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Coprime Factor Model Reduction of Multidimensional Systems

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Abstract

In this work, the right coprime factor for balancing and truncating unstable systems is extended to parameter-varying multidimensional systems employing recent works on coprime factor model reduction of one-dimensional uncertain systems. Since the balanced truncation method cannot be applied directly to unstable systems, state feedback gains should be computed and incorporated in order to stabilize the given system and to be able to apply the balanced truncation technique via defining the so-called stable coprime factor, as coprime factorization overcomes the stability condition required for model reduction. Parameter-dependent state feedback and parameter-dependent Gramians are considered in this work yielding less conservative techniques. In addition, the Gramians are defined as block diagonal matrices, which are partitioned according to the structure of the multidimensional systems. The application to a simulation example demonstrates the applicability and validity of the proposed reduction approach leading to small error bounds between the full and the reduced models.

Keywords: Multidimensional systems; Linear parameter-varying systems; Model reduction; Coprime factor; Stabilizable systems.

تقليص انظمة متعددة الابعاد بطريقة العامل كوبرايم

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الخلاصة:

في هذا العمل, تم عرض طريقة لتقليص الانظمة الغير مستقرة. حيث تم تطوير هذه الطريقة من الانظمة احادية الابعاد الى الانظمة المتعددة الابعاد والغير ثابتة. استخدمت طريقة الاستقطاع المتوازن والتي تعمل مع الانظمة المستقرة فقط, لذلك تم تصميم النظام بشكل يمكن تحويله الى نظام مستقر (يعرف بأسم النظام القابل للتحول الى حالة الاستقرارية) و بالتالي امكانية تطبيق طريقة الاستقطاع المتوازن من اجل تقليص النظام المتحول الى نظام مستقر. الانظمة المستخدمة غير ثابتة ومعتمدة على المتغيرات الموجودة في الحياة الواقعية لذلك فأن المعاملات و الممثلات الخاصة بعملية التقليص والاستقطاع كلها تعتبر غيرثابتة ومعتمدة على المتغيرات مما

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يجعل عملية الاستقطاع ادق والنتائج واقعية اكثر مع حفظ نسبة خطأ قليلة جدا بين النظام الاصلي والنظام المقلص. تم تطبيق الطريقة على مثال متعدد الابعاد واظهر نتائج جيدة.

1. Introduction

Model reduction problem is of interest in many control system applications since it simplifies the systems to be used for designing controllers or/and studying systems’ physical properties, e.g. stability and performance of systems. The most popular and applicable method for reducing stable systems is the balanced truncation method [1] as it preserves the properties (the most important property is the stability) of the original system in the reduced system. In addition, it yields the reduced system as well as the original system balanced. Moreover, the balanced truncation method removes state variables that have less effect on both the controllability and observability of the system. Many authors considered a model reduction of linear time-invariant systems via the balanced truncation method in previous works [1]. Also, some authors extended the method to the case of linear parameter-varying systems, e.g., [2], because in reality, physical coefficients (and hence the dynamics of the system) vary with respect to some varying parameters, and then the resulting system is known as linear parameter-varying system. In addition, the method has been extended to multidimensional systems in [3], while in [4] the method is applied to parameter-varying multidimensional systems defined in LFR (Linear Fractional Representation) form. The parameter-varying multidimensional systems are of interest due to their importance in many real-life applications, e.g., satellite communication systems, gyroscope systems, image processing, and several other applications. Such systems come with very high orders and are known as large-scale systems. Thus, in this work, model reduction of parameter-varying multidimensional systems is considered. Since the balanced truncation model reduction is applicable to stable systems only, which means that if the systems do not satisfy the stability condition, then the balanced truncation technique cannot be applied. Therefore, the system should be stabilized by constructing a state-feedback controller and defining a stable coprime factor such that the resulting realization is stable, thus, the balanced truncation method can be applied. This technique is known as coprime factor model reduction. A coprime factor model reduction for 1-dimensional and uncertain systems was stated in [5-8], where state feedback and coprime factorization are used in order to stabilize the considered system and define a stable coprime factor so that the balanced truncation model reduction method can be applied to the resulting stable system. The stability analysis of a nonlinear ordinary differential system is discussed in [9,10,11]. In [12] a contractive coprime factor for uncertain systems is guaranteed since it is difficult to define a normalized coprime factor for uncertain systems. In [13] the coprime factor model reduction has been applied to structured systems. The left coprime factorization was extended to parameter-varying one-dimensional systems in [14] using the so-called Linear Fractional Representation (LFR). The results of [14] can help in applying the coprime factor model reduction of parameter-varying one-dimensional systems defined in LFRs. Structured coprime factor model reduction has been considered in [15]. Here, the method of the coprime factor model reduction is extended to the case of parameter-varying multidimensional systems where all system matrices depend on a variable referred to as scheduling parameter. Parameter-dependent Gramians are considered, which are defined as diagonal blocks partitioned according to the dimension of the system. The structure of the original system is preserved in the reduced system as well. Constant-order systems are considered, i.e. the order of the system is invariant with respect to the scheduling parameter. By gridding the scheduling parameter range, a finite number of conditions using linear matrix inequalities (LMIs) can be solved to determine the solution of the model order reduction problem. The error bound of the reduction problem is computed and determined at each grid point of the scheduling parameter range.

The rest of this paper is organized as follows. The next section defines the considered parameter-varying multidimensional system as well as the main problem formulation of this work. Section 3 presents the main result of this work and gives the algorithm for reducing stabilizable parameter-varying multidimensional systems via the balanced truncation method (with guaranteed error bounds) after constructing the corresponding stable right coprime factor. Section 4 shows the validation of the proposed method on a simulation example. Finally, the paper is concluded in Section 5.

1.1 Notations

The set of all non-negative integers is represented as $\mathbb{Z}_+$. While the set of real numbers is defined as $\mathbb{R}$. The induced norm for parameter-varying multidimensional systems $G(\delta)$ which varies with the scheduling parameter δ within a given compact set Ω is defined as follows:

$$\|G(\delta)\|_{ind} = \sup_{\delta\in\Omega}\sup_{0\neq u\in l_2}\frac{\|G(\delta)u\|}{\|u\|},$$

where l_2 is the space of the square summable sequences.
Also, X^* refers to the conjugate transpose of X, for a block diagonal matrix X with blocks X_i, $i = 1,2,\dots,n$ along its diagonal, we define $X = \mathrm{diag}(X_1, X_2, \dots, X_n)$. The determinant of X is denoted as $\det(X)$.

2. Multidimensional Systems

Let a discrete-domain multidimensional system be represented as

$$\begin{aligned} x^+ &= Ax + Bu,\\ y &= Cx + Du, \end{aligned} \tag{1}$$

where x and x^+ are the multidimensional state variable of the system and its successor, respectively, which are described as follows:

$$x = \begin{bmatrix} x_1(m_1, m_2, \dots, m_l) \\ x_2(m_1, m_2, \dots, m_l) \\ \vdots \\ x_l(m_1, m_2, \dots, m_l) \end{bmatrix}, \quad x^+ = \begin{bmatrix} x_1(m_1+1, m_2, \dots, m_l) \\ x_2(m_1, m_2+1, \dots, m_l) \\ \vdots \\ x_l(m_1, m_2, \dots, m_l+1) \end{bmatrix},$$

with $m_1, m_2, \dots, m_l \in \mathbb{Z}_+$, and $x \in \mathbb{R}^n$, $x_1(\cdot) \in \mathbb{R}^{n_1}$, $x_2(\cdot) \in \mathbb{R}^{n_2}$, $\cdots$, $x_l(\cdot) \in \mathbb{R}^{n_l}$, such that $n = n_1 + n_2 + \cdots + n_l$, where n represents the order of (1). The system matrices A, B, C are partitioned accordingly as follows:

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1l} \\ A_{21} & A_{22} & \cdots & A_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ A_{l1} & A_{l2} & \cdots & A_{ll} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_l \end{bmatrix}, \quad C = [C_1 \quad C_2 \quad \cdots \quad C_l], \tag{2}$$

respectively, such that $A_{ij} \in \mathbb{R}^{n_i\times n_j}, B_i \in \mathbb{R}^{n_i\times n_u}, C_i \in \mathbb{R}^{n_y\times n_i}$, where $i,j = 1,2,\dots,l$. Moreover, $u \in \mathbb{R}^{n_u}$ and $y \in \mathbb{R}^{n_y}$ are the system input and output, respectively, which are multidimensional variables given by:

$$u = u(m_1, m_2, \cdots, m_l), \quad y = y(m_1, m_2, \cdots, m_l).$$

Moreover, let the transfer matrix

$$G = C(sI - A)^{-1}B + D, \tag{3}$$

where s is a complex number, be an equivalent representation of (1).
Next, consider linear parameter-varying multidimensional systems represented by:

$$\begin{aligned} x^+ &= A(\delta)x + B(\delta)u,\\ y &= C(\delta)x + D(\delta)u, \end{aligned} \tag{4}$$

Where $A(\delta)\colon \mathbb{R}^{n_\delta} \longrightarrow \mathbb{R}^{n\times n}$, $B(\delta)\colon \mathbb{R}^{n_\delta} \longrightarrow \mathbb{R}^{n\times n_u}$, $C(\delta)\colon \mathbb{R}^{n_\delta} \longrightarrow \mathbb{R}^{n_y\times n}$ and $D(\delta)\colon \mathbb{R}^{n_\delta} \longrightarrow \mathbb{R}^{n_y\times n_u}$, x, u, y are multidimensional state, input and output, respectively, as

introduced in (1) and
$\delta = \rho(m_1, m_2, \dots, m_l)$ is the scheduling parameter with $\rho(\cdot)$ is a continuous mapping. Let $\delta \in \Omega_\delta$, such that:

$$\Omega_\delta = \{\delta \in \mathbb{R}^{n_\delta} : \underline{\delta} \leq \delta \leq \overline{\delta}\},$$

where Ω is a compact set defining the variation range of δ. Define the variation shift of δ as $\delta_+ = \rho(m_1 + 1, m_2 + 1, \dots, m_l + 1) - \rho(m_1, m_2, \dots, m_l)$, such that $\delta_+ \in \Omega_{\delta_+}$, where:

$$\Omega_{\delta_+} = \{\delta_+ \in \mathbb{R}^{n_\delta} : \underline{\delta}_+ \leq \delta_+ \leq \overline{\delta}_+\},$$

is a compact set with $\underline{\delta}_+, \overline{\delta}_+$ define the bounds on δ_+. Moreover, we introduce the set:

$$\Omega = \{\delta \in \mathbb{R}^{n_\delta} : \delta \in \Omega_\delta,\ \delta_+ \in \mathbb{R}^{n_\delta} : \delta_+ \in \Omega_{\delta_+}\}. \tag{5}$$

We call (4) a system with constant-order when its order is invariant with respect to $\delta \in \Omega$ and $n = \sum_{i=1}^{l} n_i$. Note that we do not impose any special dependence of the system matrices on the scheduling parameter, so they can be with general dependence on the scheduling parameter, e.g., affine, fractional, polynomial, etc. The following input-output linear operator represents the LPV multidimensional state-space realization corresponding to (4):

$$G(\delta) = \begin{bmatrix} A(\delta) & B(\delta) \\ C(\delta) & D(\delta) \end{bmatrix}.$$

The problem statement of this work can be stated as follows: Given the system (4) with order $n = \sum_{i=1}^{l} n_i$, the goal is to construct a reduced order system represented by the following input-output operator

$$G^r(\delta) = \begin{bmatrix} A^r(\delta) & B^r(\delta) \\ C^r(\delta) & D(\delta) \end{bmatrix},$$

with the order $r = \left(\sum_{i=1}^{l} r_i\right) < n$, where $A^r(\delta)$, $B^r(\delta)$ and $C^r(\delta)$ are the reduced system matrices, without affecting the stability or changing the structure of the full order system and with the approximation error bound given as

$$\|G(\delta) - G^r(\delta)\|_{ind} \leq e, \qquad e = \min_{\delta \in \Omega} \|G(\delta) - G^r(\delta)\|_{ind},$$

for all $\delta \in \Omega$.

3. Balanced Truncation of Linear Parameter-Varying Multidimensional Systems

For stable systems, balanced truncation is employed for solving the considered reduction problem in Section 2. In order to present the stability condition of a parameter-varying multidimensional system, consider the following definitions:

Definition 3.1, [3]. The system (4) is stable if there exists $X(\delta) \in M$, such that:

$$A(\delta)X(\delta)A^*(\delta) - X(\delta^+) < 0, \text{ for all } \delta \in \Omega, \tag{6}$$

Where:

$$M = \{X(\delta) > 0 : X(\delta) = \operatorname{diag}(X_1(\delta), \cdots, X_l(\delta)), X_i(\delta) \in \mathbb{R}^{n_i \times n_i}, i = 1, \cdots, l, \forall\, \delta \in \Omega\} \tag{7}$$

and the shifted version of $X(\delta) \in M$ is defined as:

$$X(\delta^+) = diag\big(X_1(\rho(m_1 + 1, m_2, \dots, m_l)), X_2(\rho(m_1, m_2 + 1 \dots, m_l)), \cdots, X_l(\rho(m_1, m_2, \dots, m_l + 1))\big).$$

Definition 3.2 (Parameter-dependent Gramians), [3]. The matrices $X(\delta), Y(\delta) \in M$, where M is given in (7) are called the parameter-dependent controllability and observability Gramians, respectively, of the system (4) if they satisfy the following matrix inequalities

$$\begin{aligned} A(\delta)X(\delta)A^*(\delta) - X(\delta^+) + B(\delta)B^*(\delta) < 0, \\ A^*(\delta)Y(\delta^+)A(\delta) - Y(\delta) + C^*(\delta)C(\delta) < 0, \end{aligned} \tag{8}$$

respectively, for all $\delta \in \Omega$.

The balanced truncation model reduction method is applicable for stable systems only, i.e. the stability condition which is given in (6) should be satisfied, as stated in [3] and [4] for parameter-varying spatially interconnected systems, which are a class of multidimensional systems. One can refer to [3] and [4] for more information and details about balanced realization and truncation for stable parameter-varying multidimensional systems. The focus of this work is on the model reduction of the parameter-varying multidimensional systems that do not satisfy the stability condition (6). In this case, the balanced truncation technique cannot be applied. Therefore, the system should be stabilized (see Definition 3.3 below), i.e., by constructing a state-feedback controller and defining a stable coprime factor such that the resulting realization is stable, thus, the balanced truncation method can be applied as in [7] for one-dimensional systems and [5] for uncertain systems. This work extends the result of [5] and [7] to the case of parameter-varying multidimensional system which is presented in (4).

Definition 3.3 (Stabilizable System): The parameter-varying multidimensional system (4) is stabilizable by the parameter-dependent state feedback $F(\delta)$ for all $\delta \in \Omega$ if there exists $Q(\delta) \in M$, such that

$$\big(A(\delta) + B(\delta)F(\delta)\big)Q(\delta)\big(A(\delta) + B(\delta)F(\delta)\big)^* - Q(\delta^+) < 0, \text{ for all } \delta \in \Omega, \quad (9)$$

where:

$$Q(\delta) = \mathrm{diag}\big(Q_1(\delta), Q_2(\delta), \dots, Q_l(\delta)\big),$$

$$Q(\delta^+) = \mathrm{diag}\Big(Q_1\big(\rho(m_1+1, m_2, \dots, m_l)\big), Q_2\big(\rho(m_1, m_2+1, \dots, m_l)\big), \cdots, Q_l\big(\rho(m_1, m_2, \dots, m_l+1)\big)\Big), \quad (10)$$

3.1 Coprime factors

Consider a stabilizable system that is stabilized via a parameter-dependent state feedback $F(\delta)$ for all $\delta \in \Omega$, such that a stable right coprime factor can be constructed. Then the resulting stable coprime factor realization can be reduced by applying the standard balanced truncation procedure. The next theorem is a standard result and it is well-known in many references. So, it is given here without proof since the proof steps are similar to the result in [7,5] for 1-dimensional uncertain systems, but here it is applied for parameter-varying multidimensional systems.

Theorem 3.4. There exists a parameter-dependent state-feedback gain $F(\delta)$ and $Q(\delta) \in M$ that satisfies (9) with (10) for all $\delta \in \Omega$ if and only if there exists $P(\delta) \in M$, which satisfies:

$$A(\delta)P(\delta)A^*(\delta) - P(\delta^+) - B(\delta)B^*(\delta) < 0, \quad (11)$$

for all $\delta \in \Omega$. Then, a stabilizing choice of F is given by:

$$F(\delta) = -\big(B^*(\delta)P^{-1}(\delta^+)B(\delta)\big)^{-1}B^*(\delta)P^{-1}(\delta^+)A(\delta), \text{ for all } \delta \in \Omega. \quad (12)$$

Remark 1. Due to the general dependence of the matrices A, B on δ, the conditions in Theorem 3.4 should be applied infinitely many times along all $\delta(m_1, m_2, \dots, m_l) \in \Omega$. Therefore, to have a finite number of conditions to be solved, we grid the range of $\delta \in \Omega$, as shown below, on which the inequality conditions in Theorem 3.4 can be verified.
Now, if condition (11) is satisfied for all $\delta \in \Omega$, then one can construct a right coprime factorization (see definition 3.5, below) for the system realization as shown in Theorem 3.6 below.

Definition 3.5 (Right Coprime Factor). A stabilizable realization $G(\delta)$ has stable right coprime factors $N(\delta)$ and $S(\delta)$, for all $\delta \in \Omega$, if $G(\delta) = N(\delta)S^{-1}(\delta)$, $\forall \delta \in \Omega$, where:

$$S(\delta) = I + F(\delta)\Big(I - \big(A(\delta) + B(\delta)F(\delta)\big)\Big)^{-1}B(\delta),$$

and

$$N(\delta) = D(\delta) + \big(C(\delta) + D(\delta)F(\delta)\big)\Big(I - \big(A(\delta) + B(\delta)F(\delta)\big)\Big)^{-1} B(\delta),$$

for all $\delta \in \Omega$ and $F(\delta)$ is the parameter-dependent stabilizing feedback. Note that the left coprime factor is the dual version of the right coprime factor. Also, a right coprime factorization of $G(\delta) = N(\delta)S^{-1}(\delta)$ is said to be contractive if $N^*(\delta)N(\delta) + S(\delta)S^*(\delta) \leq I$, for all $\delta \in \Omega$.

Theorem 3.6. For a stabilizable realization $G(\delta)$ with the stabilizing feedback $F(\delta)$ (12), for all $\delta \in \Omega$, the contractive right coprime factor of $G(\delta)$ is:

$$G(\delta) = N(\delta)S^{-1}(\delta) = \left[\begin{array}{c|c} A(\delta)+B(\delta)F(\delta) & B(\delta) \\ \hline F(\delta) & I \\ C(\delta)+D(\delta)F(\delta) & D(\delta) \end{array}\right] = \left[\begin{array}{c|c} A_F(\delta) & B_F(\delta) \\ \hline C_F(\delta) & D_F(\delta) \end{array}\right], \text{ for all } \delta \in \Omega, \tag{13}$$

such that $N^*(\delta)N(\delta) + S(\delta)S^*(\delta) \leq I$, where:

$$S(\delta) = I + F(\delta)\Big(I - \big(A(\delta) + B(\delta)F(\delta)\big)\Big)^{-1} B(\delta),$$

and

$$N(\delta) = D(\delta) + \big(C(\delta) + D(\delta)F(\delta)\big)\Big(I - \big(A(\delta) + B(\delta)F(\delta)\big)\Big)^{-1} B(\delta).$$

Proof. The proof follows the same line as that of Theorem (5) in [7], and Proposition (6) in [5], which are considered for 1-dimensional uncertain systems defined in LFR (Linear Fractional Representation) form. ◻

Remark 2. Theorem 3.6 is stated for parameter-varying multidimensional system which is varying with respect to the scheduling parameter $\delta(m_1, m_2, \dots, m_l) \in \Omega$, i.e., to satisfy the conditions of the theorem, one needs to solve infinity many times LMIs, which is not possible, therefore one needs to base on gridding the range of the scheduling parameter $\delta \in \Omega$, in order to have finite conditions rather than infinite ones.

Now, the realization in (13) is clearly stable. Therefore, the balanced truncation can be applied to reduce the system. Let the resulting reduced realization with the reduced-order $r = \sum_{i=1}^{l} r_i < n = \sum_{i=1}^{l} n_i$ be denoted as:

$$G^r(\delta) = N^r(\delta)(S^{-1})^r(\delta) = \left[\begin{array}{c|c} A_F^r(\delta) & B_F^r(\delta) \\ \hline C_F^r(\delta) & D_F(\delta) \end{array}\right], \text{ for all } \delta \in \Omega. \tag{14}$$

Then the error bound between the full-order realization $G(\delta)$ and the reduced-order version $G^r(\delta)$ can be examined and determined via induced norms as stated in the following theorem.

Theorem 3.7. Let the conditions in Theorems 3.4 and 3.6 be satisfied, such that a reduced realization $G^r(\delta)$ with the reduced order $r = \sum_{i=1}^{l} r_i$ (see (14)) exists for the full realization $G(\delta)$ with the full order $n = \sum_{i=1}^{l} n_i$, see (13), for all $\delta \in \Omega$, where $r < n$, then the error bound between $G(\delta)$ and $G^r(\delta)$ satisfies:

$$\|G(\delta) - G^r(\delta)\|_{ind} \leq 2\left(\sum_{\mathrm{i}=1}^{\mathrm{l}} \sum_{\mathrm{j}=r_i+1}^{n_i} \sigma_{ij}(\delta)\right), \text{ for all } \delta \in \Omega, \tag{15}$$

where $\sigma_{ij}(\delta)$ are the diagonal entries of the truncated parts of the balanced Gramians.

Proof. The proof is similar to that in [6,7,5], so it is omitted here. ◻

Remark 3. Note that the realization here depends on the scheduling parameter $\delta(m_1, m_2, \dots, m_l) \in \Omega$, which means that the stable right coprime factor is parameter-dependent with respect to $\delta \in \Omega$, and the condition of Theorem 3.7 follows by the application infinitely many times over the scheduling parameter range. For practical implementation, by gridding the scheduling parameter range, we can apply these conditions with a finite number as shown below. In contrast to the work in [6], [7], and [5] where the considered system is a 1-dimensional uncertain system, the considered system here is a multidimensional system and all realization matrices are partitioned according to this dimension.

Next, in order to illustrate our proposed approach for reducing stabilizable parameter-varying multidimensional systems. First, we introduce an algorithm to verify the conditions in Theorems 3.4, and 3.6 on grid points over the parameter range of the scheduling parameter.

Remark 4. It is important before applying the proposed algorithms to define the functional dependence of P on δ. A plausible choice of such a function is to be similar to the functional dependence of the system matrices on δ, see [16] for other choices.

Algorithm 1: Gridding Algorithm.

1. Define grid points over the parameter range of $\delta(m_1, m_2, \dots, m_l) \in \Omega$ with a grid size of n_g, such that $\delta = \left[\delta_1, \delta_2, \dots, \delta_{(n_g)}\right]$.
2. Construct and solve all the conditions in Theorems 3.4, and 3.6 (as illustrated in Algorithm 2 below) at each grid point $\delta_k,\ k = 1, \dots, n_g$.
3. Define denser grid points $n_g r$ which considers points between the grid points n_g (for verification).
4. Check the validity of the solution from step 2 over the denser grid points, if it is valid then go to the next step. If not, then increase the grid points n_g and $n_g r$ accordingly and go to step 2.

Now, in order to stabilize the system (i.e., applying Theorem 3.4 and Theorem 3.6), construct a stabilizing feedback (at each grid point, as stated in Algorithm 1) by following the next algorithm.

Algorithm 2: Stabilizing Feedback Construction Algorithm.

1. At each grid point of $\delta \in \Omega$, solve (11) for $P(\delta) \in M$ and compute $F(\delta),\ \forall \delta \in \Omega$ from (12).
2. Given $F(\delta)$, the stable right coprime factor of $G(\delta),\ \delta \in \Omega$ as in (13), can be computed at each grid point of $\delta \in \Omega$ as:

$$G_s(\delta) = N(\delta)S^{-1}(\delta) = \begin{bmatrix} A_F(\delta) & B_F(\delta) \\ C_F(\delta) & D_F(\delta) \end{bmatrix},\ \delta \in \Omega. \tag{16}$$

Now, the resulting realization $G_s(\delta), \delta \in \Omega$ in (16) is stable, so the balanced truncation method can be applied to get the reduced system. Before applying the balanced realization and balanced truncation for the resulting stabilized realization (16), we need first to define the set of parameter-dependent balanced transformation $T(\delta) \in \mathfrak{T}$, $\forall\ \delta \in \Omega$, as presented in the next subsection.

3.2 Balanced realization

In order to transform the stabilized realization $G_s(\delta)$ (defined in (16)) into the balanced realization $G^{bal}(\delta)$ for all $\delta \in \Omega$, such that

$$G^{bal}(\delta) = \begin{bmatrix} A_F^{bal}(\delta) & B_F^{bal}(\delta) \\ C_F^{bal}(\delta) & D_F^{bal}(\delta) \end{bmatrix}, \forall\, \delta \in \Omega, \tag{17}$$

and also to transform the parameter-dependent Gramians $X(\delta), Y(\delta) \in M$ into the balanced versions $X^{bal}(\delta) = Y^{bal}(\delta) = \sum(\delta)$, $\forall\, \delta \in \Omega$, we need to define the set of parameter-dependent balanced transformation $T(\delta) \in Ʈ$, $\forall\ \delta \in \Omega$, where the set Ʈ is defined as:

$$Ʈ = \{T(\delta) \in \mathbb{R}^{n\times n}: \det(T(\delta)) \neq 0, T(\delta) = diag(T_1(\delta), \dots, T_l(\delta)), T_i(\delta) \in R^{n_i\times n_i}, i = 1, \dots, l, \forall\delta \in \Omega\}.$$

Then construct the balanced realization (at each grid point of $\delta \in \Omega$) by using the steps in the next Algorithm. Note that all steps in Algorithm 3 should be applied at each grid point of $\delta \in \Omega$ as given in Algorithm 1.

Algorithm 3: Balanced Realization Algorithm.

1. Solve the two conditions in (8) for $X(\delta), Y(\delta) \in M$, $\forall\ \delta \in \Omega$, (on the stabilizable realization (16)) by solving

$$\begin{matrix} A_F(\delta)X(\delta)A_F^*(\delta) - X(\delta^+) + B_F(\delta)B_F^*(\delta) < 0 \\ A_F^*(\delta)Y(\delta^+)A_F(\delta) - Y(\delta) + C_F^*(\delta)C_F(\delta) < 0 \end{matrix}, \forall\delta \in \Omega.$$

Note that, a trace heuristic approach [3] can be used here for solving the above conditions.

2. Factorize each block of $X(\delta) \in M$ and $Y(\delta) \in M$, $\forall\delta \in \Omega$ as:

$$X_1(\delta) = R_1^*(\delta)R_1(\delta), X_2(\delta) = R_2^*(\delta)R_2(\delta), \dots, X_l(\delta) = R_l^*(\delta)R_l(\delta),$$

and

$$Y_1(\delta) = L_1^*(\delta)L_1(\delta),\ Y_2(\delta) = L_2^*(\delta)L_2(\delta), \dots, Y_l(\delta) = L_l^*(\delta)L_l(\delta).$$

3. Decompose the blocks $L_i(\delta)R_i^*(\delta), i = 1, \dots, l, \forall\delta \in \Omega$, by using the SVD, such that
$L_1(\delta)R_1^*(\delta) = U_1(\delta)\Sigma_1(\delta)V_1(\delta), L_2(\delta)R_2^*(\delta) = U_2(\delta)\Sigma_2(\delta)V_2(\delta)$, ...,
$L_l(\delta)R_l^*(\delta) = U_l(\delta)\Sigma_l(\delta)V_l(\delta)$, $\forall\ \delta \in \Omega$.

4. Define

$$T_1(\delta) = R_1^*(\delta)V_1(\delta)\Sigma_1^{(-1/2)}(\delta), \quad T_2(\delta) = R_2^*(\delta)V_2(\delta)\Sigma_2^{(-1/2)}(\delta), \dots, \quad T_l(\delta) = R_l^*(\delta)V_l(\delta)\Sigma_l^{(-1/2)}(\delta),$$

and

$$T_1^{-1}(\delta) = \Sigma_1^{(-1/2)}(\delta)U_1^*(\delta)L_1(\delta), \quad T_2^{-1}(\delta) = \Sigma_2^{(-1/2)}(\delta)U_2^*(\delta)L_2(\delta), \dots, \quad T_l^{-1}(\delta) = \Sigma_l^{(-1/2)}(\delta)U_l^*(\delta)L_l(\delta).$$

Then, set:

$$T(\delta) = \mathrm{diag}(T_1(\delta), T_2(\delta), \dots, T_l(\delta)),$$

and

$$T^{-1}(\delta) = \mathrm{diag}\left(T_1^{-1}(\delta), T_2^{-1}(\delta), \dots, T_l^{-1}(\delta)\right), \forall\ \delta \in \Omega, \text{ where } T(\delta) \in Ʈ.$$

5. Use the parameter-dependent balanced transformation $T(\delta) \in Ʈ, \forall\delta \in \Omega$ to define the balanced realization $G^{bal}(\delta), \forall\delta \in \Omega$ (which is defined in (17) above) as follows:
$A_F^{bal}(\delta) = T^{-1}(\delta^+)A_F(\delta)T(\delta)$, $B_F^{bal}(\delta) = T^{-1}(\delta^+)B_F(\delta)$, $C_F^{bal}(\delta) = C_F(\delta)T(\delta)$, and $D_F^{bal}(\delta) = D_F(\delta)$.
Also, define the balanced Gramians as:

$$X^{bal}(\delta) = T^{-1}(\delta)X(\delta)T^{-*}(\delta) \quad \text{and} \quad Y^{bal}(\delta) = T^*(\delta)Y(\delta)T(\delta),$$

such that:
$X^{bal}(\delta) = Y^{bal}(\delta) = \Sigma(\delta) = \mathrm{diag}(\Sigma_1(\delta), \Sigma_2(\delta), \dots, \Sigma_l(\delta)), \forall\delta \in \Omega$, see Remark 5, below.

Remark 5. In Algorithm 3, note that

$$\Sigma_1(\delta) = \text{diag}\left(\sigma_{11}(\delta), \sigma_{12}(\delta), \dots, \sigma_{1n_1}(\delta)\right),$$
$$\Sigma_2(\delta) = \text{diag}\left(\sigma_{21}(\delta), \sigma_{22}(\delta), \dots, \sigma_{2n_2}(\delta)\right),$$
$$\vdots$$
$$\Sigma_l(\delta) = \text{diag}\left(\sigma_{l1}(\delta), \sigma_{l2}(\delta), \dots, \sigma_{ln_l}(\delta)\right), \forall\, \delta \in \Omega.$$

The values of $\sigma_{ij}(\delta)$, $i = 1, \cdots, l, j = 1, \cdots, n_i, \forall\, \delta \in \Omega$, (see (15) as well) are ordered in descending order along the diagonal of $\Sigma_i(\delta)$, for all i and j, i.e.,

$$\sigma_{11}(\delta) > \sigma_{12}(\delta) > \cdots > \sigma_{1n_1}(\delta),$$
$$\sigma_{21}(\delta) > \sigma_{22}(\delta) > \cdots > \sigma_{2n_2}(\delta),$$
$$\vdots$$
$$\sigma_{l1}(\delta) > \sigma_{l2}(\delta) > \cdots > \sigma_{ln_l}(\delta).$$

The smallest values represent the less significant ones.

3.3 Balanced truncation

In the previous subsection, after balancing the realization, we determine the less significant parts of the balanced system according to the singular values $\sigma_{ij}(\delta)$, $i = 1,2, \cdots, l$, $j = 1,2, \cdots, n_i, \forall\, \delta \in \Omega$.

Here, we partition the balanced Gramians and the balanced system realization according to the significant and non-significant parts, respectively. Then, we can truncate the parts which are related to the non-significant ones, such that we end up with the reduced version realization.

Therefore, one could partition $\Sigma(\delta)$ into two blocks according to the significant $\Sigma^s(\delta)$ and the non-significant ones $\Sigma^{ns}(\delta), \forall \delta \in \Omega$, such that:

$$\Sigma(\delta) = \begin{bmatrix} \Sigma^s(\delta) & \\ & \Sigma^{ns}(\delta) \end{bmatrix}, \forall\, \delta \in \Omega,$$

where**:**

$\Sigma^s(\delta) = \text{diag}\left(\Sigma_1^s(\delta), \Sigma_2^s(\delta), \dots, \Sigma_l^s(\delta)\right)$ and $\Sigma^{ns}(\delta) = \text{diag}\left(\Sigma_1^{ns}(\delta), \Sigma_2^{ns}(\delta), \dots, \Sigma_l^{ns}(\delta)\right)$,

such that:

$\Sigma_1^s(\delta) = \text{diag}\left(\sigma_{11}(\delta), \sigma_{12}(\delta), \dots, \sigma_{1r_1}(\delta)\right)$ and $\Sigma_1^{ns}(\delta) = \text{diag}\left(\sigma_{1\, r_1+1}(\delta), \dots, \sigma_{1n_1}(\delta)\right)$,

$\Sigma_2^s(\delta) = \text{diag}\left(\sigma_{21}(\delta), \sigma_{22}(\delta), \dots, \sigma_{2r_2}(\delta)\right)$ and $\Sigma_2^{ns}(\delta) = \text{diag}\left(\sigma_{2\, r_2+1}(\delta), \dots, \sigma_{2n_2}(\delta)\right)$,

$$\vdots$$

$\Sigma_l^s(\delta) = \text{diag}\left(\sigma_{l1}(\delta), \sigma_{l2}(\delta), \dots, \sigma_{lr_l}(\delta)\right)$ and $\Sigma_l^{ns}(\delta) = \text{diag}\left(\sigma_{l\, r_l+1}(\delta), \dots, \sigma_{ln_l}(\delta)\right)$,

where $r_i, i = 1,2, \dots, l$ are the order of the reduced blocks such that the resulting reduced order is $r = \sum_{i=1}^{l} r_i$, as already defined in Section 2.

Then, we give the following Algorithm for constructing the reduced version realization $G^r(\delta)$, (at each grid point of $\forall\, \delta \in \Omega$, see Algorithm 1).

Algorithm 4: Balanced Truncation and Reduced System Construction.

1. Partition $A_F^{bal}(\delta)$, $B_F^{bal}(\delta)$, and, $C_F^{bal}(\delta)$ (defined in Algorithm 3) according to the significant, and non-significant blocks of $\Sigma(\delta) = \begin{bmatrix} \Sigma^s(\delta) & \\ & \Sigma^{ns}(\delta) \end{bmatrix}, \forall \delta \in \Omega$, such that:

$$A_F^{bal} = \begin{bmatrix} A_F^r(\delta) & A_F^{12}(\delta) \\ A_F^{21}(\delta) & A_F^{22}(\delta) \end{bmatrix},\ B_F^{bal} = \begin{bmatrix} B_F^r \\ B_F^2 \end{bmatrix} \text{ and } C_F^{bal} = \begin{bmatrix} C_F^r & C_F^2 \end{bmatrix}.$$

2. Truncate the blocks which are related to the non-significant parts, and define the reduced versions as: $A_F^r(\delta), B_F^r(\delta)$, and $C_F^r(\delta)$.
3. Finally, (see equation (13)) since $A_F^r(\delta) = A^r(\delta) + B^r(\delta)F^r(\delta)$ we can compute $A^r(\delta) = A_F^r(\delta) - B^r(\delta)F^r(\delta)$, also $B^r(\delta) = B_F^r(\delta)$ and $C_F^r(\delta) = \begin{bmatrix} F^r(\delta) \\ C^r(\delta) + D(\delta)F^r(\delta) \end{bmatrix}$ can determine $C^r(\delta), \forall\, \delta \in \Omega$. So, the reduced realization $G^r(\delta),\ \forall\delta \in \Omega$ is constructed.

Now, summarize the right coprime factor reduction scheme (from Algorithm 1 to Algorithm 4) in the following algorithm, which has to be applied at each grid point of $\forall\, \delta \in \Omega$ as given in Algorithm 1.

Algorithm 5: Right Coprime Factor Reduction Scheme.
Consider the stabilizable parameter-varying multidimensional system realization $G(\delta),\ \delta \in \Omega$ given in (13) with order $n = \sum_{i=1}^{l} n_i$, a reduced realization version $G^r(\delta), \forall\, \delta \in \Omega$ with reduced order $r = \sum_{i=1}^{l} r_i$, such that $r < n$ can be constructed via balanced truncation by applying the following steps:
1. Define grid points over the parameter range of the scheduling parameter ($\delta \in \Omega$) via applying Algorithm 1.
2. At each grid point of $\delta \in \Omega$, stabilize the realization via constructing a stabilizing feedback by following the steps in Algorithm 2.
3. At each grid point of $\delta \in \Omega$, transform the stabilizable realization $G_s(\delta), \forall\, \delta \in \Omega$ (defined in (16)) to the balanced realization $G^{bal}(\delta), \forall\, \delta \in \Omega$, (defined in (17)) by the parameter-dependent balanced transformation $T(\delta) \in \mathrm{T}, \forall\, \delta \in \Omega$ as presented in Algorithm 3.
4. Finally, follow Algorithm 4 to construct the reduced system $G^r(\delta), \forall\, \delta \in \Omega$.

Following the above steps in Algorithm 5 leads to the desired reduced system with a small error bound as shown in Theorem 3.7.
Next, we show the application of our results on a simulation example, where (for simplicity) the considered system is a 2-dimensional system varying with respect to one variable.

4. Application Example

This section demonstrates the validity and applicability of our proposed method in Algorithm 5 in the previous section, where it is applied to a modified version of the numerical example of [6].
Consider the parameter-varying two-dimensional system (to be stabilizable by computing parameter-dependent state feedback):

$$\begin{bmatrix} x_1(m_1+1, m_2) \\ x_2(m_1, m_2+1) \\ y(m_1, m_2) \end{bmatrix} = \begin{bmatrix} 0.5034 & 0.1768 & -0.2340 & -0.1406 & \delta(m_1) & 0.1700 \\ 0.0096 & 0.5498 & -0.0362 & -0.6744 & 2.2496 & 0.3442 \\ 0.0337 & 0.2546 & 0.0984 & -0.4051 & 1.3599 & 0.2143 \\ -0.2709 & 0.1470 & 0.3249 & 0.0484 & 0.6356 & 0.8821 \\ -0.0909 & 0.0491 & 0.1075 & -0.1019 & 0.5681 & 0.4479 \\ 3.0396 & -0.9913 & -0.7073 & 5.2369 & -8.4887 & 0 \end{bmatrix} \begin{bmatrix} x_1(m_1, m_2) \\ x_2(m_1, m_2) \\ u(m_1, m_2) \end{bmatrix},$$

where $\delta(m_1)$, $m_1 \in \mathbb{Z}_+$, is defined such that:
$-0.6 \le \delta(m_1) \le 0.6$, and $x_1(m_1, m_2) \in \mathbb{R}^{3\times 1}$, $x_2(m_1, m_2) \in \mathbb{R}^{2\times 1}$.
So, the system has constant-dimension $n = n_1 + n_2 = 3 + 2 = 5$.

Algorithm 5 is applied to the given system by gridding the parameter range of $\delta(m_1)$, with a grid size of $n_g = 25$, such that: $\delta = [-0.60 \quad -0.55 \quad -0.50 \quad -0.45 \quad -0.40 \quad -0.35 \quad -0.30 \quad -0.25 \quad -0.20 \quad -0.15 \quad \ldots \quad 0.45 \quad 0.50 \quad 0.55 \quad 0.60]$. (18)

The order of the reduced system is $r = 2 < 5$, where $r_1 = 1$ and $r_2 = 1$.
The value of the error bound over the whole parameter range is 0.0193, which is the maximum (the worst) error bound over the whole range, where the value of the error bound at different values of δ (see (18)), e.g., $\delta = -0.45$ is 0.0022 and at $\delta = 0.05$ is 0.0010 and at $\delta = 0.45$ is 0.0193, and at $\delta = 0.55$ is 0.0151.
Figure 1 shows the impulse response of the system $G(\delta)$ and $G^r(\delta)$ at two different values of δ, where the values are taken at the boundaries of the grid points as $\delta = -0.45$ and $\delta = 0.55$. As mentioned above the error bound at the value of $\delta = -0.45$ is 0.0022 and at $\delta = 0.55$ is 0.0151 which means that the difference error between these two values is (0.0151)-(0.0022) = 0.0129.

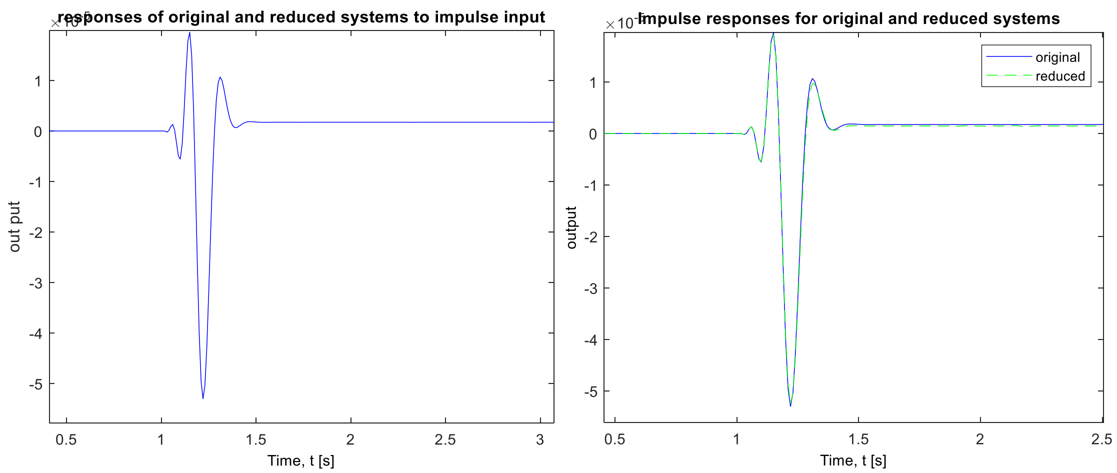

Figure 1: Impulse response for system $G(\delta)$ and $G^r(\delta)$ at $\delta = -0.45$, Left, and at $\delta = 0.55$, Right.

5. Conclusions

In this work, a coprime factor model reduction approach is applied to unstable (stabilizable) parameter-varying multidimensional systems. Thus, a balanced truncation technique can be used after stabilizing the system using parameter-dependent state feedback and by defining a stable coprime factor. Moreover, a minimized error bound over the whole parameter range is guaranteed. In order to reduce the conservatism of the reduction problem to be solved, parameter-varying Gramians are considered in this paper instead of the parameter-invariant ones. Gridding the range of the scheduling parameter allows a finite number of LMIs to be solved to determine the reduced order system or to solve the model reduction problem. A numerical example has been used to show the validation of our result, where Algorithm 5 has been applied successfully on a parameter-varying two-dimensional system.

References

[1] B. Moore, "Principle component analysis in linear systems: controllability, observability, and model reduction," *IEEE Trans. Automat. Contr*., vol. Ac-26, no. 1, pp. 17-32, 1981.
[2] H. Abbas, and H. Werner, "Frequency-weighted discrete-time LPV model reduction using structurally balanced truncation", *in Proc. 49th IEEE Conf. Decision Control*, pp. 4298-4303, 2010.
[3] F. Al-Taie and H. Werner, "Structure-preserving model reduction for spatially interconnected systems with experimental validation on an actuated beam", *International Journal of Control*, vol. 89, no. 6, pp. 1248-1268, 2015.

[4] F. Al-Taie, "Complexity of spatially interconnected systems", *Ph.D. dissertation, Hamburg university of technology, Germany,* 2016.
[5] C. Beck and P. Bendotti, "Model reduction methods for unstable uncertain systems", *in Proc. 36th IEEE Conf. Decision Control, San Diego, California USA*, pp. 3298-3303, 1997.
[6] C. Beck, "Coprime factors reduction methods for linear parameter varying and uncertain systems ", *Systems & Control Letters,* vol.55, pp. 199-213, 2006.
[7] D. Abou Jaoude and M. Farhood, "Model reduction of distributed nonstationary LPV systems", *European Journal of Control,* doi: 10.1016/j.ejcon.2017.10.007, 2017.
[8] D. Abou Jaoude and M. Farhood, "Coprime factors model reduction of spatially distributed LTV systems over arbitrary graphs", *IEEE Transactions on Automatic Control,* no.62, vol.10, pp. 5254-5261, 2017.
[9] S. Majeed and S.Ghafel, "Stability Analysis of a Prey-Predator Model with Prey Refuge and Fear of Adult Predator ", *Iraqi Journal of Science*, vol.63, no.10, pp. 4374-4387, 2022.
[10] R. Naji and A. Aaid, "The Dynamics of One Harmful Phytoplankton and Two Competing Zooplankton System ", *Iraqi Journal of Science*, vol.54, no.3, pp. 775-792, 2013.
[11] R. Zboon, and S. Altaie, "Nonlinear dynamical fuzzy control systems design with matching conditions", *Al-Nahrain Journal of Science*, vol.8, no.2, pp. 133-146, 2005.
[12] L. Li, "Coprime factor model reduction for discrete-time uncertain systems", *in Proc. 49th IEEE Conf. Decision Control, Atlanta, USA*, pp. 6213-6218,2010.
[13] A. Sootla and A. Rantzer, "Model reduction of spatially distributed systems using coprime factors and semidefinite programming", *in Proc. 18th World Congress the International Federation of Automatic Control (IFAC), Italy,* doi: 10.3182/20110828-6-IT-1002.02955, pp. 6663-6668, 2011.
[14] E. Prempain, "On coprime factors for parameter-dependent systems", *in Proc. 45th IEEE Conf. Decision Control, San Diego, USA*, pp. 5796-5800, 2006.
[15] L. Li and F. Paganini, "Structured coprime factor model reduction based on LMIs", *Automatica,* vol.41, pp. 145-151, 2005.
[16] P. Apkarian and R.J. Adams, "Advanced gain-scheduling techniques for uncertain systems", *IEEE Trans. Contr. Syst. Technol*., vol.6, pp. 21-32, 1998.