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Z-Small Monoform Modules

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Abstract:
In this paper, a new proper generalization of small monoform modules, namely Z-Small monoform modules is introduced and studied. An R-module C is called Z-small monoform module (ZSM module; for short), if every non-zero partial endomorphism of C has Z-small kernel.

Keywords: Monoform modules, small submodule, small monoform modules, Z-small monoform modules.

المقاسات ذات الصيغة المتباينة الصغيرة من النمط – Z

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الخلاصة:
في هذا البحث قدمنا ودرسنا تعميم جديد لمفهوم المقاسات ذات الصيغة المتباينة الصغيرة واطلقنا عليه اسم المقاسات ذات الصيغة المتباينة من النمط Z–. حيث ان المقاس C على R يسمى مقاس ذا صيغة متباينة صغيرة من النمط–Z (مقاسZSM– للاختصار) اذا كان كل تشاكل جزئي للمقاس C يمتلك نواة صغيرة من النمط–Z.

1.Introduction

Throughout this paper, all rings have identity and all modules are unitary right R-modules. Zelmanowitiz in [1] introduced the concept of monoform modules, where a module C is called monoform if every non-zero partial endomorphisms is monomorphisim. A partial endomorphisms of a module means an R-homomorphism if $g: A \longrightarrow C$ where A is a submodule of C [1]. Inaam Hadi and Hassan Marhoon in [2] introduced and studied the notion of small monoform, where an R-module C is named small monoform if for each nonzero submodule A of C and for each non-zero $f \in Hom\ (A, C)$, $ker\ f$ is small in A. Note that a submodule A of a module M is called small (shortly $A \ll M$), if whenever $A + B = M$ and B is a submodule of M, then $B = M$ [3]. Clearly, every monoform module is small monoform. But the converse is not true in general, see [2]. Amina and Alaa in [4] said that a submodule A of an R-module M

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is Z-small in M (briefly $A \ll_Z M$) if whenever $A + B = M$ and $Z_2(M) \subseteq B$, then $B = M$ where $Z_2(M)$ is defined by $\frac{Z_2(M)}{Z(M)} = Z\left(\frac{M}{Z(M)}\right)$ [3], where *Z(M)* is singular submodule of *M*.
A module M is called singular (respectively, non-singular) if $Z(M) = M$(respectively $Z(M) = 0$) [3].

A module M is called Z_2-torsion if $Z_2(M) = M$. Note that if M is singular, then M is Z_2-torsion, and M is non-singular if and only if $Z_2(M) = 0$ [5].
For more information, one can see [5], [6].
Obviously, if A is a small submodule of a module C, then A is Z-small, but the converse is not true in general, see [4].

These works motivate us to introduce a new concept namely Z-small monoform module (denoted by ZSM module), where we call a module C is a ZSM module if every non-zero partial endomorphism of C has a Z-small kernel.
In this paper, our concern is to study these types of modules. Next, we use the following notations. For submodules A and B of a module $C,\ A \leq B$ denotes A is a submodule of $B,\ A \leq^{\oplus} C$ denotes that A is a direct summand of C, $Hom\,(A, C)$ denotes the ring of all homomorphisms from A into C. $A \leq_{ess} C$ denote A is an essential in C, that is whenever $A \cap B = 0$ and $B \leq C$, then $B = (0)$ [3].
Recall that an R-module is uniform if all its submodules are essential. An R - module M is called a prime module if (0) is a prime submodule of *M*; that is whenever
$r \in R, m \in M, rm = 0$ implies that either $m = 0$ or $r \in annM = (0:_R M)$[7].
Some known results about monoform, small monoform modules are stated, as follows:

Remarks 1.1.
1. Let R be a commutative ring and C be a right R-module. Then C is monoform if and only if C is a uniform prime module [8].
2. Let R be a ring and C be a right R-module. Then C is a non-singular monoform if and only if C is uniform [8].
3. It is clear that every monoform module is small monoform, but the converse in general is not true, for example Z_4 is small monoform Z-module but it is not monoform [2].
4. If C is a small monoform module, then C is uniform [2].
5. The epimorphic image of small monoform module is not necessarily small monoform [2].
6. Every non-zero submodule of small monoform module is small monoform [2].
7. If C is a small monoform R-module, then C is a small monoform $\bar{R}$ -module, where $\bar{R} = \frac{R}{annM}$ [2].

The following lemma give as some properties of Z -small submodules which will be used in this paper.

Lemma 1.2.
1. Let E be an R-module, let $A \leq B \leq E$. Then $B \ll_z E$ implies $A \ll_z E$ and $\frac{B}{A} \ll_z \frac{E}{A}$ [4]. We see the converse will be satisfied under the condition, if $\frac{Z_2\,(E)+A}{A} = Z_2\left(\frac{E}{A}\right)$.
2. Let $A_1, \dots, A_n$ be submodules of a module E the $A_i \ll_Z E$ $(\forall i = 1, \dots\dots, n)$ if and only if $\sum_{i=1}^{n} A_i \ll_z E$ [4].
3. Let A and B be submodules of a module E with $A \leq B$. If $A \ll_z B$, then $A \ll_z E$ [4].
4. Let $f: E_1 \longrightarrow E_2$ be an R-homomorphism and let $A \ll_z E_1$, then $f(A) \ll_z E_2$ [4].

5. Let $E = E_1 \oplus E_2$ be an R-module, $A_1 \leq E_1, A_2 \leq E_2$. Then $A_1 \oplus A_2 \ll_z E$, if and only if $A_1 \ll_z E_1$ and $A_2 \ll_z E_2$ [4].
6. Let E be a non-singular module and $H \leq E$. Then $H \ll E$ if and only if $H \ll_z E$ [4].
   Moreover, we notice that
7. If E is a singular module, then every submodule of E is Z-small.
8. For any module E, $E \ll_z E$ if and only if $Z_2(E) = E$.

2. Z-Small monoform modules

Definition 2.1. An R-module C is called Z-small monoform module (ZSM module; for short), if every nonzero partial endomorphism of C has Z-small kernel.

Remarks and Examples 2.2.
1) Every singular module (hence Z_2-torsion) is a ZSM module.
Proof. Let C be a module with $Z_2(C) = C$, let $A \leq C, A \neq (0)$.
Then $Z_2(A) = Z_2(C) \cap A = C \cap A = A$, thus A is Z_2-torsion.
Hence by Lemma 1.2 (7), every submodule of A is Z-small. So that for any non-zero homomorphism$f: A \longrightarrow C$, $Kerf \ll_z A$. Therefore, C is a ZSM module. □
In particular for each positive integer n, $\mathbb{Z}_n$ as $\mathbb{Z}$-module is a ZSM module.
2) It is clear that every small monoform module is a ZSM module, however the following example shows that the converse may be not true:
Let $E = \mathbb{Z}_{12}$ as $\mathbb{Z}$- modules. Then E is a ZSM module by part (1). Let $A = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \overline{10}\}$, and define $f: A \longrightarrow E$ by $f(\bar{0}) = f(\bar{6}) = \bar{0}$, $f(2) = f(\bar{8}) = \bar{4}$, $f(\bar{4}) = f(\overline{10}) = \bar{8}$. Then $ker\ f = \{\bar{0}, \bar{6}\}$ is not small in A. Thus E is not small monoform.
3) If C is a non-singular module (hence $Z_2(C) = 0$), then C is a ZSM module if and only if C is a small monoform module.
Proof. Let $0 \neq A \leq C$, let $f \in Home\ (A, C), f \neq 0$. Since C is non-singular, so that A is nonsingular and hence by Lemma 1.2 (6), every submodule of A is Z-small if and only if it is small. Thus $kerf \ll_z A$ if and only if $kerf \ll A$; that is C is a ZSM module if and only if C is a small monoform module. □
We conclude that each of the $\mathbb{Z}$-modules $\mathbb{Z}$, Q, and Z_{p^∞} (where p is a prime number) is a ZSM module. Also, $\mathbb{Z}_6$ as $\mathbb{Z}_6$-module is not a ZSM module, since it is not small monoform.
4) Every non-zero submodule of a ZSM module is a ZSM module.
Proof. Assume C is a ZSM module, $0 \neq A \leq C$. To prove A is a ZSM module, let $\leq A$. If $f: B \longrightarrow A$ is a homomorphism, then $i \circ f : B \rightarrow C$ where i is the inclusion mapping. As $B \leq C$, we have ker ($i \circ f$)$\ll_z$B. But ker ($i \circ f$) = $ker\ f$, so that ker $f \ll_z B$ and A is a ZSM module. □
5) If C is a ZSM module over a ring R, then it is not necessarily that C is a ZSM as $\frac{R}{annM}$-module for example the $\mathbb{Z}$-module $\mathbb{Z}_6$ is a ZSM module and $\mathbb{Z}_6$ as $\frac{Z}{6Z} \simeq \mathbb{Z}_6$-module is not a ZSM module,

Recall that an R-module C is Z-hollow if every proper submodule of C is Z-small [4].
We have the following

Proposition 2.3. If C is a semisimple Z-hollow module, then C is a ZSM module.
Proof. Let $0 \neq A \leq C, f \in Home\ (A, C), f \neq 0$. $kerf \leq A \leq C$. As C is a Z-hollow module, $kerf \ll_z C$. But $A \leq^{\oplus} C$ and $kerf \leq A$, so that $kerf \ll_z A$ by [4, Lemma 2.8]. Thus C is a ZSM module. □

Abbas, Talebi and Hadi in [9] introduced that: A submodule A of an R-module C is called Z-essential ($A \leq_{Zes} C$, for short), if $A \cap B = 0$ and $B \leq Z_2(C)$, then $B = (0)$, [9].

We say that a submodule B of an R-module C is called a Z-complement of submodule A of C if B is a maximal submodule of C with the property $A \cap B = (0)$ and $B \leq Z_2(C)$ [9].

Proposition 2.4. An R-module C is a ZSM module if and only if for each $A \leq_{Zes} C$ and for each $f \in Hom(A, C), f \neq 0$, then $erf \ll_z A$.
Proof. (⇒) It is clear.
(⇐); Let $0 \neq A \leq C$ and $f \in Hom\ (A, C), f \neq 0$.
If $A \leq_{Zes} C$, then nothing to prove. If $A \nleq_{Zes} C$, then there exists $B \leq C$, B is a Z-complement of C. Then $A \oplus B \leq_{Zes} C$. Define $g: A \oplus B \longrightarrow C$ by $g(a + b) = f(a)$, $\forall a \in A, b \in B$.
Then $g \neq 0$ and so that $kerg \ll_z A \oplus B$. But $kerg = kerf \oplus B$. $ker\ f \oplus B \ll_z A \oplus B$, which implies $kerf \ll_z A$ by Lemma 1.2(5). □

Corollary 2.5. If C is a prime R-module with $Z_2(C) \neq 0$, then C is a ZSM module.
Proof. Let $0 \neq A \leq_{Zes} C$ and $f \in Hom(A, C), f \neq 0$. Assume that $Z_2(A) = (0)$. Since $Z_2(A) = A \cap Z_2(C)$, we have $A \cap Z_2(C) = (0)$. But $A \leq_{Zes} C$, so that $Z_2(C) = (0)$ which is a contradiction. Thus $Z_2(A) \neq (0)$. Also, A is a prime module (since $A \leq C$). Hence by [10, Proposition 2.1.11] every submodule of A is Z-small, so that $kerf \ll_z A$. Thus C is a ZSM module. □

The following is a characterization of a ZSM module in the class of Noetherian modules. But first recall that a submodule of a module is called 3-generated submodule if it is generated by 3- elements.
Theorem 2.6. Let C be a non-zero Noetherian R-module. Then C is a ZSM module if and only if each non-zero 3-generated submodule of C is a ZSM module.
Proof. (⇒) It is clear.
(⇐) Let $0 \neq A \leq C$ and let $f \in Hom(A, C), f \neq 0$. To show $kerf \ll_z A$. If $kerf = 0$, then nothing to prove. If $kerf \neq 0$, let $a \in kerf, a \neq 0$, $b \in A$ and $f(b) = c$. Put $L = < a, b, c >$, so L is a ZSM module by hypothesis. Let $H = < a, b >$ and $g = f|_H: H \longrightarrow L$, hence $kerg \ll_z H \leq A$ and so $kerg \ll_z A$. But $a \in kerf$ implies $a \in kerg$, hence $< a > \subseteq kerg \ll_z A$ for any $a \in kerf$. Since M is Noetherian, $kerf = Ra_1 + \cdots + Ra_n$ for some $a_1, \ldots, a_n \in A$. As $\langle a_i \rangle \ll_z A$ for each $i = 1, \ldots, n$, so $kerf = \sum_{i=1}^{n} Ra_i \ll_z A$ by Lemma 1.2(3). Thus C is a ZSM module. □

Recall that M is called quasi-Dedekind (respectively, small quasi-Dedekind), if for each $f \in End(M), f \neq 0, Ker\ f = 0$ ($kerf \ll M$), respectively [11], [12].
It is known that every small monoform is small quasi-Dedekind [2]. M is called Z-small quasi-Dedekind if for each $f \in End(M), f \neq 0$, $kerf \ll_z M$ [13].

Remark 2.7. Every a ZSM module is Z-small quasi-Dedekind.

Recall that an R-module C is called fully retractable module, if for every $0 \neq A \leq C$ and every $g \in Hom(A, C), g \neq 0$, then $Hom(C, A)g \neq 0$ [7].

Proposition 2.8. Let C be a fully retractable R-module such that for each$0 \neq A \leq C$, A is Z-small quasi-Dedekind. Then M is a ZSM module.

Proof. Let $0 \neq A \leq C, f \in Hom(A, C), f \neq 0$. As C is fully retractable, $Hom(C, A)\, f \neq 0$. Then there exists $g \in Hom(C, A)$ with $gof \neq 0$. As A is Z-small quasi-Dedekind, $ker\ gof \ll_z A$. But $kerf \subseteq ker\ gof$, so that $kerf \ll_z A$. □

I.M.A. Hadi and K. H. Marhoon proved that: Let M be a quasi-injective cosemisimple R-module. Then M is small quasi-Dedekind if and only if M is small monoform [14, Proposition 1.1.11]. We state and prove an analogue result, but first,

Recall that a submodule A of a module C is Z-coclosed if whenever $B \leq A, \frac{A}{B} \ll_z \frac{C}{A}$ then $A = C$ [10].

Definition 2.9. An R-module C is called Z-cosemisimple if every submodule of C is Z-coclosed.

It is clear that every Z-coclosed submodule is coclosed. Hence every Z- cosemisimple is cosemisimple, but $\mathbb{Z}_6$ as $\mathbb{Z}$ -module is cosemisimple but it is not Z-cosemisimple.

Proposition 2.10. Let C be a quasi-injective and Z-cosemisimple module. Then M is Z-small quasi-Dedekind if and only if M is a ZSM module.

Proof. (⇒) Let $0 \neq A \leq C, f \in Hom(A, C), f \neq 0$. Since M is quasi-injective, there exists $g \in EndC$, such that $g \circ i = f$, where i is the inclusion mapping $i: A \longrightarrow C$. Hence $g(a) = f(a), \forall a \in A$, which implies that $kerf \subseteq ker\, g$. But C is Z-small quasi-Dedekind, so $kerg \ll_z C$. This implies $kerf \ll_z C$. As $kerf \subseteq A$ and A is Z-coclosed (since C is Z-cosemisimple), so that by [10, Proposition, 2.2.17], $kerf \ll_z A$. Thus M is a ZSM module.

(⇐) It follows from Remark 2.7. □

Recall that an R-module C is called retractable if for each $0 \neq A \leq C, Hom(C, A) \neq 0$.

Proposition 2.11. Let C be a nonsingular retractable R-module. Then the following statements are equivalent.

1) C is a monoform module.
2) C is a small monoform module.
3) C is a uniform module.
4) C is compressible (i.e., for each $A \leq C, A \neq 0$, there exists a monomorphism $f: C \longrightarrow A$) [15].
5) C is a ZSM module.

Proof. (1)⇔(2) ⇔(3) ⇔(4), see [14, Proposition, 1.2.9].

(2) ⇔(5) it follows from Remarks and Examples 2.2(3).

Recall that an R-module M is called multiplication R-module if for each $N \leq M$, $N = MI$ for some $I \leq R$ [16]. □

Proposition 2.12. If M is a faithful finitely generated multiplication module over a principle ideal ring R. If M is a ZSM module, then R is a ZSM ring.

Proof. Let $0 \neq I \leq R$, $f \in Hom(I, R), f \neq 0$. Since R is a principle ideal ring, $I = \langle a \rangle$ for some $a \in R$. Let $N = Ma$. Define $g: N \longrightarrow M$ by $g(ma) = mf(a)$, g is a well-defined and homomorphism. It is easy to see that $M\ kerf \subseteq kerg$. But $kerg \ll_z N$, since M is a ZSM module. Hence $M\ kerf \ll_z N$. To prove $kerf \ll_z I = \langle a \rangle$. Let $kerf + \langle b \rangle = Ra$ and $\langle b \rangle \supseteq Z_2(R)$. Then $M\ kerf + M < b >= M < a >$. But $M\langle b \rangle \supseteq MZ_2(R) = Z_2(M)$. Hence $M\langle b \rangle = Ma$, since $M\ kerf \ll_z Ma$. As M is a faithful finitely generated R-module, then $\langle b \rangle = \langle a \rangle$. Thus $kerf \ll_z \langle a \rangle$. □

Corollary 2.13. Let M be a cyclic faithful module over a principle ideal ring R. If M is a ZSM module, then R is ZSM ring.

Remark 2.14. The direct sum of a ZSM modules need not be a ZSM module, for example: $M = \mathbb{Z}_4$ as $\mathbb{Z}_4$-module is a ZSM module, let $L = \mathbb{Z}_4 \oplus \mathbb{Z}_4$ as $\mathbb{Z}_4$-module and let $f: \mathbb{Z}_4 \oplus < \underline{2} > \longrightarrow L$, defined by $f\left(\underline{x}, \underline{y}\right) = \left(\underline{0}, \underline{y}\right), \forall \left(\underline{x}, \underline{y}\right) \in \mathbb{Z}_4 \oplus < \underline{2} >$, then $f \neq 0$, and $ker\, f = \mathbb{Z}_4 \oplus \left(\underline{0}\right)$. But $\mathbb{Z}_4 \oplus (0) \ll_z \mathbb{Z}_4 \oplus < \underline{2} >$, since $\mathbb{Z}_4$ is not Z-small in $\mathbb{Z}_4$, since $Z_2(\mathbb{Z}_4) \neq \mathbb{Z}_4$ (see Lemma 1.2(8)). Thus L is not a ZSM module.

Recall that an R-module is called fully stable if for each $N \leq M$, N is stable; that is for each $f \in Hom(N, M), f(N) \subseteq N$, see [17].

Theorem 2.15. Let M be a fully stable R-module such that $M = M_1 \oplus M_2$, M_1 and M_2 are submodules of M, and for each R-homomorphism. $f: H_1 \oplus H_2 \longrightarrow M$, $f \neq 0$ $(H_1 \leq M_1, H_2 \leq M_2), f(H_1) \neq 0, f(H_2) \neq 0$. Then M_1 and M_2 are ZSN modules if and only if M is a ZSM module.

Proof. Let $H \leq M, H \neq 0, f \in Hom(H, M), f \neq 0$. To prove $Kerf \ll_z H$. Since M is fully stable, H is stable and so that $H = (H \cap M_1) \oplus (H \cap M_2)$ [17, Proposition 4.5, p 29].

Consider $H \cap M_1 \ i_1 \rightarrow \ H \ f \rightarrow \ M \ \rho_1 \rightarrow \ M_1$

$H \cap M_2 \ i_2 \rightarrow \ H \ f \rightarrow \ M \ \rho_2 \rightarrow \ M_2$

Where i_1, i_2 are inclusion mappings and ρ_1, ρ_2 are projection mappings. Then $\rho_1 \circ f \circ i_1: H \cap M_1 \longrightarrow M_1$ and $\rho_2 \circ f_2 \circ i_2: H \cap M_2 \longrightarrow M_2$. Put $H_1 = H \cap M_1$, $H_2 = H \cap M_2$. By hypothesis, $f(H_1) \neq 0$, so there exists $x_1 \in H \cap M_1, x_1 \neq 0$, $f(x_1) \neq 0$. Similarly, there exists $x_2 \in H \cap M_2, x_2 \neq 0$ and $f(x_2) \neq 0$. On the other hand, $f \circ i_1(x_1) = f(x_1) \neq 0$ and $f \circ i_2(x_2) = f(x_2) \neq 0$. Since H_1 and H_2 are stable, $f(H_1) \subseteq H_1$ and $f(H_2) \subseteq H_2$. But $f(x_1) \in H_1, f(x_1) \neq 0$, so that $\rho_1 \circ f \circ i_1(x_1) = f(x_1) \neq 0$. Similarly, $\rho_2 \circ f \circ i_2(x_2) = f(x_2) = 0$. Thus $\rho_1 \circ f \circ i_1 \neq 0$ and $\rho_2 \circ f \circ i_2(x_1) \neq 0$. As M_1 and M_2 are ZSM modules, then $ker(\rho_1 \circ f \circ i_1) \oplus ker(\rho_2 \circ f \circ i_2) \ll_z H_1 \oplus H_2 = H$. Let $x = \acute{x}_1 + \acute{x}_2 \in kerf$ where $\acute{x}_1 \in H_1$ and $\acute{x}_2 \in H_2$, hence $f(\acute{x}_1) + f(\acute{x}_2) = 0$, and so $f(\acute{x}_1) = -f(\acute{x}_2) \in H_1 \cap H_2 = 0$ and so $\rho_1 \circ f \circ i_1(\grave{x}_1) = \rho_1 \circ f(\acute{x}_1) = f(\acute{x}_1) = 0$. Also $\rho_2 \circ f \circ i_2(\grave{x}_2) = f(\acute{x}_2) = 0$. Hence $\acute{x}_1 + \acute{x}_2 = x \in ker(\rho_1 \circ f \circ i_1) \oplus ker(\rho_2 \circ f \circ i_2) \ll_z H$. So that, $kerf \subseteq ker(\rho_1 \circ f \circ i_1) \oplus ker(\rho_2 \circ f \circ i_2) \ll_z H$. Thus $kerf \ll_z H$ and M is a ZSM module.

(⇐) It is clear from Remarks and Examples 2.2(4). □

Conclusion

Most of properties of a ZSM module are analogous to that of small monoforms. However , if *C* is a small monoform *R*-module, then *C* is a small monoform *R*/ann *M*-module by [14, Remarks and Examples 1.1.2(5)], but this property can't be transfer to ZSM modules, see Remarks and Examples 2.2.(5).

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