. Thus M' is A -regular.

Corollary (3.11)]

Every submodule of A -regular module is A -regular module.

Proof

Let N be a submodule of an A -regular M . To show that N is A -regular R -module, let K be any submodule of N . Since M is A -regular. Hence we have:

$$\begin{aligned} K \cap J(R)N &= K \cap (N \cap J(R)M) && \text{since } N \text{ is } A\text{-pure in } M \\ &= (K \cap N) \cap J(R)M \\ &= K \cap J(R)M && \text{since } K \text{ is } A\text{-pure in } M \\ &= J(R)N \end{aligned}$$

Therefore K is A -pure in N which implies that N is A -regular.

Recall that an R -module M is called F -regular if and only if $\forall x \in M$ and $\forall r \in R$, $\exists t \in R$ such that $rx = rtrx$, [6].

Proposition (3.12):

If M is A -regular R -module, then for every nonzero element x in M and for each $r \in J(R)$, there exist $t \in R$ such that $rx = rtrx$.

Proof

Let $0 \neq x \in M$ and $r \in J(R)$. Since $rx \in J(R)M$ and $rx \in \langle rx \rangle$ implies $rx \in J(R)M \cap \langle rx \rangle$. But M is A -regular. $\langle rx \rangle \cap J(R)M = J(R)\langle rx \rangle$. Hence $rx \in J(R)\langle rx \rangle$, which implies that $rx = rtrx$ where $t \in R$.

Proposition (3.13):

Let M be an R -module is each nonzero element x in M and for every $r \in J(R)$, $rx = rtrx$ for some $t \in R$, then M is A -regular.

Proof

Let N be a submodule of a module M , to show $N \cap J(R)M = J(R)N$. Let $y \in N \cap J(R)M$, then $y \in N$ and $y \in J(R)M$, hence $y = rm$ where $r \in J(R)$ and $m \in M$, $y = rm = rtrm$, for some $t \in R$. Thus $y \in J(R)N$, which implies that M is A -regular.

corollary (3.14)

Let R be an A -regular ring, then for each $r \in J(R)$, $r = rtr$ for some $t \in R$.

Proposition (3.15)

If M is A -regular R -module, then $\forall 0 \neq x \in M$ such that $\text{ann}(x) \subseteq J(R)$, then $R/\text{ann}(x)$ is A -regular ring.

Proof

Let $0 \neq x \in M$ and $\bar{r} \in J(R/\text{ann}(x)) = \varphi(J(R))$ where $\varphi: R \rightarrow R/\text{ann}(x)$ is an epimorphism, then there exist $r \in J(R)$ such that $\varphi(r) = \bar{r}$. Since M is A -regular R -module, $r \in J(R)$, $\exists t \in R$ such that $rx = rtrx$, then $(r-rtr)x = 0$, then $r-rtr \in \text{ann}(x)$, hence $r + \text{ann}(x) = rtr + \text{ann}(x)$, which implies that $\bar{r} = \bar{r} \bar{t} \bar{r}$. Thus $R/\text{ann}(x)$ is A -regular ring.

Recall that $T(M) = \{m \in M : \text{there exist } r \in R \text{ such that } r.m = 0\}$, $T(M)$ = the set of all torsion elements. If $T(M) = M$, then M is called torsion R -module, [10].

Proposition (3.16)

Let M be an A -regular R -module contains a non-torsion element, then R is A -regular ring.

Proof

Let $x \in M$ be a non-torsion element, then $[0:x] = 0 = \text{ann}(x)$, so by (3.15), $R/\text{ann}(x) \cong R$ is A -regular by (3.10).

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